Edgeworth's conjecture that as the number of traders in an exchange economy increases the core approaches the set of competitive equilibria has been formalized both as a theorem about a sequence of finite economies, and as a theorem about an economy having an infinite number of agents.

This paper, using nonstandard analysis, provides a synthesis of these two approaches. It is shown that the core and the set of competitive equilibria are equivalent within a nonstandard exchange economy. This theorem implies an asymptotic theorem concerning the core and competitive equilibria of sequences of finite economies.

1. INTRODUCTION

An exchange economy consists of a set of traders, each of whom is characterized by an initial endowment and a preference relation. In addition, one usually assumes that the set of traders is finite. However, in order to state theorems precisely concerning the asymptotic or limiting properties of the core (such theorems will be called limit theorems), economies have been studied which have an infinite number of traders. For instance, there are the denumerable economies of Debreu-Scarfi [3] and the nondenumerable economies of Aumann [1].

The concepts of interest, here the core and competitive equilibrium, can be defined even in infinite economies. Hence, one has the option of taking an imprecise statement about the core, such as Edgeworth's conjecture that "the core approaches the set of competitive allocations as the number of traders increases" and translating it into a theorem, $T'$, about the core of an infinite economy or expressing it as a limit theorem, $T$, about the cores of large but finite economies.

Of course, these two approaches need not be incompatible in that one might be able to define a limiting process such that a statement $T$ about the asymptotic behavior of a family of large but finite economies is reflected in a statement $T'$ about a certain infinite economy, in the sense that $T$ is true if and only if $T'$ is true. In this context Edgeworth's conjecture gives rise to two theorems, $T$ and $T'$.

Although we are primarily interested in the truth of $T$, we shall establish $T$ by proving $T'$ and by showing that $T$ is a consequence of a general metamathematical argument. Because of this argument, it is not surprising that an area of mathematical logic, model theory, provided an appropriate framework for defining the limiting process described above.

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1 This research was supported in part by the National Science Foundation (GP18728) and the Office of Naval Research. We are happy to acknowledge several stimulating conversations with H. Scarfi. Also we would like to acknowledge Aumann's seminal paper [1] in that the idea of the proof of Theorem 1 is based on his paper.
The fundamental result that we will need from model theory is the existence of a particular kind of extension of the real numbers, $\mathbb{R}$, called the nonstandard numbers and denoted $\ast \mathbb{R}$. $\ast \mathbb{R}$ has all the formal properties of $\mathbb{R}$ in that any mathematical property of $\mathbb{R}$ which can be described in some given language, which may be the first order predicate calculus or some higher language, can be translated into a mathematical property of $\ast \mathbb{R}$. Moreover, the sentence, $\varphi$, expressing the fact that this property holds for $\mathbb{R}$ is true if and only if its interpretation in $\ast \mathbb{R}$. In addition, $\ast \mathbb{R}$ contains nonzero infinitesimals, i.e., nonzero numbers whose absolute value is less than any positive real number, and infinite integers, i.e., integers which are greater than every real integer.

It is the existence of infinitesimals and infinite integers that permits us to formulate precisely the notion of perfect competition, the economic concept which underlies the Edgeworth conjecture.

The idealized notion of perfect competition is that the action of any finite set of traders in terms of their willingness to buy or sell at the competitive prices should have a negligible effect on these prices. Clearly this cannot be the case for any finite economy, but perfect competition is a meaningful concept for an economy having $\omega$ traders, where $\omega$ is an infinite integer, and where the endowment for each trader relative to that of the whole economy is infinitesimal.

After defining a nonstandard exchange economy and the notions of the core and competitive equilibrium in such an economy, we will prove that Edgeworth's conjecture is true in the following sense:

**Theorem 1:** If $\mathcal{E}$ is a nonstandard exchange economy, then the allocation $X$ is in the core of $\mathcal{E}$ if and only if there exists a price vector, $\bar{p}$, such that $\langle \bar{p}, X \rangle$ is a competitive equilibrium.

We then define competitive equilibrium and core allocations for a family of large but finite exchange economies, $\mathcal{G}$. Allocations, price vectors, and preference relations for $\mathcal{G}$ consist of sequences of allocations, price vectors, and preference relations belonging to the economies which make up $\mathcal{G}$.

We show that:

**Theorem 2:** The allocation $X$ is in the core of $\mathcal{G}$ if and only if there exists a price vector, $\bar{p}$, such that $\langle \bar{p}, X \rangle$ is a competitive equilibrium for $\mathcal{G}$.

Theorems 1 and 2 correspond to $T'$ and $T$ discussed earlier. Theorem 2 is another formulation of Edgeworth's conjecture, but in terms of the relationship between core allocations and equilibria of large but finite economies.

2. Definitions and Assumptions

The basic mathematical notions that we will need will come from an area of model theory which is termed nonstandard analysis. An informal introduction to these ideas follows. The interested reader may consult [6, 7, and 8] for details.
Let $R$ be the field of real numbers. We consider the properties of $R$ in a higher-order language, $L$, which includes symbols for all individuals (that is, numbers) of $R$, all subsets of $R$, all relations of two, three, $\ldots$, variables between individuals of $R$, all functions from individuals to individuals of $R$, and, more generally, all relations and functions of a finite type (for example, functions from sets of numbers into relations between numbers) that can be defined beginning with the numbers of $R$. Let $K$ be the set of all sentences formulated in $L$ which hold (are true) in $R$. Then there exists a structure $*R$ with the following properties:

2.1: $*R$ is a model of $K$ "in Henkin's sense." That is, all sentences of $K$ hold in $*R$, provided, however, that we interpret all quantifiers other than those referring to individuals in a nonstandard fashion, as follows. Within the class of all entities of any given type other than 0 (the type of individuals) there is distinguished a certain subclass of entities called internal. And, for such a type, the quantifiers "for all $x$" and "there exists an $x$" are to be interpreted as "for all internal $x$," "there exists an internal $x$" (of the given type). Such sentences are said to be true in $*R$ by transfer. $R$ can be injected into $*R$, and so $*R$ may, and will, be regarded as an extension of $R$.

2.2: Every concurrent binary relation $S(x, y)$ in $R$ possesses a bound in $*R$.

The relation $S(x, y)$, of any type is called concurrent if for any $a_1, \ldots, a_n \geq 1$, for which there exists $b_1, \ldots, b_n$ such that $S(a_1, b_1), \ldots, S(a_n, b_n)$ hold in $R$, there also exists a $b$ such that $S(a_1, b), \ldots, S(a_n, b)$ hold in $R$. The entity $b$ in $*R$ is called a bound for $S(x, y)$ if $S(a, b)$ holds in $*R$ for every $a$ for which there exists a $b$ such that $S(a, b)$ holds in $R$.

2.3: For any mapping $f(y)$ from a set $A$ (of any type) in $R$ into the extension $*B$ of a set $B$ in $R$, there exists an internal mapping $\varphi(y)$ from $*A$, the extensions of $A$, into $*B$ which coincides with $f(y)$ on $A$.

Any structure $*R$ which satisfies 2.1, 2.2, and 2.3 is called a comprehensive enlargement of $R$. In particular $*R$ can be constructed as an ultrapower of $R$. (See [4] and [5] for the notions of an ultrapower.) It is shown in [7] that $*R$ has the following properties:

(i) The set of individuals of $*R$ is an ordered non-archimedean field. In this paper we shall call all individuals of $*R$ non-standard numbers.

(ii) $R$, the real numbers, is a proper ordered subfield of the non-standard numbers. Elements of $R$ will be called standard numbers. Observe that according to our present usage every standard number is a non-standard number.

(iii) There exist nonstandard numbers, in particular nonstandard integers, which are greater than every standard number. These numbers are called infinite nonstandard numbers and infinite integers respectively.

(iv) There exist numbers in $*R$ which are in absolute value less than any positive real number, in fact they, except zero, are just the reciprocals of the infinite non-standard numbers. These numbers are called infinitesimals.
(v) The system of internal entities in \( *R \) has the following property: If \( S \) is an internal set of relations, then all elements of \( S \) are internal. More generally, if \( S \) is an internal \( n \)-ary relation, \( n \geq 1 \) and the \( n \)-tuple \((S_1, \ldots, S_n)\) satisfies (belongs to) \( S \), then \( S_1, \ldots, S_n \) are internal.

A nonstandard number is said to be finite if in absolute value it is less than some standard number. If \( r \) is a finite nonstandard number, then there exists a unique standard number called the standard part of \( r \), denoted by \( 0^*r \), such that \( r - 0^*r \) is an infinitesimal. Hence there are three kinds of nonstandard numbers: (i) the infinitesimals which in absolute value are less than every standard number, (ii) the finite nonstandard numbers which in absolute value are less than some standard number, and (iii) the infinite nonstandard numbers which are greater than every standard number. Note that according to this taxonomy every infinitesimal is a finite nonstandard number.

