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THE CORE OF A MARKET GAME WITH EXOGENOUS RISK AND INSURANCE*

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1. INTRODUCTION

This note deals with the existence and nature of the core of a market game with exogenous uncertainty present. The traders, instead of holding bundles of actual goods may be regarded as holding contingent claims. These can be regarded in the same way as they are handled in the constructs of Arrow [1] or Debreu [3]. We may enormously enlarge the number of goods so that a good is not only a physical item such as a box of matches, but it has a state attached to it, such as "a box of matches if it rains" or "a box of matches if the sun shines".

A simple example illustrates this approach. Consider two traders, where the first has a probability of ½ of having 2 apples and a probability of ½ of having 4 apples. Similarly the second has a probability of ½ of having 4 oranges and a probability of ½ of having 8 oranges.

We must distinguish two cases. The probabilities are either correlated or they are not. For example, suppose that "Nature" or the referee tosses one coin. If it comes down heads there are 2 apples and 4 oranges, or if tails, there are 4 apples and 8 oranges. In this world there are only 2 possible states of the system given by H or T. If the referee tosses a coin once for each trader there are four states given by HH, HT, TH or TT. In the first instance there are 4 goods *in toto;* in the second there are 8 goods, where the number of Arrow-Debreu goods equals the actual goods multiplied by the number of states of the system. Thus in the first case the holdings of Trader 1 can be described by (2,4,0,0) and Trader 2 by (0,0,4,8), where the first good is "apples if heads", the second good "apples if tails", the third good "oranges if heads" and the fourth "oranges if tails".

When the random events are not correlated then the holdings of Trader 1 are described by (2,2,4,4,0,0,0,0) and of Trader 2 by

(0,0,0,0,4,8,4,8), where, for example, the first good is "apples if HH".

If we assume that all traders are risk-averters (i.e. they are not attracted by a fair gamble) then it is known that this exchange economy has a price system. If we do not assume that traders are risk-averters this is not necessarily true. An investigation of the properties of the system where some traders welcome risk has been discussed elsewhere [5].

In the remainder of this note we consider the existence and properties of the core of the exchange economy with exogenous risk.

2. The Core of an Exchange Economy with Exogenous Risk

Regardless of whether risk is correlated or non-correlated, by the device of enlarging the number of commodities it follows immediately that the new market with the inflated number of commodities and the holdings of each player given in these terms is a market game and hence has a non-empty core.

Furthermore it can be seen trivially that in general, for few players the core may be large. The Edgeworth bilateral monopoly model immediately illustrates this. The whole contract curve is the core. The Edgeworth box can be used to illustrate the case of one commodity and two contingencies. Consider two traders whose fates are correlated. There is a chance of ½ that the first obtains 2 apples and the second obtains nothing, and vice versa. Thus their holdings of the contingent commodities are given by (2,0) and (0,2), where the first commodity can be described as "apples if heads" and the second as "apples if tails".

In general the Arrow-Debreu utility functions must be separable and additive with respect to the states of the economy. If, for instance, in the simple example above an individual has a utility function for apples of the form $U_i = f_i(x)$ then his utility function for the two commodities "apples in state 1" and "apples in state 2" must be of the form $U_i = p_1 f_i(x) + p_2 f_i(y)$.

In Figure 1 the diagram is drawn for the instance where Trader 1 has $f_1(x) = \sqrt{x}$ and Trader 2 has $f_2(x) = x$. Thus

$$U_1(x_1, y_1) = \tfrac{1}{2}\sqrt{x_1} + \tfrac{1}{2}\sqrt{y_1}$$

$$U_2(x_2, y_2) = \tfrac{1}{2}(2-x_1) + \tfrac{1}{2}(2-y_1)$$

where $x_1+x_2 = 2$, $y_1+y_2 = 2$ and $x_1, x_2, y_1, y_2 \geq 0$.

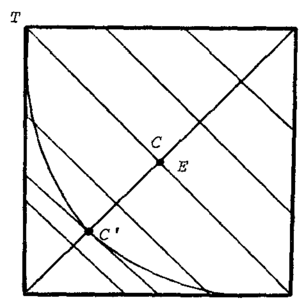

FIGURE 1.

The core indicated by $C'C$ stretches from where Trader 1 obtains $(\tfrac{1}{2}, \tfrac{1}{2})$ to (1, 1). In terms of utility this is from .707 to 1.0. The initial point T has a utility to each of U_1 = .707 and U_2 = 1.0. We can see from the diagram that $TC(=TE)$ is the price ray that determines the competitive equilibrium. All the advantage accrues to traders of the first type.

