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DISTRIBUTION OF THE CIRCULAR SERIAL CORRELATION COEFFICIENT FOR RESIDUALS FROM A FITTED FOURIER SERIES[1,2]

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CONTENTS

Summary. In this paper the observations are considered to be normally distributed with constant variance and means consisting of linear combinations of certain trigonometric functions. The likelihood ratio criterion for testing the independence of the observations against the alternatives of circular serial correlation of a given lag is found to be a function of the circular serial correlation coefficient for residuals from the fitted Fourier series (Section 4). The exact distribution (Section 5), the moments (Section 6), and approximate distributions

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(Section 7) are given for the cases of greatest interest. From these results significance levels have been found (Section 3). The use of these levels is indicated (Section 2), and an example of their use is given (Section 3).

1. Introduction. Two mathematical models have been used extensively in time-series analysis. In one model the observation is the sum of a "systematic part" and a random error. The cyclical properties of this model result from the cyclical properties of the systematic part, which is usually taken to be a short Fourier series. The stochastic element is superimposed on the non-stochastic part, and the error at one time point does not affect a later observation. The other model is the stochastic difference equation or "autoregressive model." An observation is the sum of a linear function of previous observations and a random element. The cyclical properties follow from the properties of the difference equation (i.e., the linear combination of observations), but are disturbed by the random disturbance that is integrated into the system. A more general model can be constructed that includes both of the two mentioned. The observation can be taken as a linear combination of past observations and Fourier terms plus a random element.

In this paper, the linear combination will be only a multiple of some preceding observation. For lag 1, the model is of the form

$$x_i - \mu_i = \rho(x_{i-1} - \mu_{i-1}) + u_i, \qquad i = 1, 2 \cdots, N, \tag{1}$$

where $x_0 \equiv x_n$ and $\mu_0 \equiv \mu_n$. In (1), the $\{x_i\}$ are the N observations; the $\{u_i\}$ are N random disturbances, each assumed normally and independently distributed with zero mean and variance σ^2; the means $\{\mu_i\}$ are linear combinations of some of the N functions of i: $\cos \dfrac{2\pi i g}{N}$ and $\sin \dfrac{2\pi i h}{N}$. For N odd, $g = 0, 1, \cdots, \frac{1}{2}(N - 1)$; $h = 1, \cdots, \frac{1}{2}(N - 1)$. For N even, $g = 0, 1, \cdots, \frac{1}{2}N$; $h = 1, \cdots, \frac{1}{2}N - 1$. Hence,

$$\mu_i = \sum_{g'} \alpha_{g'} \cos \frac{2\pi i g'}{N} + \sum_{h'} \beta_{h'} \sin \frac{2\pi i h'}{N}, \tag{2}$$

where g' and h' run over certain values of the ranges of g and h, respectively. Let K' be the number of terms in (2). Usually the constant term, α_0, is included (in this case $g = 0$ and $\cos \dfrac{2\pi i g}{N} = 1$). Of the N trigonometric functions available, the terms in (2) are usually chosen so that terms with certain periods are included and terms with other periods are excluded. It should be noted that (1) defines a circular model.

The sample estimates of $\alpha_{g'}$ and $\beta_{h'}$ are the usual regressions of x_i on $\cos \dfrac{2\pi i g'}{N}$ and $\sin \dfrac{2\pi i h'}{N}$, respectively. Because of the orthogonality of these trigonometric terms, the estimates are

$$
(3) \qquad \begin{aligned}
a_{g'} &= \sum_{i=1}^{N} x_i \cos \frac{2\pi i g'}{N} \Big/ \frac{N}{2}, \qquad g \neq 0, \tfrac{1}{2}N, \\
b_{h'} &= \sum_{i=1}^{N} x_i \sin \frac{2\pi i h'}{N} \Big/ \frac{N}{2}, \\
a_0 &= \sum_{i=1}^{N} x_i \Big/ N, \\
a_{\frac{1}{2}N} &= \sum_{i=1}^{N} x_i \cos \pi i \Big/ N = \sum_{i=1}^{N} (-1)^i x_i \Big/ N.
\end{aligned}
$$

The fitted series is

$$
(4) \qquad m_i = \sum_{g'} a_{g'} \cos \frac{2\pi i g'}{N} + \sum_{h'} b_{h'} \sin \frac{2\pi i g'}{N}.
$$

where the sums on g' and h' are over the ranges in (2).

The serial correlation coefficient suitable for this model is

$$
(5) \qquad R = \frac{\sum_{i=1}^{N} (x_i - m_i)(x_{i-1} - m_{i-1})}{\sum_{i=1}^{N} (x_i - m_i)^2},
$$

where $m_0 \equiv m_n$. This statistic can be used to estimate ρ, or it can be used to test hypotheses about ρ. In fact, for the circular model this statistic leads to the best tests [3].

It is hoped that the mathematical model studied in this paper can be used in the treatment of certain problems in economic time series. For example, the seasonal variation in a series of data may be considered as a "systematic part" made up of trigonometric components. In the next section we discuss in a more detailed way how the use of this model may arise in the field of economics.

We have considered circular serial correlation, although in most statistical problems it is non-circular serial correlation that is involved. The reason for treating the circular case is the inherent mathematical simplicity. The circular coefficient and Fourier series of the type (2) are "naturally" related. The relevant fact is that the vectors

$$
\left(\cos \frac{2\pi g}{N}, \cos \frac{4\pi g}{N}, \cdots, \cos \frac{2N\pi g}{N}\right) \text{ and } \left(\sin \frac{2\pi h}{N}, \sin \frac{4\pi h}{N}, \cdots, \sin \frac{2N\pi h}{N}\right)
$$

are characteristic vectors of the matrix of the quadratic form in $(x_i - m_i)$ of the numerator of R. For this reason the distribution and significance points are easily obtained.

In the usual applications the circular coefficient can be used even if the hypothesis alternative to independence of observations is non-circular serial correlation. The circular coefficient may not have as good power against non-circular alternatives as non-circular coefficients, such as

$$\frac{\sum_{i=2}^{N} (x_i - m_i)(x_{i-1} - m_{i-1})}{\sum_{i=1}^{N} (x_i - m_i)^2}. \tag{6}$$

However, the difference between these two statistics is $(x_1 - m_1)(x_N - m_N)/\Sigma(x_i - m_i)^2$, and it can be shown that this converges stochastically to zero (as N increases and ρ remains fixed).

2. The use of fitted Fourier series. A linear combination of trigonometric terms may be used as a regression function when there is a "systematic part" (or "trend") that is periodic. For instance, it may be reasonable to assume that a series of agricultural data has a systematic component with certain periodicities due to variation in weather. Then one may ask whether this regression function "explains" all of the interrelations in the series.