We will denote the \( n \)-dimensional vector space over \( *R \) by \( *R^n \). A commodity bundle \( \bar{x} \) is a point in the nonnegative orthant \( *\Omega^n \) of \( *R^n \). Let \( *\rho \) be the nonstandard extension of the Euclidean metric \( \rho \). Then a set of points \( S(\bar{x}, r) \) in \( *R^n \) will be called an \( S \)-ball if there exists \( \bar{y} \in *R^n \) and a positive standard number \( r \) such that \( S(\bar{x}, r) = \{ \bar{y} : *\rho(\bar{x}, \bar{y}) < r \} \). It is shown in [7] that the \( S \)-balls may serve as a basis for a topology in \( *R^n \). This topology will be called the \( S \)-topology.

The monad of \( \bar{x}, \mu(\bar{x}) \), is the set of points whose distance from \( \bar{x} \) is an infinitesimal. If \( \bar{y} \in \mu(\bar{x}) \), we shall write \( \bar{x} \simeq \bar{y} \). \( \bar{x} \succ \bar{y}, \bar{x} \succeq \bar{y}, \) and \( \bar{x} \succeq \bar{y} \) will have their conventional meanings: \( \bar{x} \succeq \bar{y} \) means that \( \bar{x} \) is greater than \( \bar{y} \) or \( \bar{y} \) exceeds \( \bar{x} \) by at most an infinitesimal amount in each coordinate; and \( \bar{x} \succeq \bar{y} \) means that \( \bar{x} \) is greater than \( \bar{y} \) by a noninfinitesimal amount in each coordinate.

The \( S \)-convex hull of a set \( B \) is defined as the set of all finite convex combinations of elements of \( B \), i.e., vectors of the form \( \bar{y} = \alpha_1 \bar{x}_1 + \cdots + \alpha_n \bar{x}_n \) where \( \bar{x}_i \in B \), \( \alpha_i \geq 0 \), \( \sum_{i=1}^n \alpha_i = 1 \), the \( \alpha_i \) are nonstandard numbers, and \( n \) is a standard integer. \( B \) is defined as \( S \)-convex if \( B \) contains its \( S \)-convex hull.

A nonstandard exchange economy is defined as a pair of indexed sets, \( \{\bar{x}_t\}_{t=1}^\omega \) and \( \{\succ_t\}_{t=1}^\omega \) where for all \( t, \bar{x}_t \in *\Omega^n \), \( \succ_t \subseteq *\Omega^n \times *\Omega^n \), and \( \omega \) is an infinite (nonstandard) integer. Interpret \( \bar{x}_t \) as the initial endowment of the \( t \)-th trader and interpret \( \succ_t \) as his preference relation. The nonstandard exchange economies which we will consider are assumed to have the following properties:

(i) The function indexing the initial endowments, \( I(t) \), is internal.
(ii) \( I(t) \) is standardly bounded, i.e., there exists a standard vector \( \bar{r}_0 \) such that for all \( t, I(t) \leq \bar{r}_0 \).
(iii) \( (1/\omega) \sum_{t=1}^{\omega} I(t) \geq 0 \).
(iv) The relation, \( Q \), where \( Q = \{ (t, \succ_t) : t \in T, \succ_t \subseteq *\Omega^n \times *\Omega^n \} \) is internal.
(v) \( \succ_t \) is irrelexive.
(vi) \( \succ_t \) is monotonic.
(vii) For all finite \( \bar{x}, \bar{y} \in *\Omega^n \), if \( \bar{x} \simeq 0 \), then for every \( \bar{w} \simeq \bar{x} + \bar{y}, \bar{z} \simeq \bar{y}, \bar{w} \succeq \bar{z} \), where \( \bar{x} \simeq 0 \) means that \( \bar{x} \) is greater than \( 0 \) by a noninfinitesimal amount in at least one component.

An assignment is an internal function from \( T \); the set of traders, into \( *\Omega^n \).

An allocation or final allocation is a standardly bounded assignment \( Y(t) \) from
the set of traders, \(\{1, 2, \ldots, \omega\}\), into \(\Omega_n^*\) such that

\[
\frac{1}{\omega} \sum_{t=1}^{\omega} Y(t) \approx \frac{1}{\omega} \sum_{t=1}^{\omega} I(t).
\]

Condition (iv) implies for all internal \(X(t)\), \(Y(t) \in \Omega_n^T\), where \(T\) is the set of traders, that:

(v) \(\{t | X(t) \gg Y(t)\}\) is an internal set of traders.

A coalition, \(S\), is defined as an internal set of traders. It is said to be negligible if \(|S|/\omega \approx 0\). Note that if \(S\) is negligible then for all allocations

\[
X(t), \frac{1}{\omega} \sum_{t \in S} X(t) \approx 0.
\]

A coalition, \(S\), is effective for an allocation \(Y\) if \((1/\omega) \sum_{t \in S} Y(t) \approx (1/\omega) \sum_{t \in S} I(t)\).

\[
\bar{x} \gg \bar{y} \text{ iff } (\forall \bar{w} \in \mu(\bar{x})) \bar{w} \gg \bar{y}.
\]

An allocation \(Y\) dominates an allocation \(X\) via a coalition \(S\) if for all \(t \in S, Y(t) \gg X(t)\), and \(S\) is effective for \(Y\).

The core is defined as the set of all allocations which are not dominated by any non-negligible coalition.

A price vector, \(\bar{p}\), is a finite nonstandard vector such that \(\bar{p} \gg 0\).

The \(r\)th traders budget set, \(B_r(t)\), is \(\{\bar{x} \in \Omega_n^* | \bar{p} \cdot \bar{x} \leq \bar{p} \cdot I(t)\}\).

We say that \(\bar{y}\) is maximal in \(B_r(t)\) if \(\bar{y} \in B_r(t)\) and there does not exist an \(\bar{x} \in B_r(t)\) such that \(\bar{x} \gg \bar{y}\).

A competitive equilibrium is defined as a pair \((\bar{p}, X)\), where \(\bar{p}\) is a price vector and \(X\) an allocation such that there exists an internal set of traders \(K\) for which \(|K|/\omega \approx 1\) and \(X(t)\) is maximal in \(B_r(t)\) for all \(t \in K\).

3. THEOREMS

We shall first demonstrate that there exist nonstandard economies satisfying the assumptions we wish to make concerning the initial endowments and preferences of the traders.

Let \(f\) be a monotonic continuous function of \(\Omega_n\) into \(R\) and define the relation \(\gg_f\) on \(\Omega_n\) as \(\bar{x} \gg_f \bar{y}\) if \(f(\bar{x}) > f(\bar{y})\) for all \(\bar{x}, \bar{y} \in \Omega_n^*\). Then it is shown in [7] that \(f: D \rightarrow R\) is \(n\)-continuous where \(f\) is the nonstandard extension of \(f\) and \(D\) a standardly bounded subset of \(\Omega_n^*\). Now \(f\) induces a preference relation \(\gg_f\) on \(\Omega_n^*\) where for all \(\bar{x}, \bar{y} \in \Omega_n^*\), \(\bar{x} \gg_f \bar{y}\) if \(f(\bar{x}) > f(\bar{y})\). Note that \(\gg_f = \approx(\gg_f)\).

It is obvious that \(\gg_f\) is irreflexive and monotonic. Suppose \(\bar{x}, \bar{y}\) are finite vectors in \(\Omega_n^*\) and \(\bar{x} \gg 0\). Then \(f(\bar{x} + 0\bar{y}) = f(\bar{x} + 0\bar{y}) > f(0\bar{y}) = f(0\bar{y})\).

Since \(f(0\bar{x} + 0\bar{y})\) and \(f(0\bar{y})\) are standard numbers and \(f(0\bar{y}) \approx f(0\bar{y}) \approx f(0\bar{y} + \bar{y})\); the continuity of \(f\) implies that assumption (iv) \((\gamma)\) is satisfied.

An example of a function having all of the above properties is \(f(\bar{x}) = x_1 + \ldots + x_n\) for all \(x \in \Omega_n\). Hence \(f(\bar{x}) = x_1 + \ldots + x_n\) for all \(x \in \Omega_n^*\).
CONSISTENCY LEMMA: Let $G$ be a countably infinite standard economy which has the following properties for all traders, $t$, in $G$:

(i) There exists a standard vector $\bar{r}_0$ such that $I(t) \leq \bar{r}_0$.

(ii) $I(t) > 0$.

(iii) $\succ_t = \succ_f$, where $f$ is a monotonic continuous function of $\Omega_n$ into $R$ and for all $\bar{x}, \bar{y} \in \Omega_n$ we use the same $f$ for every $t$. If $\bar{x} \succeq \bar{y}$, then $f(\bar{x}) > f(\bar{y})$.