Only one iteration is required to show the convergence of the core. Consider the market with two traders of each type. Suppose that the distribution $(1-\varepsilon, 1-\varepsilon)$ for traders of the first type and $(1+\varepsilon, 1+\varepsilon)$ for traders of the second type were in the core, then consider the coalition (2, 1) with resources (4, 2). Give the trader of the second type $(2+\frac{3\varepsilon}{2}, \frac{3\varepsilon}{2})$; then the traders of the first type can obtain

$(1-\frac{3\varepsilon}{4},\ 1-\frac{3\varepsilon}{4})$ each, but the payoffs in the latter instance are $(\sqrt{1-\frac{3\varepsilon}{4}},\ 1+\frac{3\varepsilon}{2})$ which are greater than the payoffs $(\sqrt{1-\varepsilon}, 1+\varepsilon)$ in the former instance if $\varepsilon > 0$. The limit core is the competitive equilibrium at the point (1, 1) with payoffs of 1 each for traders of both types.[1] (It is known that the core of a market game yields equal treatment for identical traders, hence the argument above completely locates the core.)

In this model the resources of the individuals were negatively correlated. Either traders of one type or the other had the commodity. This obscures the important difference between correlated and non-correlated risk. An equally simple example illustrates the difficulties which can arise.

Suppose that a single coin is tossed and the endowments are as follows:

For traders of type 1 and 2: 2 if H and 0 if T.

Hence each has (2, 0) and no trade will take place regardless of the number of traders. The payoffs to players of each type are $\frac{1}{2}\sqrt{2}$ (=.707) and 1.0 respectively.

If two coins are tossed, then there are 4 states, hence 4 "contingent goods" and the endowments are (2,2,0,0) and (2,0,2,0) respectively and the utility functions become:

$$U_1 = \tfrac{1}{4}\sqrt{x} + \tfrac{1}{4}\sqrt{y} + \tfrac{1}{4}\sqrt{z} + \tfrac{1}{4}\sqrt{w}$$

$$U_2 = \tfrac{1}{4}(4-x) + \tfrac{1}{4}(2-y) + \tfrac{1}{4}(2-z) + \tfrac{1}{4}(0-w)$$

where $0 \le x \le 4,\ 0 \le y,\ z \le 2$ and $w = 0$.

When we consider replications of these two different markets, one with correlated and the other with noncorrelated endowments, we observe that in the first case the market is always inessential and no trade takes place. In the second case as the numbers increase the Pareto optimal surface moves out and the number of contingent commodities increases as 2^{2n}. A two dimensional slice of the Pareto optimal surface is shown in Figure 2. For $n = 1$ the maximum obtainable by the trader of type 1 is 1.207. For $n = 2$ the maximum even split by traders of type 1 is 1.895. Finally as n becomes large the amount obtainable by traders of type 1 approaches 2 per capita. The law of large numbers enables us to replace the uncertain supply by its certainty equivalent of 1 per capita for all $2n$ traders which means that in the limit the Pareto optimal surface stretches from (0,0,...; 2,2,...) to (2,2,...; 0,0,...) when all traders of the same type receive the same amounts. This enables the group as a whole to insure at actuarial "fair bet" risk.

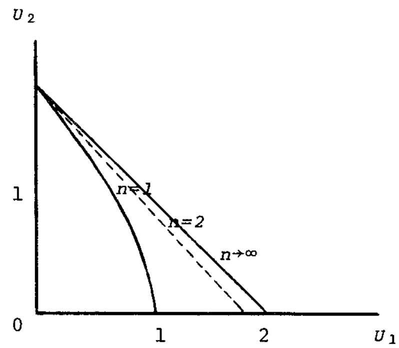

FIGURE 2.

It has been shown that for an exchange economy with correlated or uncorrelated risk the core exists and it can be proved that in either case the core converges to the competitive equilibrium. When there is correlated risk the proof follows immediately from the proof of Debreu and Scarf [4] by merely introducing the contingent commodities. As these do not increase in numbers with replication this model immediately satisfies the conditions of Debreu and Scarf [4]. When risk is not correlated the commodity space increases in size with the number of traders and a somewhat different proof is required [2].

3. INTERPRETATION

A cooperative society can arrive at a pricing system to ensure against either correlated or noncorrelated risk. In the former, however, the risk to society as a whole nevertheless remains regardless of numbers. In the latter the risk to society as a whole decreases with the numbers. Earthquakes or large crop failures are examples of the former. Fire and accident insurance are examples of the latter.

In this cooperative model, as society as a whole implicitly does the insuring no distinction between agencies or institutions which are willing and capable to insure noncorrelated risks, and those who can insure major societal disasters, appears. In a noncooperative model we would expect this distinction to appear. In several economies the insurance function for the former is reasonably covered by private agencies, and for the latter by governmental agencies.

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FOOTNOTES

* A preliminary investigation into the core of an exchange economy with exogenous risk and risk-neutral or risk-averse traders was carried out by Shapley and Shubik, but was never fully written up. M. Yaari called my attention to the work done independently by Y. Caspi in 1971.

1. This immediate convergence for $n = 2$ depended upon U_2 being a linear function. In general the "shrinking" of the core would be slower.

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