An example taken from agricultural economics is a development of that given by Koopmans [8]. Suppose p_t and q_t are the price and supply, respectively, of a given farm product at time t. Let $Q_t^{(d)}$ be the quantity demanded at time t if $p_t = P$, and $Q_t^{(s)}$ be the quantity supplied at time t if $p_{t-L} = P$, where P is an arbitrarily selected point of reference on the price scale, serving to define the Q's. Let the market equations be defined as follows:

$$p_t - P = -\beta(q_t - Q_t^{(d)}) + u_t, \tag{7}$$

$$q_t - Q_t^{(s)} = \delta(p_{t-L} - P) + v_t, \tag{8}$$

where u and v are random disturbances. The first equation expresses the price depressing tendency of an abnormally large supply; the second expresses the supply-stimulating influence of abnormally high prices L time units earlier (the time between planning the product and selling it). We can substitute from (7) at time $(t - L)$ into (8) and obtain

$$q_t - Q_t^{(s)} = \rho(q_{t-L} - Q_{t-L}^{(d)}) + w_t, \tag{9}$$

which is of the form (1) for general lag L ($i - 1$ is replaced by $t - L$) if $Q_t^{(s)} - \rho Q_{t-L}^{(d)} = \mu_t - \rho\mu_{t-L}$. Now we may wish to test the null hypothesis, $H_0 : \rho = 0$. If we assume that our alternative hypothesis is $H_a : \rho > 0$, we can test the null hypothesis by use of the positive tail of the distribution of R. Similarly for $H_a : \rho < 0$, we would use the negative tail of the distribution of R. In other cases, if we believe $\rho \neq 0$, we might wish to estimate ρ.

It is of particular interest to consider using the Fourier series for seasonal variation. The most important cases are given below with indications of the appropriate tables of significance points for testing the hypothesis $\rho = 0$. (a) *Annual data.* Here only a constant is fitted; this is the sample mean. The tables given in [2] or [5] are to be used. (b) *Semi-annual data.* To "correct" for variation of period two we fit a constant and $\cos \pi t = (-1)^t$. The table given in Section 3 for $P = 2$ is to be used. (c) *Quarterly data.* The four terms to be fitted are 1, $\cos \pi t = (-1)^t$, $\cos \frac{\pi t}{2}$, and $\sin \frac{\pi t}{2}$. The table given in Section 3 for $P = 2$ and 4 is to be used. (d) *Bimonthly data.* The six terms to be fitted are 1, $\cos \pi t$, $\cos \frac{2\pi t}{3}$, $\sin \frac{2\pi t}{3}$, $\cos \frac{\pi t}{3}$, and $\sin \frac{\pi t}{3}$. The table given in Section 3 for $P = 2, 3$, and 6 is to be used. (e) *Monthly data.* The twelve terms to be fitted are 1, $\cos \frac{\pi t}{6}$, $\sin \frac{\pi t}{6}$, $\cos \frac{\pi t}{3}$, $\sin \frac{\pi t}{3}$, $\cos \frac{\pi t}{2}$, $\sin \frac{\pi t}{2}$, $\cos \frac{2\pi t}{3}$, $\sin \frac{2\pi t}{3}$, $\cos \frac{5\pi t}{6}$, $\sin \frac{5\pi t}{6}$, and $\cos \pi t = (-1)^t$. The table given in Section 3 for $P = 2$, 12/5, 3, 4, 6, and 12 is to be used. It is assumed here that the data are given for each time interval in a certain number of years. Then the residuals are the same as the residuals taken from means for each month or season. That is, if the data are monthly, one may compute the sample means for January, February, etc., and residuals are to be taken from the corresponding monthly means. The fitted Fourier coefficients are certain linear functions of these means.

3. Tables of significance points of R.

3.1. *Significance points of R using a seasonal trend for annual, semi-annual, bimonthly, and monthly data.* The calculations of significance points of R (lag 1 only) have been subdivided according to the number of terms included in the estimating equations, m_i. The significance points for only a constant in m_i have been tabulated in [2] and [5]. Since the main use for m_i equations involving sine and cosine terms seems to be for semi-annual, quarterly, bimonthly, and monthly data, for which N is even, the results presented in this paper are for N even. Then we will have all of the sine and cosine terms in pairs except for $\cos \pi i = (-1)^i$ and the constant term. We shall find it convenient to refer to the period $P_{g'} = N/g'$ or $P_{h'} = N/h'$ of the terms in (2).

We have calculated significance points R' exact to 3 decimal places, for $Pr\{R > R'\} = \alpha = .01, .05, .95$, and .99. The values of R' corresponding to $\alpha = .01$ and .05 are usually indicated as the positive significance points and those corresponding to $\alpha = .95$ and .99, the negative significance points. In all of these cases, except for annual data, the distribution of R is symmetrical. Hence only the positive significance points need be given, since the negative points are simply the corresponding positive points with opposite sign; that is, $R'(.95) = -R'(.05)$, $R'(.99) = -R'(.01)$.

The significance points were calculated from the exact distribution of R given in Section 5 for all N up to the values where the approximate significance points using an Incomplete Beta distribution (Section 7) were the same as the exact significance points. The Incomplete Beta significance points were used up to the value of N for which a normal approximation was satisfactory. For some of the results, the normal points became sufficiently accurate to be used following the exact points.

The values of R' are given in Table 1 except for (a), for the following values of N:

(a) *Annual data*—see the tables in [2] or [5].

(b) *Semi-annual data* ($P = 2$): $N = 6(2)60$. The exact points were needed for N through 10($\alpha = .05$) and N through 22 ($\alpha = .01$). The normal points could be used for $N = 60$ ($\alpha = .05$) but were still too large by .003 for $N = 60$ ($\alpha = .01$)

(c) *Quarterly data* ($P = 2, 4$): $N = 8(4)100$. The exact points were needed for N through 20 ($\alpha = .05$) and N through 32 ($\alpha = .01$). The normal points were adequate for all N above 20 ($\alpha = .05$) but were still too large by .001 for $N = 100$ ($\alpha = .01$).

(d) *Bimonthly data* ($P = 2, 3, 6$): $N = 12(6)150$. The exact points were needed for N through 24 ($\alpha = .05$) and N through 30 ($\alpha = .01$). Again the normal points were adequate for all N above 24 ($\alpha = .05$) but were still too large by .0005 for $N = 150$ ($\alpha = .01$).