Then there exists a nonstandard exchange economy $G$ satisfying the assumptions of Section (2).

PROOF: Let $N$ be the nonnegative standard integers; then $I: N \to R_n$. Thus its nonstandard extension $\ast I$ maps $\ast N$ into $\ast R_n$. $\ast I$ restricted to $\{1, 2, \ldots, \omega\}$ for any infinite integer $\omega$ is a function which satisfies assumptions (i), (ii), and (iii) in Section (2). We then assign the same preference relation $\succ_\gamma$ to each of the traders $1, 2, \ldots, \omega$. This provides the desired model.

In our analysis of a nonstandard exchange economy the existence of an equilibrium price vector will be established by invoking a separating hyperplane theorem. Since our notions of convexity and topology are not internal notions, we cannot prove separating hyperplane theorems in $\ast R_n$ by simple "transfer" from $R_n$.

A set $D$ in $\ast R_n$ is said to be nearstandard if every vector in $D$ is finite. We shall establish a separation theorem for nearstandard $S$-convex sets. The $S$-interior of a subset, $A$, of $\ast R_n$ is the union of all the subsets of $A$ which are open in the $S$-topology. The $S$-closure of $A$ is the complement of the $S$-interior of the complement of $A$.

LEMMA 1: If $A$ is a subset of $\ast R_n$ and $\bar{0} \notin S$-interior of $A$, $S$-int$(A)$, then for all $\bar{x} \in S$-int$(A)$, $\bar{\mu}(0) \cap \bar{\mu}(x) = \emptyset$.

PROOF: Suppose there exists an $\bar{x} \in S$-int$(A)$ such that $\mu(0) \cap \mu(\bar{x}) \neq \emptyset$. Then $\ast \mu(0, \bar{x}) \simeq 0$. But $\mu(\bar{x}) \subseteq S(\bar{x}, r) \subseteq A$ for some standard $r$. Hence $S(0, r) \subseteq A$, a contradiction.

LEMMA 2: If $A$ a subset of $\ast R_n$ and $\bar{0} \notin S$-int$(A)$, then $\bar{0} \notin S$-int$(A)$, $S$-int$(A) = \{0\}$.

PROOF: If $\bar{0} \in S$-int$(A)$, then there exists an $\bar{x} \in S$-int$(A)$ such that $\bar{0} \in \mu(\bar{x})$, i.e., $\mu(\bar{x}) = \mu(0)$. But this contradicts Lemma 1.

To show that the $S$-interior of an $S$-convex set in $\ast R_n$ is $S$-convex, we will adapt a well-known proof that the interior of a convex set in $R_n$ is convex [9].

Let $S_\delta(\bar{x})$ denote the closed $S$-ball centered on $\bar{x}$ with radius $\delta$ and $B = S(0, 1)$; then $S_\delta(\bar{x}, \bar{y}) = \bar{x} + \delta B$. The $S$-closure, $S$-cl$(A)$, and $S$-int$(B)$ can then be expressed as $S$-cl$(A) = \cap\{A + \delta B|\delta \geq 0\}$ and $S$-int$(A) = \{\bar{x} \text{ there exists a } \delta \geq 0, x + \delta B \subseteq A\}$. 
LEMMA 3: Let $A$ be a $S$-convex set in $R_n$. Let $\bar{x} \in S\text{-}int(A)$ and $\bar{y} \in S\text{-}cl(A)$. Then $(1 - \lambda)\bar{x} + \lambda\bar{y}$ belongs to $S\text{-}int(A)$ for $\lambda$ such that $0 \leq \lambda \leq 1$.

PROOF: It is sufficient to prove the theorem for standard $\lambda$. Suppose $\lambda$ standard and $\epsilon \in \{0, 1\}$; then we must show that $(1 - \lambda)\bar{x} + \lambda\bar{y} + \delta B \subseteq A$ for some standard $\delta > 0$. We have $\bar{y} \in A + \delta B$ for all standard $\delta > 0$, because $\bar{y} \in S\text{-}cl(A)$. Thus, for every standard $\delta > 0$, $(1 - \lambda)\bar{x} + \lambda\bar{y} + \delta B \subseteq (1 - \lambda)\bar{x} + \lambda(A + \delta B) + \delta B = (1 - \lambda)[\bar{x} + \delta(1 + \lambda)(1 - \lambda)^{-1}B] + \lambda A$. But $\bar{x} + \delta(1 + \lambda)(1 - \lambda)^{-1}B \subseteq A$ for sufficiently small $\delta$. Hence, $(1 - \lambda)\bar{x} + \lambda\bar{y} + \delta B \subseteq (1 - \lambda)A + \lambda A = A$ for some standard $\delta > 0$.

LEMMA 4: If $A$ is a neareststandard $S$-convex set in $R_n$, then $S\text{-}int(A)$ is $S$-convex.

PROOF: Take $\bar{y}$ to be in $S\text{-}int(A)$ in Lemma 3, for $0 \leq \lambda \leq 1$. For $1 \approx \lambda, (1 - \lambda)\bar{x} + \lambda\bar{y} \approx \bar{y}$. But $y \in S\text{-}int(A)$; hence we are done.

LEMMA 5: If $A$ is an $S$-convex set in $R_n$, then $0^\cdot A$ is a convex set in $R_n$.

PROOF: Suppose $\bar{x}, \bar{y} \in 0^\cdot A$, and $\alpha$ a standard number such that $0 \leq \alpha \leq 1$. Then there exist infinitesimals $\varepsilon_1, \varepsilon_2$ such that $\bar{x} + \varepsilon_1, \bar{y} + \varepsilon_2 \in A$. But $A$ is $S$-convex which implies that $\alpha(\bar{x} + \varepsilon_1) + (1 - \alpha)(\bar{y} + \varepsilon_2) \in A$. We can express $\alpha(\bar{x} + \varepsilon_1) + (1 - \alpha)(\bar{y} + \varepsilon_2)$ as $\alpha\bar{x} + (1 - \alpha)\bar{y} + \varepsilon_3$, where $\varepsilon_3 = \alpha\varepsilon_1 + (1 - \alpha)\varepsilon_2 \approx 0$. Thus $\alpha\bar{x} + (1 - \alpha)\bar{y} \in 0^\cdot A$.

S-SEPARATION LEMMA: If $A$ is a neareststandard $S$-convex set in $R_n$ and $0 \notin S\text{-}int(A)$, then there exists a standard $\bar{p} \neq 0$ such that for all $y \in S\text{-}int(A), \bar{p} \cdot y \geq 0$.

PROOF: By Lemma 4 $S\text{-}int(A)$ is $S$-convex. By Lemma 5 $0^\cdot (S\text{-}int(A))$ is a convex set in $R_n$. By Lemma 2 $0 \notin 0^\cdot (S\text{-}int(A))$. Hence, there exists a $\bar{p} \in R_n$ such that $\bar{p} \neq 0$ and for all $\bar{x} \notin 0^\cdot (S\text{-}int(A)), \bar{p} \cdot \bar{x} \geq 0$ (see [2] for a proof of this result). Every $y \in S\text{-}int(A)$ can be expressed as $\bar{y} = \bar{x} + \varepsilon$ where $\bar{x} \in 0^\cdot (S\text{-}int(A))$ and $\varepsilon \approx 0$. Thus, for all $y \in S\text{-}int(A)$, we have that $\bar{p} \cdot y = \bar{p} \cdot \bar{x} + \bar{p} \cdot \varepsilon \geq 0 + \bar{p} \cdot \varepsilon \approx 0$, i.e., $\bar{p} \cdot y \geq 0$.

Let $\delta$ be the nonstandard exchange economy where the set of traders is an internal set $T$ and $|T| = \omega$, an infinite integer. A set of traders $U$ is said to be full if $U$ is internal and $|U|/\omega \approx 0$. Let $X(t)$ be a fixed allocation in the core of $\delta$; then $F^{10}_o = \{x \in *\Omega_o \mid x$ finite, $(\forall \bar{w} \in S_{1,0}(\bar{x} + I(t)\bar{w} \succ_i X(t))\}$. $G_o(t) = F_o(t) - I(t)$, $G(t) = \bigcup_{x \in X} G_o(t)$, and $F(t) = \bigcup_{x \in X} F_o(t)$. Let $A(U)$ denote the $S$-convex hull of the union $\bigcup_{t \in T} G(t)$. Note that $F(t)$ and $G(t)$ are $S$-open.

PRINCIPAL LEMMA: There exists a full set of traders $U$ such that $0 \notin S\text{-}int(A(U))$.

PROOF: For each finite $\bar{x} \in *R_n$, let $G^{-1}(\bar{x})$ be the set of all traders $t$ for which $G(t)$ contains $\bar{x}$. Then $G^{-1}(\bar{x}) = \bigcup_{x \in X} G^{-1}(\bar{x})$, where $G^{-1}(\bar{x}) = \{t \in T \mid (\forall \bar{w} \in S_{1,0}(\bar{x} + I(t)\bar{w} \succ_i X(t))) \text{ and } \bar{x} + I(t) \in *\Omega_o \}$. 


It follows from assumption (v) that for each $n$ and every $\bar{x}$, $G_n^{-1}(\bar{x})$ is internal. Let $M$ be the set of all those standard rational points $\bar{r} \in \mathcal{R}_n$ (i.e., points with standard rational coordinates) such that for all $n \in N$, $G_n^{-1}(\bar{r})$ is negligible. Since the standard rational points can be put into a one-to-one correspondence with the standard integers, we can express $M$ as $\{\bar{r}_i\}_{i \in N}$. For each $\bar{r}_i$ there exists a $n_i \in N$ such that $G_n^{-1}(\bar{r}_i) \subseteq G_{n_i}^{-1}(\bar{r}_i)$, where $G_{n_i}^{-1}(\bar{r}_i)$ is internal and negligible.

This follows from the assumption that we are working in a comprehensive enlargement and from the following theorem proved in [7]: Let $\{S_n\}_{n \in N}$ be an internal sequence such that $S_n$ is infinitesimal for all finite $n$. Then there exists an infinite integer $v$ such that $S_n$ is infinitesimal for all $n < v$.

The same argument can be used to show that if

$$B_n = \bigcup_{i=0}^{n} G_n^{-1}(\bar{r}_i), \hspace{1cm} n \in N,$$

then there exists an infinite integer $\rho$ such that $B_{\rho}$ is negligible and $B_n \subseteq B_{\rho}$ for all $n \in N$. Define $U = T - B_{\rho}$; then $U$ is full.

Suppose $0 \in S$-int $(A(U))$. Then there exists an $\bar{x} \geq 0$ such that $-\bar{x} \in A(U)$. By the definition of $A(U)$, $-\bar{x}$ is a $S$-convex combination of $k$ points in $\bigcup_{i \in T} G(t_i)$, where $k$ is finite. That is, there are traders $t_1, \ldots, t_k \in U$ (not necessarily distinct), points $\bar{x}_i \in G(t_i)$, and positive standard numbers $\beta_1, \ldots, \beta_k$ such that $\sum_{i=1}^{k} \beta_i \bar{x}_i = -\bar{x} \leq 0$. We may assume that for all $i$, $\beta_i \bar{x}_i \geq 0$. Then $\bar{x}_i \in G(t_i)$ implies that for some $n_i \in N$, $\{\forall \bar{w} \in S_{n_i}(\bar{x}_i + I(t_i)) \bar{w} > \omega, X(t_i)\}$. Hence, in all cases there exist standard rational points $\bar{r}_i = G(t_i)$ sufficiently close to the $\bar{x}_i$, and positive standard rational numbers $\gamma_i$ sufficiently close to the $\bar{r}_i$ so that we still have

$$\sum_{i=1}^{k} \gamma_i \bar{r}_i \leq 0 \quad \text{and} \quad \sum_{i=1}^{k} \gamma_i = 1.$$

Without loss of generality, we can assume that the $t_i$ are noninfinitesimal.

Let $-\bar{r} = \sum_{i=1}^{k} \gamma_i \bar{r}_i$ and pick an arbitrary trader $t_0$ in $U$. Since $\bar{r} \geq 0$, we have $\bar{x} + I(t_0) \geq X(t_0)$ for sufficiently large positive finite rational $\bar{x}$. Hence, by the monotonic assumption $\bar{x} + I(t_0) \nless X(t_0)$, i.e., $\bar{x} \in G(t_0)$. Now set $\bar{r}_0 = \bar{x} \bar{r}$, $x_0 = 1/(\alpha + 1), x_i = \alpha x_i/(\alpha + 1)$ for $i = 1, \ldots, k$. Then $x_i > 0$ for all $i$ and $\sum x_i = 1$; furthermore $\sum_{i=1}^{k} x_i \bar{r}_i = (\sum x_i \bar{x} + (1 + 1)) \bar{x} \bar{r}_i = (\sum x_i \bar{x} + 1)(\bar{x} - \bar{r}) = 0$, and for all $i$, $r_i \in G(t_i)$. Then $t_i \in G_n^{-1}(\bar{r}_i)$, and since $t_i \in U$, it follows that $\bar{x} = 0$.

Hence for all $i$, $\exists t_i \in N$ such that $|G_n^{-1}(\bar{r}_i)|/\omega \approx 0$.

We shall show that for a sufficiently small positive standard number $\delta$, we can find disjoint subsets $A_i^\prime$ of $G_n^{-1}(\bar{r}_i)$ such that $|A_i^\prime|/\omega \approx \delta x_i$. Let

$$\delta = \min_{1 \leq i \leq k} \left\{ \frac{|G_n^{-1}(\bar{r}_i)|}{\omega x_i} \right\}$$

and $\delta = \theta/k$. Clearly, this $\delta$ will do. Let $A^\prime = \bigcup_{i=1}^{k} A_i^\prime$ and define the following internal function:

$$Y(t) = \begin{cases} \bar{r}_i + I(t) & \text{for} \quad t \in A_i^\prime, \\ I(t) & \text{for} \quad t \notin A^\prime \end{cases}.$$
It is obvious that \( Y(t) \in \Omega_n^* \) for \( t \notin A' \), and for \( t \in A' \) it follows from \( \tilde{r}_i \in G(t) \), i.e., \( A_i \subset G_{n_i}^{-1}(\tilde{r}_i) \); hence \( \tilde{r}_i + I(t) \in F_n(t) \subset \Omega_n^* \). Next,

\[
\frac{1}{\omega} \sum_{t \in A'} Y(t) = \frac{1}{\omega} \sum_{i=0}^{k} \sum_{t \in A_i} Y(t) = \frac{1}{\omega} \sum_{i=0}^{k} \sum_{t \in A_i} \{\tilde{r}_i + I(t)\} = \frac{1}{\omega} \sum_{i=0}^{k} \sum_{t \in A_i} \tilde{r}_i
\]

\[
+ \frac{1}{\omega} \sum_{i=0}^{k} \sum_{t \in A_i} I(t) = \frac{1}{\omega} \sum_{i=0}^{k} |A_i| \tilde{r}_i + \frac{1}{\omega} \sum_{t \in A'} I(t) \geq \frac{1}{\omega} \sum_{i=0}^{k} a_i \tilde{r}_i
\]

\[
+ \frac{1}{\omega} \sum_{t \in A'} I(t) \simeq \bar{0} + \frac{1}{\omega} \sum_{t \in A'} I(t) = \frac{1}{\omega} \sum_{t \in A'} I(t)
\]

and therefore \( A' \) is effective for \( Y \). Since \( Y(t) = I(t) \) for \( t \notin A' \), it follows that \( Y \) is an allocation. Finally, from \( A_i \subset G_{n_i}^{-1}(\tilde{r}_i) \) it follows that \( \tilde{r}_i + I(t) \succ X(t) \) for \( t \in A' \), i.e., \( Y(t) \succ X(t) \) for \( t \in A' \). Since \( A' \) is not negligible, we have shown that \( X \) is not in the core, contrary to assumption.

**Theorem 1:** If \( \mathcal{E} \) is a nonstandard exchange economy satisfying the assumptions of Section 2, then an allocation \( X \) is in the core of \( \mathcal{E} \) if and only if there exists a price vector, \( \bar{p} \), such that \( \langle \bar{p}, X \rangle \) is a competitive equilibrium of \( \mathcal{E} \).