(e) *Monthly data* ($P = 2, 12/5, 3, 4, 6, 12$): $N = 24(12)300$. The exact points were needed for $N = 24$ ($\alpha = .05$) and $N = 24, 36$ ($\alpha = .01$). The normal points were adequate for $N > 24$ ($\alpha = .05$) and $N > 300$ ($\alpha = .01$).[4]

Significance points for the Incomplete Beta approximation (See Section 7) are tabulated in terms of $2p$ and $2q$. The values of $2p$ and $2q$ are the same when $\mu_1'(R) = 0$; for (c), (d), and (e) above these values are simply $N - 3$, $N - 5$, and $N - 11$, respectively. Hence, for two-tailed significance points for these cases, the ordinary correlation tables can be used with $N - 3$, $N - 5$, and $N - 11$ degrees of freedom, respectively. Also, our one-tailed significance points can be approximated by use of the 10% and 2% significance points for the ordinary correlation coefficient. 10%, 5%, 2%, 1%, and 0.1% two-tailed significance points have been tabulated by Fisher and Yates [6]. These significance points are accurate to three decimal places for the serial correlation coefficients as follows:[5]

(c) $n = N - 3$ degrees of freedom: $N \geq 24$ ($\alpha = .05$); $N \geq 36$ ($\alpha = .01$),

(d) $n = N - 5$ degrees of freedom: $N \geq 24$ ($\alpha = .05$); $N \geq 30$ ($\alpha = .01$),

(e) $n = N - 11$ degrees of freedom: $N \geq 24$ ($\alpha = .05$ and $\alpha = .01$), where α is the one-tailed significance point. For semi-annual data (b), $2p = 2q = \dfrac{N^2 - 3N + 4}{N - 4}$, which is not an integer for $N > 12$. When $N = 12$, $2p = 2q = 14$, for which the ordinary correlation significance point is adequate for $\alpha = .05$.

[4] It should be noted that for (c), (d), and (e), an approximation given by Cochran [4] is easily computed and is more accurate than the normal approximation for the $\alpha = .01$ significance points.

[5] In [6] n is 2 less than the number of pairs used in computing the ordinary correlation coefficient when the sample means are first subtracted.

Details of computing techniques using the exact distribution are given by R. L. Anderson [1] for computing values of R' when $\mu_i = 0$.

3.2. *Significance points of R for other single-period trends.* Significance points have also been obtained for $P = 3$, $P = 4$, $P = 6$, and $P = 12$, for which $K' = 3$.

TABLE 1

*Exact significance points, R', for different fitted series**

	$P = 2$			$P = 2,4$			$P = 2, 3, 6$			$P = 2, 12/5, 3, 4, 6, 12$	
$N\diagdown\alpha$	.05	.01	$N\diagdown\alpha$	.05	.01	$N\diagdown\alpha$	.05	.01	$N\diagdown\alpha$	.05	.01
6	.495	.499	8	.636	.693	12	.592	.744	24	.441	.592
8	.484	.607	12	.515	.661	18	.442	.592	36	.323	.445
10	.453	.601	16	.439	.582	24	.369	.504	48	.267	.371
12	.426	.572	20	.388	.523	30	.323	.445	60	.233	.325
14	.402	.544	24	.351	.478	36	.291	.403	72	.209	.293
16	.382	.519	28	.323	.441	42	.267	.371	84	.191	.268
18	.364	.496	32	.300	.414	48	.248	.346	96	.177	.249
20	.348	.476	36	.282	.391	54	.233	.325	108	.166	.234
22	.334	.458	40	.267	.371	60	.220	.308	120	.157	.221
24	.321	.442	44	.254	.354	66	.209	.293	132	.149	.210
26	.310	.427	48	.243	.338	72	.200	.280	144	.142	.200
28	.300	.414	52	.233	.325	78	.191	.268	156	.136	.192
30	.290	.402	56	.224	.313	84	.184	.258	168	.131	.184
32	.282	.390	60	.216	.302	90	.177	.249	180	.126	.178
34	.274	.380	64	.209	.293	96	.172	.241	192	.122	.172
36	.266	.370	68	.202	.284	102	.166	.234	204	.118	.166
38	.260	.361	72	.197	.276	108	.161	.227	216	.115	.162
40	.254	.353	76	.191	.268	114	.157	.221	228	.111	.157
42	.248	.345	80	.186	.261	120	.153	.215	240	.108	.153
44	.242	.338	84	.182	.255	126	.149	.210	252	.105	.149
46	.237	.331	88	.177	.249	132	.145	.205	264	.103	.146
48	.233	.324	92	.173	.243	138	.142	.200	276	.101	.142
50	.228	.318	96	.170	.238	144	.139	.196	288	.099	.140
52	.224	.313	100	.166	.234	150	.136	.192	300	.097	.136
54	.220	.307									
56	.216	.302									
58	.212	.297									
60	.209	.292									

* P = Periods Used in Fitted Series.

In these cases, the distribution of R is asymmetrical. The Incomplete Beta approximation is symmetrical for $P = 3$, with $2p = 2q = N - 2$, even though the exact distribution is not.

The significance points for these single-period trends are given in Table 2.

The exact distribution was required to compute the $\alpha = .01$ and .99 significance points for N through 48 in all cases and also for most cases with $\alpha = .05$ and .95. For $N > 48$, the Cochran approximation [4] gave the same results as the Incomplete Beta approximation. Since this Cochran approximation can be computed more rapidly, it should be used if other significance points are desired. The normal approximation is not recommended because it is less accurate than the Cochran approximation and requires almost as much calculation. For $\alpha = .01$ and .99, the significance points using the normal approximation were too large (in absolute value) by from .0005 to .001 for the last entries in Table 2. The two-

TABLE 2

Exact significance points, R', for single periods > 2

$P = 3$					$P = 6$				
N	α				N	α			
	.99	.95	.05	.01		.99	.95	.05	.01
6	−.970	−.854	.496	.500					
12	−.690	−.522	.475	.619	12	−.766	−.651	.296	.506
18	−.558	−.409	.392	.526	18	−.630	−.509	.277	.440
24	−.480	−.348	.340	.463	24	−.540	−.427	.254	.393
30	−.428	−.309	.304	.417	30	−.482	−.373	.236	.359
36	−.389	−.280	.277	.382	36	−.438	−.335	.220	.332
42	−.360	−.257	.256	.356	42	−.403	−.306	.207	.311
48	−.336	−.240	.240	.334	48	−.375	−.283	.197	.294
54	−.316	−.226	.226	.316	54	−.352	−.264	.188	.279
60	−.300	−.214	.214	.300	60	−.333	−.248	.180	.266
66	−.286	−.204	.204	.286	66	−.316	−.235	.173	.255
72	−.274	−.195	.195	.274	72	−.301	−.224	.167	.246
78	−.263	−.187	.187	.263	78	−.288	−.214	.161	.237
84	−.254	−.181	.181	.254	84	−.277	−.205	.156	.229
90	−.245	−.175	.175	.245	90	−.267	−.197	.151	.222
96	−.237	−.169	.169	.237	96	−.258	−.190	.147	.216
102	−.230	−.164	.164	.230	102	−.250	−.184	.143	.210
108	−.224	−.159	.159	.224	108	−.242	−.178	.140	.205
114	−.218	−.155	.155	.218	114	−.235	−.173	.137	.200
120	−.212	−.151	.151	.212	120	−.229	−.168	.134	.195
126	−.207	−.147	.147	.207	126	−.223	−.163	.131	.191
132	−.202	−.144	.144	.202	132	−.218	−.159	.128	.187
138	−.198	−.141	.141	.198	138	−.213	−.155	.125	.183
144	−.194	−.138	.138	.194	144	−.208	−.152	.123	.180
150	−.190	−.135	.135	.190	150	−.203	−.148	.121	.177

TABLE 2—*Continued*

P = 4					P = 12				
N	α .99	α .95	α .05	α .01	N	α .99	α .95	α .05	α .01
8	−.889	−.768	.503	.637	12	−.778	−.671	.096	.245
12	−.742	−.608	.420	.585	24	−.555	−.444	.197	.330
16	−.643	−.502	.369	.522	36	−.447	−.348	.188	.298
20	−.576	−.441	.333	.474	48	−.383	−.293	.175	.270
24	−.519	−.396	.306	.437	60	−.339	−.257	.163	.249
28	−.477	−.361	.285	.407	72	−.307	−.231	.153	.231
32	−.445	−.334	.268	.383	84	−.283	−.212	.145	.217
36	−.418	−.312	.253	.363	96	−.263	−.196	.138	.206
40	−.395	−.293	.241	.345	108	−.247	−.183	.132	.196
44	−.375	−.277	.230	.330	120	−.233	−.173	.126	.187
48	−.358	−.264	.221	.317	132	−.221	−.164	.121	.180
52	−.343	−.252	.213	.305	144	−.211	−.156	.117	.173
56	−.330	−.242	.206	.294	156	−.202	−.149	.113	.167
60	−.319	−.233	.199	.285	168	−.194	−.143	.110	.162
64	−.308	−.225	.193	.277	180	−.187	−.138	.107	.157
68	−.298	−.218	.188	.269	192	−.181	−.133	.104	.153
72	−.289	−.211	.183	.262	204	−.175	−.128	.101	.149
76	−.281	−.205	.178	.255	216	−.170	−.124	.099	.145
80	−.274	−.199	.174	.249	228	−.165	−.121	.097	.141
84	−.267	−.194	.170	.243	240	−.161	−.117	.094	.138
88	−.261	−.189	.166	.238	252	−.157	−.114	.092	.135
92	−.255	−.184	.162	.233	264	−.153	−.111	.091	.132
96	−.249	−.180	.159	.228	276	−.149	−.109	.089	.130
100	−.244	−.176	.156	.223	288	−.146	−.106	.087	.127
108	−.234	−.169	.150	.215	300	−.143	−.104	.086	.125
120	−.221	−.160	.143	.205					
132	−.210	−.152	.136	.196					
144	−.201	−.145	.131	.187					

tailed significance points cannot be obtained from the ordinary correlation tables except for $P = 3$.

3.3. *Example of use of significance points.* As an example of the use of these significance points, R', we shall consider the following data [17] on the receipts of butter (in units of 1,000,000 pounds) at five markets (Boston, Chicago, San Francisco, Milwaukee, and St. Louis). The figures in parentheses are deviations from the average of the given months over the 3 years.

Month	Year			Total T_i	Average
	1935	1936	1937		
Jan.	48.9(2.4)	48.3(1.8)	42.4(−4.1)	139.6	46.5
Feb.	43.4(−0.6)	47.1(3.1)	41.4(−2.6)	131.9	44.0
March	43.8(−4.6)	52.4(4.0)	49.0(0.6)	145.2	48.4
April	50.8(−1.5)	55.3(3.0)	50.8(−1.5)	156.9	52.3
May	67.6(1.6)	64.7(−1.3)	65.8(−0.2)	198.1	66.0
June	83.7(0.7)	79.5(−3.5)	85.9(2.9)	249.1	83.0
July	82.7(10.7)	62.6(−9.4)	70.6(−1.4)	215.9	72.0
Aug.	60.8(4.8)	51.3(−4.7)	55.8(−0.2)	167.9	56.0
Sept.	55.4(3.6)	51.0(−0.8)	49.1(−2.7)	155.5	51.8
Oct.	48.4(−1.0)	54.0(4.6)	45.7(−3.7)	148.1	49.4
Nov.	37.7(−4.5)	45.2(3.0)	43.8(1.6)	126.7	42.2
Dec.	41.0(−3.2)	44.9(0.7)	46.7(2.5)	132.6	44.2
Total	664.2(8.4)	656.3(0.5)	647.0(−8.8)	1967.5	655.8
Average	55.35(0.70)	54.69(0.04)	53.92 (−0.73)	163.96	54.65

We assume that the trend is composed of the 12 terms having periods that divide 12. We shall test the null hypothesis that the deviations from the trend are independently distributed against the alternative that there is positive serial correlation. The fitted series is of the form

$$m_i = b_0^* + \sum_{j=1}^{5} \left(b_{2j-1}^* \cos \frac{\pi i j}{6} + b_{2j}^* \sin \frac{\pi i j}{6} \right) + b_{11}^* \cos \pi i; \tag{10}$$

here we find it convenient to use the notation, $b_0^*, b_1^*, \cdots, b_{11}^*$, for the coefficients (with a different relationship between the subscripts and the trigonometric functions than in (4)). We find that the m_i are simply the average receipts given for each month in the above table (46.5, 44.0, $\cdots$, 44.2). Hence the deviations $(x_i - m_i)$ are given by the figures in parentheses (2.4, −0.6, $\cdots$, 2.5). The calculated lag 1 circular serial correlation coefficient is

$$\begin{aligned} R_0 &= \frac{(2.4)(-0.6) + (-0.6)(-4.6) + \cdots + (1.6)(2.5) + (2.5)(2.4)}{(2.4)^2 + (-0.6)^2 + \cdots + (2.5)^2} \\ &= \frac{232.18}{474.51} = 0.489. \end{aligned} \tag{11}$$

Entering Table 1 for $P = 2, 12/5, 3, 4, 6$, and 12 and $N = 36$, we find that $R'(.05) = 0.323$ and $R'(.01) = .445$. Hence, at either the 5% or 1% level the null hypothesis of zero serial correlation ($\rho = 0$) is to be rejected (against the alternative single-tail hypothesis, $\rho > 0$). If we had been interested in the two-tailed alternative hypothesis, $\rho \neq 0$, we would use the ordinary correlation tables with $N - 11 = 25$ degrees of freedom and we would find that for the two-tailed test $R'(.01) = 0.487$. Our value is significant at the 5% level and barely significant at the 1% level.

The values of b^* in (10) are computed as follows

$$b_0^* = \sum_{i=1}^{12} T_i/36,$$

$$(12) \qquad b_{2j-1}^* = \sum_{i=1}^{12} T_i \cos \frac{\pi i j}{6} \Big/ 18,$$

$$b_{2j}^* = \sum_{i=1}^{12} T_i \sin \frac{\pi i j}{6} \Big/ 18,$$

$$b_{11}^* = \sum_{i=1}^{12} T_i \cos \pi i/36.$$

The computed values of b_0^* to b_{11}^* are 54.65, −14.82, −2.02, 6.60, 1.23, −3.98, 0.30, 2.21, 1.73, −0.61, 0.60, 0.15, respectively. However, it is not necessary to compute these values in order to obtain m_i. The problem of estimating the variances of these b's will be discussed in Section 4.