**Proof:** Suppose \( \langle \bar{p}, X \rangle \) is a competitive equilibrium for \( \mathcal{E} \), i.e., there exists an internal set \( K \) such that \( |K|/\omega \simeq 1 \) and \( X(t) \) are maximal in \( B_{\bar{p}}(t) \) for all \( t \in K \). If \( X(t) \) is not in the core of \( \mathcal{E} \), then there exists an allocation \( Y(t) \) and a nonnegligible coalition \( S \) such that \( (1/\omega) \sum_{t \in S} Y(t) \simeq (1/\omega) \sum_{t \in S} I(t) \) and for all \( t \in S \), \( Y(t) \succ X(t) \). Let \( R = S \cap K \); then \( R \) is internal and we see that: (i) \( |R|/\omega \simeq |S|/\omega \); (ii) for all \( t \in R \), \( Y(t) \succ X(t) \); and (iii) for every \( t \in R \), \( \bar{p} \cdot Y(t) \simeq \bar{p} \cdot I(t) \). The third result implies that for every \( t \in R \), \( \bar{p} \cdot (Y(t) - I(t)) \simeq 0 \). Since \( R \) is an internal set, there exists \( t_0 \in R \) such that for all \( t \in R \), \( \bar{p} \cdot (Y(t_0) - I(t)) \geq \bar{p} \cdot (Y(t_0) - I(t)) \). But \( \bar{p} \cdot (Y(t_0) - I(t_0)) \simeq 0 \); hence there exists \( \varepsilon \simeq 0 \) such that \( \bar{p} \cdot (Y(t_0) - I(t_0)) \simeq 0 \). Consequently

\[
\frac{1}{\omega} \sum_{t \in R} \bar{p} \cdot (Y(t) - I(t)) \geq \frac{1}{\omega} \sum_{t \in R} \bar{p} \cdot (Y(t_0) - I(t_0)) \geq \frac{|R|}{\omega} \varepsilon \simeq 0.
\]

Therefore,

\[
\bar{p} \cdot \sum_{t \in R} Y(t)/\omega = \frac{1}{\omega} \sum_{t \in R} \bar{p} \cdot Y(t) \simeq \bar{p} \cdot \sum_{t \in R} I(t)/\omega
\]

which contradicts the effectiveness of \( S \). Since

\[
\frac{1}{\omega} \sum_{t \in S} Y(t) = \frac{1}{\omega} \sum_{t \in R} Y(t) + \frac{1}{\omega} \sum_{t \in S - R} Y(t) \geq \frac{1}{\omega} \sum_{t \in R} I(t) + \frac{1}{\omega} \sum_{t \in S - R} I(t)
\]

\[
= \frac{1}{\omega} \sum_{t \in S} I(t),
\]
$S$ is effective. But
\[
\sum_{t \in S - R} Y(t)/\omega = \sum_{t \in S - R} I(t)/\omega \simeq 0;
\]
hence,
\[
\bar{p} \cdot \sum_{t \in R} Y(t)/\omega \simeq \bar{p} \cdot \sum_{t \in R} I(t)/\omega.
\]

Suppose $X$ is in the core of $\delta$ and let $U$ be the full set of traders in the principal lemma. Then by the principal lemma $0 \notin S\text{-int}(DAU)$; hence by the $S$-separation lemma there exists a standard $\bar{p} \neq 0$ such that for all $\bar{x} \in S\text{-int}(DAU)$, $\bar{p} \cdot \bar{x} \geq 0$. Therefore for all $\bar{y} \in G(t)$, $\bar{p} \cdot \bar{y} \geq 0$ since $S\text{-int}(DAU) \cong S\text{-int}(G(t)) \cong G(t)$. This is equivalent to saying that $\bar{p} \cdot \bar{x} \geq \bar{p} \cdot I(t)$ for all $\bar{x} \in F(t)$. Suppose there exists a standard vector such that $\bar{z} \geq 0$. Then $\bar{z} + X(t) \in F(t)$ for all $t \in T$, by assumption (iv) (p). Therefore $\bar{p} \cdot (\bar{z} + X(t)) \geq \bar{p} \cdot I(t)$. But if for some $i$, $p_i = 0$, we can choose a $\bar{z} \in \Omega_n$ such that $\bar{p} \cdot (\bar{z} + X(t)) \not\leq \bar{p} \cdot I(t)$. Thus $\bar{p} \not\geq 0$.

We will now show that for all $t \in T$, $\bar{p} \cdot X(t) \simeq \bar{p} \cdot I(t)$. Suppose for some $t$ that $\bar{p} \cdot X(t) \not\leq \bar{p} \cdot I(t)$. Then there exists $\bar{z} \in \Omega_n$ such that $\bar{z} \geq 0$, $X(t) + \bar{z} \not\succeq_t X(t)$, and $\bar{p} \cdot (X(t) + \bar{z}) \not\leq \bar{p} \cdot I(t)$. Since $X(t) + \bar{z} \in F(t)$, this is a contradiction.

We can now show that except for at most a negligible set of $t$, $\bar{p} \cdot X(t) \simeq \bar{p} \cdot I(t)$. If for some non-negligible internal set, $S$, we have $\bar{p} \cdot X(t) \geq \bar{p} \cdot I(t)$, then $(1/\omega) \sum_{t \in S} \bar{p} \cdot X(t) \geq (1/\omega) \sum_{t \in S} \bar{p} \cdot I(t)$, which contradicts the assumption that $X$ is in the core. Thus $\bar{p} \cdot X(t) \simeq \bar{p} \cdot I(t)$ except for at most a negligible internal set of traders.

To complete the proof we must show that $X(t)$ is maximal in $t$’s budget set, i.e., that $\bar{p} \cdot \bar{x} \geq \bar{p} \cdot I(t)$ for $\bar{x} \in F(t)$. We first show that $\bar{p} \not\geq 0$. Suppose not; let $p^1 \not\geq 0$, say. Since $\bar{p}$ is standard and $\bar{p} \not\geq 0$, some coordinate of $\bar{p}$ is not infinitesimal; say $p^1 \geq 0$. But $(1/\omega) \sum_{t \in T} I^2(t) \geq 0$. Since $X$ is an allocation it follows that $(1/\omega) \sum_{t \in T} X^2(t) \geq 0$, so there must be a nonnegligible internal set of traders, $S$, for whom $X^2(t) \geq 0$. Now for any trader $t$, it follows from assumption (iv) (g) that $X(t) + (1, 0, \ldots, 0) \not\succ_t X(t)$. Choosing $t \in S$, we see that for some sufficiently small $\epsilon \geq 0$, $X(t) + (1, -\epsilon, 0, \ldots, 0) \not\succ_t X(t)$. Hence, $X(t) + (1, -\epsilon, 0, \ldots, 0) \in F(t)$.

Thereby, $\bar{p} \cdot I(t) \leq \bar{p} \cdot [X(t) + (1, -\epsilon, 0, \ldots, 0)] = \bar{p} \cdot X(t) + p^1 - \epsilon p^1 \leq \bar{p} \cdot X(t)$, i.e., $\bar{p} \cdot I(t) \leq \bar{p} \cdot X(t)$ for all $t \in S$. Since $|S|/\omega \neq 0$, this contradicts $\bar{p} \cdot X(t) \simeq \bar{p} \cdot I(t)$ except for at most a negligible internal set of $t$. Therefore, $\bar{p} \not\geq 0$.

Now suppose $\bar{x} \in F(t)$ and that $I(t) \not\geq 0$; then $\bar{p} \cdot I(t) \not\geq 0$ because $\bar{p} \not\geq 0$. Since $\bar{p} \cdot \bar{x} \geq \bar{p} \cdot I(t)$, it follows that $\bar{p} \cdot \bar{x} \geq 0$; hence there is a $j$ such that $x^j \geq 0$; let $j = 1$. Say. It then follows that $\bar{x} - (e, 0, \ldots, 0) \in F(t)$ for sufficiently small $\epsilon \geq 0$.

Then $\bar{p} \cdot I(t) \leq \bar{p} \cdot [\bar{x} - (e, 0, \ldots, 0)] = \bar{p} \cdot \bar{x} - \epsilon p^1 \leq \bar{p} \cdot \bar{x}$. I.e., $\bar{p} \cdot I(t) \leq \bar{p} \cdot \bar{x}$. If $I(t) \not\geq 0$ and $\bar{x} \not\geq 0$, then clearly $\bar{p} \cdot \bar{x} \geq 0 \not\simeq 0 \bar{p} \cdot I(t)$. Finally, suppose $I(t) \not\succeq 0$ and $\bar{x} \not\succeq 0$. Since $\bar{x} \in F(t)$, this means that $I(t) \in F(t)$. If the set of traders $t$ for whom $I(t) \in F(t)$, $S$, is negligible, then it can be ignored; if, on the other hand, it is non-negligible, then $I(t)$ dominates $X$ via $S$, contradicting the membership of $X$ in the core. This completes the proof of the theorem.
Theorem 1 poses Edgeworth's conjecture as an equivalence theorem for core allocations and competitive allocations in an infinite (more particularly, non-standard) economy. It is well-known that the conjecture is not generally true for finite economies. In view of the unfamiliar nature of nonstandard exchange economies, it seems worthwhile to restate the main theorem as an equivalent result on a family of finite economies. We represent each trader by a sequence of traders in the given finite economies in keeping with the idea that the ultimate behavior of the sequence of finite economies should reflect the behavior of traders selected from each of the economies. The details are as follows.