4. Testing the hypothesis of lack of serial correlation.

4.1. *Statement of the problem.* Consider the N random variables $u_1, \cdots, u_N$, each normally and independently distributed with mean 0 and variance σ^2. Define the N variables $x_1, \cdots, x_N$ by the equations

$$(13) \qquad x_i - \mu_i = \rho(x_{i-L} - \mu_{i-L}) + u_i \qquad (i = 1, \cdots, N),$$

where

$$(14) \qquad x_{-j} \equiv x_{N-j},\ \mu_{-j} \equiv \mu_{N-j} \qquad (j = 0, 1, \cdots, N-1)$$

and μ_i is the linear combination of trigonometric functions given in (2). If L and N are relatively prime (in particular, if $L = 1$), the Jacobian of the transformation from $\{u_i\}$ to $\{x_i\}$ is $1 - \rho^N$, and the probability density of $\{x_i\}$ is

$$(15) \qquad \frac{1 - \rho^N}{(2\pi\sigma^2)^{\frac{1}{2}N}} e^{-\frac{1}{2}Q/\sigma^2},$$

where $Q = (1 + \rho^2) \sum_{i=1}^{N} (x_i - \mu_i)^2 - 2\rho \sum_{i=1}^{N} (x_i - \mu_i)(x_{i-L} - \mu_{i-L})$. If $L = 1$, the covariance between x_i and x_j is $\sigma^2[\rho^{|i-j|} + \rho^{N-|i-j|}]/[(1 - \rho^N)(1 - \rho^2)]$. If $L = q\alpha$ and $N = p\alpha$, where p, q, and α are positive integers and q and p are relatively prime, then the Jacobian is $(1 - \rho^p)^\alpha$ and the density of $\{x_i\}$ is

$$(16) \qquad \frac{(1 - \rho^p)^\alpha}{(2\pi\sigma^2)^{\frac{1}{2}N}} e^{-\frac{1}{2}Q/\sigma^2}.$$

We shall now obtain the likelihood ratio test of the hypothesis $H_0 : \rho = 0$ on the basis of a sample consisting of one observation on each x_i.

4.2. *Preliminary transformations.* We shall find it convenient to express μ_i in terms of fixed variates ϕ_{ij}, having certain properties. Later we will verify that the ϕ's are simply constant multiples of the trigonometric terms in (2). We suppose now that

$$\mu_i = \sum_{j=1}^{K'} \phi_{ij}\gamma_j \qquad (i = 1, \cdots, N), \tag{17}$$

where $K' < N$, the $\{\gamma_j\}$ are parameters, and the ϕ_{ij} are known functions of i and j satisfying

$$\phi_{i-L,j} + \phi_{i+L,j} = 2\lambda_{Lj}\phi_{ij} \qquad (i = 1, \cdots, N;\ j = 1, \cdots, K'), \tag{18}$$

$$\sum_{i=1}^{N} \phi_{ij}\phi_{ik} = \delta_{jk} \qquad (j, k = 1, \cdots, K'), \tag{19}$$

$$\phi_{-i,j} = \phi_{N-i,j} \qquad (i = 0, 1, \cdots, N - 1), \tag{20}$$

and δ_{jk} is the Kronecker delta. Let

$$m_i = \sum_{j=1}^{K'} \phi_{ij}c_j, \tag{21}$$

where

$$c_j = \sum_{i=1}^{N} x_i\phi_{ij}. \tag{22}$$

Then by usual regression theory we have

$$\sum_{i=1}^{N} (x_i - m_i)\phi_{ij} = 0, \tag{23}$$

$$\sum_{i=1}^{N} (x_i - \mu_i)^2 = \sum_{i=1}^{N} (x_i - m_i)^2 + \sum_{j=1}^{K'} (c_j - \gamma_j)^2 \tag{24}$$

because c_j is the least squares estimate of γ_j. Let us evaluate

$$\begin{aligned} {}_L\bar{C} &= \sum_{i=1}^{N} (x_i - \mu_i)(x_{i-L} - \mu_{i-L}) \\ &= \sum_{i=1}^{N} [(x_i - m_i) + (m_i - \mu_i)][(x_{i-L} - m_{i-L}) + (m_{i-L} - \mu_{i-L})] \\ &= \sum_{i=1}^{N} (x_i - m_i)(x_{i-L} - m_{i-L}) + \sum_{i=1}^{N}\sum_{j=1}^{K'} \phi_{i-L,j}(c_j - \gamma_j)(x_i - m_i) \\ &\quad + \sum_{i=1}^{N}\sum_{j=1}^{K'} \phi_{ij}(c_j - \gamma_j)(x_{i-L} - m_{i-L}) \\ &\quad + \sum_{i=1}^{N}\sum_{j,k=1}^{K} \phi_{ik}\phi_{i-L,j}(c_k - \gamma_k)(c_j - \gamma_j). \end{aligned} \tag{25}$$

Call the first term on the right hand side of (25) $_L C$. In view of (20) the next two terms are

$$\sum_{j=1}^{K'} \sum_{i=1}^{N} (x_i - m_i)(\phi_{i-L,j} + \phi_{i+L,j})(c_j - \gamma_j). \tag{26}$$

This is seen to be zero by consideration of (18) and (23). The last term can be written

$$\frac{1}{2} \sum_{i=1}^{N} \sum_{j,k=1}^{K'} (\phi_{ik}\phi_{i-L,j} + \phi_{i+L,j}\phi_{ik})(c_k - \gamma_k)(c_j - \gamma_j) = \sum_{j=1}^{K'} \lambda_{Lj}(c_j - \gamma_j)^2 \tag{27}$$

by use of (18), (19), and (20). Thus

$$_L\bar{C} = \sum_{i=1}^{N} (x_i - m_i)(x_{i-L} - m_{i-L}) + \sum_{j=1}^{K'} \lambda_{Lj}(c_j - \gamma_j)^2. \tag{28}$$

It follows that

$$\begin{aligned} Q = (1 + \rho^2) \sum_{i=1}^{N} (x_i - m_i)^2 - 2\rho \sum_{i=1}^{N} (x_i - m_i)(x_{i-L} - m_{i-L}) \\ + \sum_{j=1}^{K'} (1 + \rho^2 - 2\rho\lambda_{Lj})(c_j - \gamma_j)^2. \end{aligned} \tag{29}$$

We can complete the matrix $\Phi = (\phi_{ij})$ so that Φ is an N-th order square matrix with elements satisfying (18), (19), and (20). If we make the transformation

$$x_i = \sum_{j=1}^{N} \phi_{ij} c_j \qquad (i = 1, \cdots, N), \tag{30}$$

then

$$\sum_{i=1}^{N} (x_i - m_i)^2 = \sum_{j=K'+1}^{N} c_j^2, \tag{31}$$

$$\sum_{i=1}^{N} (x_i - m_i)(x_{i-L} - m_{i-L}) = \sum_{j=K'+1}^{N} \lambda_{Lj} c_j^2. \tag{32}$$