Let \( \mathcal{G} \) be a countable family of large but finite economies, i.e., \( \mathcal{G} = \{ \mathcal{E}_n \} \), where for any \( n \), \( |\mathcal{E}_n| \), the number of traders in \( \mathcal{E}_n \), \( < |\mathcal{E}_{n+1}| \); \( |\mathcal{E}_n| \) is finite and \( |\mathcal{E}_1| > 0 \). \( \mathcal{E}_n \) is completely specified by the initial endowments and preferences of the traders in \( \mathcal{E}_n \). Hence let

\[
\mathcal{E}_n = \{ I_n(t), P_n(t) \}
\]

where for all \( n \), \( I_n(t) \in \Omega_\mathcal{E}_n \) and \( P_n(t) \in \mathcal{E}(\Omega_m \times \Omega_n)^{\mathcal{E}_n} \). We shall assume that \( \mathcal{E} \) has the following properties.

1. There exists \( \bar{r}_0 \in \Omega_m \) such that for all \( n \) and every \( t \in \mathcal{E}_n \), \( I_n(t) \leq \bar{r}_0 \).
2. \( \lim \inf_{n \to \infty} \frac{1}{|\mathcal{E}_n|} \sum_{t \in \mathcal{E}_n} I_n(t) > 0 \).
3. For all \( n \) and every \( t \in \mathcal{E}_n \), we shall assume that: (a) \( P_n(t) \) is reflexive. (b) if \( \bar{x} \geq \bar{y} \), then \( \bar{x}P_n(t)\bar{y} \). All sequences \( \{ \bar{x}_n \} \), \( \{ \bar{y}_n \} \), are convergent and take their values in \( \Omega_\mathcal{E}_n \). (y) for all \( \{ \bar{x}_n \} \), \( \{ \bar{y}_n \} \), \( \{ t_n \} \), if \( \lim \inf_{n \to \infty} \bar{x}_n \bar{y}_n > 0 \), then there exists \( \delta > 0 \), \( n_0 \in N \), such that \( n \geq n_0 \), \( \bar{z}_n \in S(\bar{x}_n + \bar{y}_n, \delta) \), and \( \bar{w}_n \in S(\bar{x}_n, \delta) \) imply \( \bar{z}_nP(t_n)\bar{w}_n \).

Let \( \mathcal{H}, \{ \mathcal{E}_n \} \), be any subsequence of economies of \( \mathcal{E} \), \{ \mathcal{E}_n \} \), then the following notions are defined with respect to \( \mathcal{H} \).

\( \{ Y_n(t) \} \) is an allocation for \( \mathcal{H} \), if for all \( n \), \( Y_n(t) \in \Omega_m^{\mathcal{E}_n} \), and there exists an \( \bar{r}_1 \in \Omega_m \) such that for all \( n \) and every \( t \in \mathcal{E}_n \), \( Y_n(t) \leq \bar{r}_1 \). Also

\[
\lim_{n \to \infty} \frac{1}{|\mathcal{E}_n|} \sum_{t \in \mathcal{E}_n} I_n(t) - \frac{1}{|\mathcal{E}_n|} \sum_{t \in \mathcal{E}_n} Y_n(t) = 0.
\]

\( \{ \mathcal{S}_n \} \) is a coalition of \( \mathcal{H} \), if for all \( n \), \( \mathcal{S}_n \subseteq \mathcal{E}_n \). We say that \( \{ \mathcal{S}_n \} \) is negligible if \( \lim_{n \to \infty} (|\mathcal{S}_n|/|\mathcal{E}_n|) = 0 \). Note that if \( \{ \mathcal{S}_n \} \) is negligible, then for all allocations \( \{ Y_n(t) \} \),

\[
\lim_{n \to \infty} \frac{1}{|\mathcal{E}_n|} \sum_{t \in \mathcal{S}_n} Y_n(t) = 0.
\]

A coalition, \( \{ \mathcal{S}_n \} \), is effective for an allocation \( \{ Y_n(t) \} \) if

\[
\lim_{n \to \infty} \frac{1}{|\mathcal{E}_n|} \sum_{t \in \mathcal{S}_n} I_n(t) - \frac{1}{|\mathcal{E}_n|} \sum_{t \in \mathcal{S}_n} Y_n(t) = 0.
\]

An allocation \( \{ Y_n(t) \} \), dominates an allocation \( \{ X_n(t) \} \) via a coalition \( \{ \mathcal{S}_n \} \), if for all \( \{ t_n \} \), such that \( t_n \in \mathcal{S}_n \), \( \lim \inf_{n \to \infty} Y_n(t_n) - X_n(t_n) > 0 \), and for some \( \delta > 0 \) and \( n_0 \in N \) if \( n \geq n_0 \), \( \bar{w} \in S(Y_n(t_n), \delta) \), then \( \bar{w}P(t_n)X_n(t_n) \), where \( \{ \mathcal{S}_n \} \) is effective for \( \{ Y_n(t) \} \).
The core of $\mathscr{H}$ is the set of all allocations which are not dominated via any nonnegligible coalition.

An allocation for $\mathscr{G}$ is said to be in the core of $\mathscr{G}$ if for every subsequence of economies $\mathscr{H}$, the allocation for $\mathscr{G}$ restricted to $\mathscr{H}$ is in the core of $\mathscr{H}$.

$\{\bar{p}_n\}_{n}^\infty$ is a price vector if for all $n$, $\bar{p}_n \in \Omega_m$, and $\lim_{n \to \infty} \bar{p}_n > 0$.

$\langle \{X_n(t)\}_{t=1}^\infty, \{\bar{p}_n\}_{n}^\infty \rangle$ is a competitive equilibrium for $\mathscr{H}$ if the following conditions hold: (i) $\{X_n(t)\}_{t=1}^\infty$ is an allocation; (ii) $\{\bar{p}_n\}_{n}^\infty$ is a price vector; and (iii) there does not exist a coalition $\{\mathcal{S}_n\}_{n}^\infty$ such that $\lim_{n \to \infty} \lambda(\mathcal{S}_n, \bar{p}_n) > 0$, and for all $\{t_n\}_{n}^\infty$, where $t_n \in \mathcal{S}_n$, there exists $\{\bar{v}_n\}_{n}^\infty$ for which $\lim_{n \to \infty} \lambda(\mathcal{S}_n, \bar{p}_n \cdot I_{\mathcal{S}_n}(t_n) - \bar{v}_n \cdot \bar{p}_n) > 0$, $\lim_{n \to \infty} |\bar{v}_n - X_n(t_n)| > 0$, and for some $\delta > 0$, $n_0 \in N$, if $n > n_0$, $\bar{w} \in S(\bar{v}_n, \delta)$, then $\bar{w} P_{\mathcal{S}_n}(t_n) X_{\mathcal{S}_n}(t_n)$; (iv) there does not exist a coalition $\{\mathcal{S}_n\}_{n}^\infty$ such that $\lim_{n \to \infty} \lambda(\mathcal{S}_n, \bar{p}_n \cdot X_{\mathcal{S}_n}(t_n) - \bar{p}_n \cdot I_{\mathcal{S}_n}(t_n)) > 0$, and for all $\{t_n\}_{n}^\infty$, where $t_n \in \mathcal{S}_n$, for which $\lim_{n \to \infty} \lambda(\mathcal{S}_n, \bar{p}_n \cdot X_{\mathcal{S}_n}(t_n) - \bar{p}_n \cdot I_{\mathcal{S}_n}(t_n)) > 0$.

$\langle \{X_n(t)\}_{t=1}^\infty, \{\bar{p}_n\}_{n}^\infty \rangle$ is a competitive equilibrium for $\mathscr{G}$ if $\langle \{X_n(t)\}_{t=1}^\infty, \{\bar{p}_n\}_{n}^\infty \rangle$ restricted to $\mathscr{H}$, where $\mathscr{H}$ is any subsequence of economies of $\mathscr{G}$, is a competitive equilibrium for $\mathscr{H}$.

Since $\mathscr{G}$ is a pair of sequences of functions, $\{I_n(t)\}_{n=1}^\infty$ and $\{P_n(t)\}_{n=1}^\infty$, we may define $^*\mathscr{G}$ as the nonstandard extensions of these two sequences. Hence $^*\mathscr{G} = \langle \{^*I_n(t)\}_{n=1}^\infty, \{^*P_n(t)\}_{n=1}^\infty \rangle$, where $n$ now ranges over the nonstandard integers, $^*N$.

For each infinite integer $\omega \in ^*N - N$, we define the nonstandard exchange economy $\mathcal{E}_\omega = \langle \{^*I_n(t)\}, \{^*P_n(t)\} \rangle$.

**Lemma 6**: $\{X_n(t)\}_{t=1}^\infty$ is in the core of $\mathscr{G}$, $\mathcal{E}(\mathscr{G})$, if and only if for all infinite integers $\omega$, $X_\omega(t)$ is in the core of $\mathcal{E}_\omega$.

**Proof**: Suppose there exists an infinite integer $\omega$ such that $X_\omega(t) \notin \mathcal{E}(\mathcal{E}_\omega)$. Then there exists an allocation $Y_\omega(t)$, a coalition $\mathcal{S}_\omega \subseteq \mathcal{E}_\omega$, and $\epsilon_1, \epsilon_2 > 0$ such that $(1/\omega) \sum_{t \in \mathcal{S}_\omega} Y_\omega(t) - (1/\omega) \sum_{t \in \mathcal{S}_\omega} I_\omega(t)$, $(1/\omega) \sum_{t \in \mathcal{S}_\omega} Y_\omega(t) - (1/\omega) \sum_{t \in \mathcal{S}_\omega} I_\omega(t)$, and $|\mathcal{S}_\omega| / |\mathcal{E}_\omega| \geq \epsilon_2$ for all $t \in \mathcal{S}_\omega$, for all $\bar{w} \in S(Y_\omega(t), \epsilon_1)$, $\bar{w} P_{\mathcal{S}_\omega}(t) X_{\mathcal{S}_\omega}(t)$. Therefore for all $n \in N$ and every positive $\delta \in R$, the following sentence is true in our nonstandard universe:

$^*U : (\exists \nu \in ^*N) (\exists Y_\nu \in ^*\Omega_m^\infty) (\exists \mathcal{S} \subseteq \mathcal{E}_\nu)$

$\left[ v > n \land \left| \frac{1}{\nu} \sum_{t \in \mathcal{S}} Y_\nu(t) - \frac{1}{\nu} \sum_{t \in \mathcal{S}} I_\nu(t) \right| < \delta \land \left| \frac{1}{\nu} \sum_{t \in \mathcal{S}} Y_\nu(t) - \frac{1}{\nu} \sum_{t \in \mathcal{S}} I_\nu(t) \right| < \delta \land (\forall t \in \mathcal{S})(\forall \bar{w} \in S(Y_\nu(t), \epsilon_1)) \bar{w} P_{\mathcal{S}}(t) X_{\mathcal{S}}(t) \land |\mathcal{S}| / |\mathcal{E}| \geq \epsilon_2 \right] .$

Hence this sentence is true when translated in $U$, our standard universe. Therefore, for every $n \in N \land \delta > 0$, there exists $m \in N$, $Y_m \in \Omega_{m-l}$, and $\mathcal{S}_m \subseteq \mathcal{E}_m$ such that $m > n$,

$\left| \frac{1}{m} \sum_{t \in \mathcal{S}_m} Y_m(t) - \frac{1}{m} \sum_{t \in \mathcal{S}_m} I_m(t) \right| < \delta \land \left| \frac{1}{m} \sum_{t \in \mathcal{S}_m} Y_m(t) - \frac{1}{m} \sum_{t \in \mathcal{S}_m} I_m(t) \right| < \delta \land (\forall t \in \mathcal{S}_m)(\forall \bar{w} \in S(Y_m(t), \epsilon_1)) \bar{w} P_m(t) X_m(t) \land |\mathcal{S}_m| / |\mathcal{E}_m| \geq \epsilon_2 .}$
Consequently there exists a subsequence of economies, $\mathcal{H}$, which has the allocation $\{X_m(t)\}_{m=1}^{\infty}$ and a nonnegligible coalition $\{\mathcal{S}^*_m\}_{m=1}^{\infty}$ such that $\{X_m(t)\}_{m=1}^{\infty}$ dominates $\{X_n(t)\}_{n=1}^{\infty}$ via $\{\mathcal{S}^*_m\}_{m=1}^{\infty}$; hence $\{X_n(t)\}_{n=1}^{\infty} \notin \mathcal{G}(\mathcal{S})$.

Suppose $\{X_m(t)\}_{m=1}^{\infty} \notin \mathcal{G}(\mathcal{S})$; then there exists a subsequence of economies $\{\mathcal{S}_m\}_{m=1}^{\infty}$, $\mathcal{H}$, a coalition $\{\mathcal{S}^*_m\}_{m=1}^{\infty}$, and an allocation $\{Y_m\}_{m=1}^{\infty}$ such that

$$\lim_{m \to \infty} \frac{1}{|\mathcal{S}_m|} \left( \sum_{x \in \mathcal{S}_m} Y_m(x) - \sum_{x \in \mathcal{S}_m} I_m(x) \right) = 0,$$

$$\lim_{m \to \infty} \frac{1}{|\mathcal{S}^*_m|} \left( \sum_{x \in \mathcal{S}^*_m} Y_m(x) - \sum_{x \in \mathcal{S}^*_m} I_m(x) \right) = 0,$$

$$\liminf_{m \to \infty} \frac{|\mathcal{S}^*_m|}{|\mathcal{S}_m|} > 0.$$

For all $\{t_m\}_{m=1}^{\infty}$ such that for every $m$, $t_m \in \mathcal{S}_m$,

$$\liminf_{m \to \infty} |Y_m(t_m) - X_m(t_m)| > 0,$$

and for some $\delta > 0$ and $m_0 \in N$ if $m \geq m_0$, $\bar{w} \in S(Y_m(t_m), \delta)$, then $\bar{w}P_{\mathcal{S}_m}(t_m)X_m(t_m)$.

Hence there exists an infinite integer $\omega$ for which the following statements hold about the nonstandard extensions of $\{X_m(t)\}_{m=1}^{\infty}$, $\{Y_m(t)\}_{m=1}^{\infty}$, $\{P_{\mathcal{S}_m}(t)\}_{m=1}^{\infty}$, $\{\mathcal{S}^*_m\}_{m=1}^{\infty}$, and $\{I_m\}_{m=1}^{\infty}$:

$$\frac{1}{\omega} \sum_{t \in \mathcal{S}_\omega} X_\omega(t) \simeq \frac{1}{\omega} \sum_{t \in \mathcal{S}_\omega} I_\omega(t),$$

$$\frac{1}{\omega} \sum_{t \in \mathcal{S}_\omega} Y_\omega(t) \simeq \frac{1}{\omega} \sum_{t \in \mathcal{S}_\omega} I_\omega(t),$$

$$\frac{1}{\omega} \sum_{t \in \mathcal{S}^*_\omega} Y_\omega(t) \simeq \frac{1}{\omega} \sum_{t \in \mathcal{S}^*_\omega} I_\omega(t),$$

$|\mathcal{S}_\omega|/|\mathcal{S}_\omega| \neq 0$, for all $t \in \mathcal{S}_\omega$, $Y_\omega(t) \gg X_\omega(t)$. Hence $X_\omega(t) \notin \mathcal{G}(\mathcal{S}_\omega)$.

**Lemma 7:** $\langle \{X_\omega(t)\}_{m=1}^{\infty}, \{\bar{p}_\omega\}_{m=1}^{\infty} \rangle$ is a competitive equilibrium for $\mathcal{S}$ if and only if for all infinite integers $\omega$, $\langle X_\omega(t), \bar{p}_\omega \rangle$ is a competitive equilibrium for $\mathcal{S}_\omega$.

**Proof:** Suppose there exists an infinite integer $\omega \in N^*, N$ such that $\langle X_\omega(t), \bar{p}_\omega \rangle$ is not a competitive equilibrium for $\mathcal{S}_\omega$. Then there exists an internal set of traders, $\mathcal{S}_\omega$, and $e_1, e_2 \gg 0$ such that $|\mathcal{S}_\omega|/|\mathcal{S}_\omega| \gg e_2$, and for all $t \in \mathcal{S}_\omega$, there exists a $\bar{y}_t \in B_{\bar{p}_\omega}(t)$ such that $\bar{w}P_{\mathcal{S}_\omega}(t)X_\omega(t)$ for all $\bar{w} \in S(\bar{y}_t, e_1)$. Hence for all $n \in N$ and for all $\delta \in R, \delta > 0$, the following sentence is true in $^*U$, our nonstandard universe:

$$(\exists v \in *N)(\exists \bar{p}_t \in *\Omega_\omega)(\exists \mathcal{S}_\omega \subseteq \mathcal{S}_\omega)[v > n \wedge (\forall t \in \mathcal{S}_\omega)(\exists \bar{y}_t \in *\Omega_\omega)(\forall (\bar{p}_t \cdot \bar{y}_t - \bar{p}_t \cdot I_\omega(t)) \delta \vee \bar{p}_t \cdot \bar{y}_t \leq \bar{p}_t \cdot I_\omega(t))$$

if $\bar{w} \in S(\bar{y}_t, e_1)$ then $\bar{w}P_{\mathcal{S}_\omega}(t)X_\omega(t) \wedge |\mathcal{S}_\omega|/|\mathcal{S}_\omega| \gg e_2].$

Therefore, this sentence is true when translated in $U$, our standard universe.
Hence, for every $n \in N$ and $\delta \in R$, $\delta > 0$, there exists $m > n$, $\bar{p}_m \in \Omega_m$, $\mathcal{S}_m \subseteq \mathcal{E}_m$ such that $m > n$

$$(\forall t \in \mathcal{S}_m)(\exists \bar{y}_m \in \Omega_m)((\bar{p}_m \cdot \bar{y}_m - \bar{p}_m \cdot I_m(t)) < \delta \lor \bar{p}_m \cdot \bar{y}_m \leq \bar{p}_m \cdot I_m(t))$$

if $\bar{w} \in S(\bar{y}_m, e_1)$ then $\bar{w} P_m(t) X_m(t) \land |\mathcal{S}_m|/|\mathcal{E}_m| \geq \varepsilon_2$.