4.3. *The likelihood ratio criterion*. To obtain the likelihood ratio test of the hypothesis $H_0 : \rho = 0$ against alternative hypotheses $H_a : \rho \neq 0$, we divide the maximum of the likelihood assuming H_0 by the maximum of the likelihood assuming H_a. It is clear from (15) and (29) that if H_0 is true, the maximum likelihood estimates of γ_j and σ^2 are c_j and

$$s_0^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - m_i)^2, \tag{33}$$

respectively. If H_a is true, the maximum likelihood estimate of γ_j is c_j. To state the maximum likelihood estimates of σ^2 and ρ under H_a it is convenient to define $_L R$, the sample serial coefficient of lag L, as

$$_L R = \frac{1}{N s_0^2} \sum_{i=1}^{N} (x_i - m_i)(x_{i-L} - m_{i-L}). \tag{34}$$

Then the maximum likelihood estimate of σ^2 under H_a is

$$s^2 = s_0^2(1 + \hat{\rho}^2 - 2\hat{\rho}_L R), \tag{35}$$

where $\hat{\rho}$ is the maximum likelihood estimate of ρ and satisfies

$$_LR(1 + \hat{\rho}^N) - \hat{\rho}(1 + \hat{\rho}^{N-2}) = 0, \tag{36}$$

if L and N are relatively prime and satisfies

$$_LR(1 + \hat{\rho}^p) - \hat{\rho}(1 + \hat{\rho}^{p-2}) = 0, \tag{37}$$

if $L = q\alpha$, $N = p\alpha$, and p and q are relatively prime.

Upon substituting these estimates into the likelihood function we find that the likelihood ratio criterion is

$$\lambda = \frac{(1 + \hat{\rho}^2 - 2\hat{\rho}_L R)^{\frac{1}{2}N}}{1 - \hat{\rho}^N}, \tag{38}$$

if L and N are relatively prime and

$$\lambda = \left[\frac{(1 + \hat{\rho}^2 - 2\hat{\rho}_L R)^{\frac{1}{2}p}}{1 - \hat{\rho}^p}\right]^\alpha, \tag{39}$$

if $L = q\alpha$, $N = p\alpha$ and p and q are relatively prime. The maximum likelihood estimate of ρ is the root of (36) or (37) that makes (38) or (39), respectively, a minimum. It should be noticed that throughout this section ρ could be replaced by $1/\rho$ (and changing σ^2 by a factor $1 + \rho^2$). To make the maximum likelihood estimate unique, we require that $|\hat{\rho}| \leq 1$. It can be shown that there exists one and only one root of (36) or (37) that satisfies this requirement and minimizes λ. (There is a peculiarity to this solution in that if N is odd, $L = 1$, and $_1R < -1 + 2/N$, then $\hat{\rho} = -1$ is the root minimizing λ). In any case, λ is a function of $_LR$. We have shown that for $0 < {_LR} < 1$, it is a monotonic decreasing function; and for $-1 < {_LR} < 0$, it is a monotonic increasing function. A critical region defined by $\lambda \leq \lambda_0$ can, therefore, be defined by $_LR \leq R_1 < 0$ and $0 < R_2 \leq {_LR}$. (The probability that $_LR = -1$ or $+1$ is 0.) Thus we can use $_LR$ to test the null hypothesis $H_0 : \rho = 0$ instead of the likelihood ratio criterion (against one-sided alternatives they are equivalent). The strongest justification for the use of $_LR$ in testing $H_0 : \rho = 0$ is that for circular distributions the uniformly most powerful tests against one-sided alternatives and the B_1 test against two-sided alternatives are given in terms of inequalities on $_LR$ [3].

We can also use $_LR$ as an estimate of ρ. In fact, $_LR$ is asymptotically a root of (36) or (37). This is proved by showing that $_LR(1 + {_LR}^N) - {_LR}(1 + {_LR}^{N-2}) = {_LR}^{N-1}(1 - {_LR}^2)$ converges stochastically to zero. We shall use $_LR$ both to estimate ρ and to test hypotheses about this parameter.[6]

Now we shall define ϕ_{ij} used in Section 4.2 in terms of the trigonometric terms indicated in Section 1. In the rest of the paper we shall let the index g run from

[6] W. J. Dixon [5] arrived at $_LR$ as the maximum likelihood estimate for μ_i a constant by neglecting the Jacobian in (15).

0 to $\frac{1}{2}N$ for N even and from 0 to $\frac{1}{2}(N - 1)$ for N odd; we let the index h run from 1 to $\frac{1}{2}N - 1$ for N even and from 1 to $\frac{1}{2}(N - 1)$ for N odd. We shall use a prime to denote an index running over those values corresponding to fitted terms and a double prime to denote an index running over those values corresponding to terms not fitted.

Let the N trigonometric functions of i, namely $\cos \frac{2\pi i g}{N}$ and $\sin \frac{2\pi i h}{N}$, be numbered from 1 to N such that the fitted terms are numbered from 1 to K' and the non-fitted terms from $K' + 1$ to N. According to this numbering we define ϕ_{i1} to ϕ_{iN} as

$$\phi_{ij} = \sqrt{\frac{2}{N}} \cos \frac{2\pi i g}{N}, \tag{40}$$

or

$$\phi_{ij} = \sqrt{\frac{2}{N}} \sin \frac{2\pi i h}{N}. \tag{41}$$

Defined this way, the ϕ_{ij} satisfy (18) and (19) and (20). It can be shown by using the addition formulas for sines and cosines that

$$\lambda_{L,j} = \cos \frac{2\pi L f}{N}, \tag{42}$$

where $f = g$ or $f = h$ depending on whether j refers to a term (40) or (41). We shall assume that the numbering of trigonometric functions is such that

$$\lambda_{L,K'+1} \geq \lambda_{L,K'+2} \geq \cdots \geq \lambda_{L,N}. \tag{43}$$

It can easily be seen that (2) is of the form (17) except that $\alpha_{g'}$ and $\beta_{h'}$ must be multiplied by $\sqrt{\frac{1}{2}N}$ unless $g' = 0$ or $\frac{1}{2}N$ and by $\sqrt{N}$ for $g' = 0, \frac{1}{2}N$ to obtain γ_j. The regression coefficients $a_{g'}$ and $b_{h'}$ are similarly related to the c_j.

It can be seen from (29) that the a_f and b_f are independently distributed with variance $\frac{1}{2}N\sigma^2 \Big/ \left(1 + \rho^2 - 2\rho \cos \frac{2\pi L f}{N}\right)$ for $f \neq 0, \frac{1}{2}N$ and variance $N\sigma^2/(1 - \rho)^2$ for $f = 0$ and for $f = \frac{1}{2}N$ if L is even and $N\sigma^2/(1 + \rho)^2$ for $f = \frac{1}{2}N$ if L is odd. In these variance formulas we can estimate σ^2 from (35) using ${}_LR$ for $\hat{\rho}$ and ρ.