Consequently there exists a subsequence of economies, $\mathcal{H}$, such that $\langle \{X_a(t)\}_i, \{\bar{p}_a\}_i \rangle$ restricted to $\mathcal{H}$ is not a competitive equilibrium for $\mathcal{H}$; hence $\langle \{H_a(t)\}_i, \{\bar{p}_a\}_i \rangle$ is not a competitive equilibrium for $\mathcal{G}$.

Suppose $\langle \{X_a(t)\}_i, \{\bar{p}_a\}_i \rangle$ is not a competitive equilibrium for $\mathcal{G}$; then there exists a subsequence $\{X_a(t)\}_i, \{\bar{p}_a\}_i, \{\mathcal{S}_m\}_i, \{\mathcal{E}_m\}_i$, a sequence $\{\mathcal{S}_m\}_i, \{\mathcal{E}_m\}_i$, and $\varepsilon_1, \varepsilon_2 > 0$ such that $\liminf_{n \to \infty} |\mathcal{S}_m|/|\mathcal{E}_m| \geq \varepsilon_2$ and for all $\{t_m\}_i$, where $\liminf_{n \to \infty} |\bar{p}_m - X_m(t_m)| > 0$, $\liminf_{m \to \infty} \bar{p}_m \cdot I_m(t) - \bar{p}_m \cdot \bar{y}_m > 0$, and for some $m_0, \bar{w} P_m(t) X_m(t_m)$ for all $m \geq m_0$ and $\bar{w} \in S(\bar{y}_m, e_1)$. Hence, there exists an infinite integer $\omega$ for which the following statements hold about the nonstandard extensions of $\{X_a(t)\}_i, \{I_a(t)\}_i$, $\{\mathcal{S}_m\}_i$, and $\{\mathcal{E}_m\}_i$:

$$\frac{1}{\omega} \sum_{t \in \mathcal{S}_m} I_a(t) \geq \frac{1}{\omega} \sum_{t \in \mathcal{E}_m} X_a(t), \quad |\mathcal{S}_m|/|\mathcal{E}_m| \geq \varepsilon_2,$$

for all $t \in \mathcal{S}_m$, there exists $\bar{y}_m \in B_{\bar{p}_m}(t)$ such that $\bar{y}_m \gg X_a(t)$. Hence, $\langle X_a(t), \bar{p}_a \rangle$ is not a competitive equilibrium for $\mathcal{E}_m$.

**Theorem 2:** $\{X_a(t)\}_i$ is in the core of $\mathcal{G}$ if and only if there exists $\{\bar{p}_a\}_i$ such that $\langle \{X_a(t)\}_i, \{\bar{p}_a\}_i \rangle$ is a competitive equilibrium for $\mathcal{G}$.

**Proof:** We need only show that every $\mathcal{E}_m$ is a nonstandard exchange economy satisfying the assumptions of Theorem 1. Then the proof is an immediate consequence of Lemmas 6 and 7. All the assumptions are obviously met except possibly (iv) (y), which we shall show holds also. Suppose (iv) (y) is false in some $\mathcal{E}_m$; then there exists a trader $t$, and standard $\bar{x}, \bar{y} \in \mathcal{E}_m$, such that for all $x \geq 0$ there exists $\bar{z} \in S(\bar{x}, \bar{y}, x)$. Therefore, for all $n \in N$ and every positive $x$ in $R$, the following sentence is true in our nonstandard universe $\star U$:

$$(\exists v \in *N)(\exists \alpha \in \mathcal{E}_m)(\exists \bar{x} \in \mathcal{E}_m)[(v > n \land \bar{x} \in S(\bar{x} + \bar{y}, x) \land \bar{x} \gg \bar{y})].$$

Hence this sentence is true when translated in $U$, our standard universe. Therefore, for every $n \in N; x \in R; x > 0$, there exists $l > n, t_l \in \mathcal{E}_m$ in $\mathcal{E}_m$ such that $\bar{x}_l \in S(\bar{x} + \bar{y}, x)$ and $\bar{x}_l$ not preferred to $\bar{y}$. But this contradicts our assumption (3) (y) about the preference of $\mathcal{G}$. To complete the proof, suppose $\langle \{X_a(t)\}_i, \{\bar{p}_a\}_i \rangle$ is a competitive equilibrium for $\mathcal{G}$; then by Lemma 7 $\langle X_a(t), \bar{p}_a \rangle$ is a competitive equilibrium for $\mathcal{E}_m$ for every infinite integer $\omega$. Hence by Theorem 1 for all infinite $\omega, X_a$ is in the core of $\mathcal{E}_m$ and consequently by Lemma 6 $\langle X_a(t) \rangle_i$ is in the core of $\mathcal{G}$. Suppose $\{X_a(t)\}_i$ is in the core of $\mathcal{G}$; then by Lemma 6 $X_a$ is in the core of $\mathcal{E}_m$ for all infinite integers $\omega$. Hence, by Theorem 1, there exists $\bar{p}_a$ such that $\langle X_a(t), \bar{p}_a \rangle$ is a competitive equilibrium for $\mathcal{E}_m$ for every infinite integer $\omega$. Consequently for
every positive $\delta_1, \delta_2 \in R$ the following sentence is true in $*U$:

$$(\exists v \in *N)(\forall \theta \in *N)(0 \geq \nu \Rightarrow (\exists \bar{p}_0 \in *\Omega_m)((\forall t \in \mathcal{S}_t)(\exists \bar{y}_t \in *\Omega_m)$$

$$\times \{(\bar{p}_0 \cdot \bar{y}_t \leq \bar{p}_0 \cdot I_0(t) \lor \bar{y}_t \cdot I_0(t) < \delta_2) \land \bar{y}_t \cdot P_0(t) X_0(t) \}}$$

$$\land \bar{y}_t - X_0(t) \geq \delta_1 \Rightarrow [\mathcal{S}_t]_{\|\delta_0\|} < \delta_2$$].

Translating this sentence in $U$ for $\delta_1 = \delta_2 = 1/n$, $n = 1, 2, \ldots$, we generate a sequence of prices $\{\bar{p}_j\}_{j=1}^\infty$ and integers $\{n_j\}_{j=1}^\infty$ where $n_j$ is the first integer such that the sentence is true for $\delta_1 = \delta_2 = 1/j$, $\bar{p}_j$ is a corresponding price, and $n_j < n_{j+1}$. Let $n_0 = 0$ and assign economies $n_j + 1$ through $n_{j+1}$ prices $\bar{p}_j$ where $j = 0, 1, 2, \ldots$. We claim that $\langle X_n(t) \rangle_{t=1}^\infty$, $\langle P_n \rangle_{t=1}^\infty$, $\langle \bar{y}_n \rangle_{t=1}^\infty$ is a competitive equilibrium for $\mathcal{S}$. Suppose not: then there exists $\langle \mathcal{S}_n \rangle_{t=1}^\infty$, $\langle \bar{y}_n \rangle_{t=1}^\infty$, $\langle \bar{n}_n \rangle_{t=1}^\infty$, $\nu_1, \nu_2$ such that $\inf_{n \to \infty} (\|\nu_1\|/\|\nu_2\|) > \epsilon_2$, all $n, \{\bar{s}_n\}_{n=1}^\infty \in \mathcal{S}_n$ and $\bar{y}_n \in \Omega_n$, and $\inf_{n \to \infty} (\bar{p}_n \cdot I_n(t_n) - \bar{n}_n \cdot \bar{y}_n) > 0$, $\inf_{n \to \infty} [\bar{y}_n - X_n(t_n)] > 0$, for some $n_0$, $\bar{w} P_n(t_n) X_n(t_n)$ for all $n \geq n_0$ and $\bar{w} \in S(\bar{y}_n, \epsilon)$. But for all $j$ such that $(1/j) < \min (\inf_{n \to \infty} (\|\nu_1\|/\|\nu_2\|), \inf_{n \to \infty} (\bar{p}_n \cdot I_n(t_n) - \bar{n}_n \cdot \bar{y}_n), \inf_{n \to \infty} [\bar{y}_n - X_n(t_n)] )$, this contradicts the properties of $\langle X_n(t), \bar{p}_n \rangle$ for economy $\delta_n$.

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REFERENCES