5. The exact distribution of ${}_LR$.

5.1. *Introduction*. Under the null hypothesis $H_0 : \rho = 0$ the observations $\{x_i\}$ are normally and independently distributed with variance σ^2 and means $Ex_i = \mu_i$. The variables c_j defined by (22) and (29) are normally and independently distributed with variance σ^2 and means γ_j. For $j > K'$, $\gamma_j = 0$. It follows from (31), (32), (33), and (34) that

$$ {}_LR = \frac{\sum_{j=K'+1}^{N} \lambda_{Lj} c_j^2}{\sum_{j=K'+1}^{N} c_j^2}, \tag{44}$$

where the $\lambda_{L,j}$ are given by (42) corresponding to the $K'' = (N - K')$ trigonometric terms not fitted. Thus to obtain the distribution of ${}_LR$ we need only consider the joint distribution of $\{c_j\}$, $j = K' + 1, \cdots, N$. If H_a is true, the joint density of all the c_j is (15), where

$$Q = (1 + \rho^2)V - 2\rho\, {}_LC + \sum_{j=1}^{K'} (1 + \rho^2 - 2\rho\lambda_{Lj})(c_j - \gamma_j)^2, \tag{45}$$

and

$$V = \sum_{j=K'+1}^{N} c_j^2 \quad \text{and} \quad {}_LC = \sum_{j=K'+1}^{N} \lambda_{Lj} c_j^2 .$$

5.2. *Some special distributions of* ${}_1R = R$. If the constant term ($g' = 0$) is fitted and the other terms are fitted in pairs $\left(\cos \frac{2\pi i f}{N} \text{ and } \sin \frac{2\pi i f}{N}\right)$, then K' is odd. If N is odd, then K'' is even; the λ_{1j} occur in pairs and we can define λ_k'' as

$$\begin{aligned} \lambda_{1,K'+1} = \lambda_{1,K'+2} = \lambda_1'' > \lambda_{1,K'+3} = \lambda_{1,K'+4} \\ = \lambda_2'' > \cdots > \lambda_{1,N-1} = \lambda_{1N} = \lambda_{\frac{1}{2}K''}'' . \end{aligned} \tag{46}$$

This also holds if N is even and if, in addition to the constant term and paired cosines and sines, we fit $\cos \pi i = (-1)^i (g' = N/2)$. If N is even and we do not fit $\cos \pi i$, we have K'' odd. Then

$$\begin{aligned} \lambda_{1,K'+1} = \lambda_{1,K'+2} = \lambda_1'' > \lambda_{1,K'+3} = \lambda_{1,K'+4} = \lambda_2'' > \cdots > \lambda_{1,N-2} \\ = \lambda_{1,N-1} = \lambda_{\frac{1}{2}(K''-1)}'' > \lambda_{1N} = \lambda_{\frac{1}{2}(K''+1)}'' = -1. \end{aligned} \tag{47}$$

The general expression for the distribution of R in these cases has been found by one of the authors [2]. In this case the cumulative distribution function is 1 minus

$$\begin{aligned} Pr\{R > R'\} = \sum_{k=1}^{m} (-1)^{k+1} \mid V_k \mid (\lambda_k'' - R')^{\frac{1}{2}K''-1}, \\ \lambda_{m+1}'' \leq R' \leq \lambda_m'', \end{aligned} \tag{48}$$

where V_k is found from a result of Lehmann [9] to be

$$V_k = \frac{2^{\frac{1}{2}(N+1)}}{N} \sin \frac{2\pi f''}{N} \sin \frac{\pi f''}{N} \prod_{j'} \sqrt{(\lambda_k'' - \lambda_{1j'})}, \tag{49}$$

where f'' is such that $\lambda_k'' = \cos \frac{2\pi f''}{N}$ and the product on j' is over the K' terms $\lambda_{1j'}$, excluding $\lambda_{1j'} = 1$. Hence, $\lambda_{1j'}$ takes on $K' - 1$ values in $\frac{1}{2}(K' - 1)$ pairs if K' is odd and in $\frac{1}{2}(K' - 2)$ pairs plus a single $\lambda_{1,j'} = -1$ if K' is even. We can also write V_k as

$$V_k = \frac{2^{\frac{1}{2}(N+K')}}{N} \sin \frac{2\pi f''}{N} \sin \frac{\pi f''}{N} \prod_{g' \neq 0} \sqrt{\sin \frac{\pi(g' + f'')}{N} \sin \frac{\pi(g' - f'')}{N}} \cdot \prod_{h'} \sqrt{\sin \frac{\pi(h' + f'')}{N} \sin \frac{\pi(h' - f'')}{N}}. \tag{50}$$

5.3. *Some special distributions of ${}_LR$ for $L > 1$.* We have noted in (44) above that $\lambda_{L,j} = \cos \frac{2\pi L f''}{N}$, where f'' corresponds to a term not used in the estimation equations for m_i, which was a function of $\left\{\cos \frac{2\pi i g'}{N}, \sin \frac{2\pi i h'}{N}\right\}$. If L, the lag, is relatively prime to N, the distribution is the same as that given above for $L = 1$, except for the re-evaluating of the λ_k''. In the article by R. L. Anderson [2], where only the constant term in m_i was used, the λ_k'' for lag L were exactly the same as the λ_k'' for lag 1. However, this will not be the case for other terms used in m_i. For example, consider lag 2 and N odd with m_i consisting of the constant term plus terms in $\cos \frac{2\pi i}{N}$ and $\sin \frac{2\pi i}{N}$. In this case the λ_k'' for lag 1 are $\left\{\cos \frac{4\pi}{N}, \cos \frac{6\pi}{N}, \cdots, \cos \frac{(N-1)\pi}{N}\right\}$ and the λ_k'' for lag 2 are

$$\left\{\cos \frac{2\pi}{N}, \cos \frac{6\pi}{N}, \cos \frac{8\pi}{N}, \cdots, \cos \frac{(N-1)\pi}{N}\right\}.$$

Next suppose the highest common factor of L and N is α (as before, $L = q\alpha$ and $N = p\alpha$, with p and q relatively prime). In this case

$$\lambda_{L,j} = \cos \frac{2\pi q f''}{p}. \tag{51}$$

Since p and q are relatively prime, the results are the same as for q replaced by 1 and L replaced by α. Each root is repeated α times.

$N = 2L (p = 2)$

If we let $N = 2L$, $\lambda_k'' = \cos \pi k = +1$ or -1. $\lambda'' = +1$ corresponds to these fitted terms in m_i: $\left\{1, \cos \frac{2\pi i g'}{N}, \sin \frac{2\pi i h'}{N}\right\}$ for g', h' even. $\lambda'' = -1$ corresponds to these terms: $\left\{\cos \frac{2\pi i g'}{N}, \sin \frac{2\pi i h'}{N}\right\}$ for g', h' odd. Let $L - n_1$ be the number of terms pertaining to $\lambda'' = +1$ and $L - n_2$ be the number of terms for $\lambda'' = -1$. Then, as in [2], we have the density

$$D({}_LR_2) = \frac{(1 - {}_LR_2)^{\frac{1}{2}(n_2 - 2)}(1 + {}_LR_2)^{\frac{1}{2}(n_1 - 2)}}{2^{\frac{1}{2}(n_1 + n_2) - 1}\beta(\frac{1}{2}n_1, \frac{1}{2}n_2)}, \tag{52}$$

where ${}_LR_2$ was the notation used for lag L and $p = 2$. The cumulative function is the Incomplete Beta function, found by setting $x = \frac{1}{2}(1 - R')$.

$N = 3L (p = 3)$

If we let $N = 3L$, $\lambda_k'' = \cos \frac{2\pi f''}{N} = +1, -\frac{1}{2}$. The fitted terms in m_i corresponding to $\lambda'' = 1$ are $\left\{1, \cos \frac{2\pi i g'}{N}, \sin \frac{2\pi i h'}{N}\right\}$ for $g', h' = 3m$. Similarly, those corresponding to $\lambda'' = -\frac{1}{2}$ have $g', h' = 3m - 1$ or $3m - 2$. Let the number of fitted terms with $\lambda'' = +1$ be $L - n_1$ and with $\lambda'' = -\frac{1}{2}$ be $2L - n_2$. Then

$$D({}_LR_3) = \frac{(1 - {}_LR_3)^{\frac{1}{2}(n_2-2)}(\frac{1}{2} + {}_LR_3)^{\frac{1}{2}(n_1-2)}}{(3/2)^{\frac{1}{2}(n_1+n_2)-1}\beta(\frac{1}{2}n_1, \frac{1}{2}n_2)}, \tag{53}$$

where ${}_LR_3 \geq -\frac{1}{2}$. This cumulative function is also an Incomplete Beta function, found by setting $x = 2(1 - R')/3$.

$N = 4L(p = 4)$

If $N = 4L$, $\lambda_k'' = \cos \frac{2\pi f''}{N} = +1, 0, -1$. The fitted terms in m_i corresponding to $\lambda'' = 1$ have $f'' = 4m$, those for $\lambda'' = -1$ have $f'' = 4m - 2$; and those for $\lambda'' = 0$ have $f'' = 4m - 1$ or $4m - 3$. Let the number of terms in m_i of each sort be $L - n_1$, $L - n_2$, and $2L - n_3$, respectively. Then

$$D(R) = c\begin{cases} (1 + R)^{\frac{1}{2}(n_1+n_3-2)} \int_{y=0}^{1} y^{\frac{1}{2}(n_3-2)} (1 - y)^{\frac{1}{2}(n_1-2)} \\ \qquad \cdot [(1 - R) - y(1 + R)]^{\frac{1}{2}(n_2-2)}\, dy, & \text{for } R \leq 0, \\ (1 - R)^{\frac{1}{2}(n_2+n_3-2)} \int_{y=0}^{1} y^{\frac{1}{2}(n_3-2)}(1 - y)^{\frac{1}{2}(n_2-2)} \\ \qquad \cdot [(1 + R) - y(1 - R)]^{\frac{1}{2}(n_1-2)}\, dy, & \text{for } R \geq 0, \end{cases} \tag{54}$$

where R is ${}_LR_4$ and $c = \Gamma(\frac{1}{2}[n_1 + n_2 + n_3])/[\Gamma(\frac{1}{2}n_1)\Gamma(\frac{1}{2}n_2)\Gamma(\frac{1}{2}n_3)2^{\frac{1}{2}(n_1+n_2-2)}]$.

5.4. *The exact distribution of ${}_LR$ when $\rho \neq 0$.* The joint distribution of the observations for lag 1 when the null hypothesis is not true ($\rho \neq 0$) is (15), where Q is given by (45) with $L = 1$ and ${}_1C = RV$. V, R, $\{c_j\}(j = 1, \cdots, K')$ are a sufficient set of statistics for estimating σ^2, ρ, and $\{\gamma_j\}(j = 1, \cdots, K')$. Using the results given by Madow [11], it can be shown that the simultaneous distribution of V and R is

$$\frac{1 - \rho^N}{2^{\frac{1}{2}K''}\Gamma(\frac{1}{2}K'')\sqrt{\prod_{j'=1}^{K'} (1 + \rho^2 - 2\rho\lambda_{1j'})}} V^{\frac{1}{2}K''-1}e^{-V(1+\rho^2-2\rho R)/2\sigma^2} D(R), \tag{55}$$

where $D(R)$ is the density function corresponding to (48). Integrating V from 0 to ∞, we obtain as the density for R

$$\frac{(1 - \rho^N)(\frac{1}{2}K'' - 1)}{\sqrt{\prod_{j'=1}^{K'} (1 + \rho^2 - 2\rho\lambda_{1j'})}} (1 + \rho^2 - 2\rho R)^{\frac{1}{2}K''} \cdot \sum_{k=1}^{m} (-1)^{k+1}(\lambda_k'' - R)^{\frac{1}{2}(K''-4)} \mid V_k \mid, \tag{56}$$

for $\lambda''_{m+1} \leq R \leq \lambda''_m$, where V_k are given by (50). In the same way, one obtains the distribution of ${}_LR$ for $\rho \neq 0$ when $N = 2L$, $N = 3L$, and $N = 4L$ by multiplying (52), (53), and (54), respectively, by

$$(57) \qquad (1 - \rho^P)^L \frac{(1 + \rho^2 - 2\rho R)^{\frac{1}{2}K''}}{\sqrt{\prod_{j'=1}^{K'} (1 + \rho^2 - 2\rho\lambda_{L j'})}},$$

where $K'' = n_1 + n_2$ or $n_1 + n_2 + n_3$. This method was used by Madow for residuals from the sample mean [12].

6. Moments.

6.1. *The exact moments of R.* Most of the results of this section are straightforward adaptations of earlier results for the case of μ_i constant. Hence, we shall omit the details of derivations. The moment generating function of V and C for $\sigma^2 = 1$ is

$$(58) \qquad \phi(t_0, t) = E(e^{t_0 V + tC}) = \frac{1 - \rho^N}{\prod_{j''=K'+1}^{N} \left[1 + \rho^2 - 2t_0 - 2(\rho + t)\lambda_{1,j''}\right]^{\frac{1}{2}}}.$$

The h^{th} moment of $R = C/V$ is given by

$$(59) \qquad \mu_h'(R) = \int_{-\infty}^{0} \int_{-\infty}^{y_{h-1}} \cdots \int_{-\infty}^{y_1} \frac{\partial^h \phi}{\partial t^h}\bigg]_{t=0} dt_0 \prod_{i=1}^{h-1} dy_i,$$

with the $\{y_i\}$ restricted from being too large (not more than a certain amount larger than zero). In the case of independence, ($\rho = 0$), we have the following first two moments of R:

$$(60) \qquad \begin{aligned} \mu_1'(R) &= \frac{1}{K''} \sum_{j''=K'+1}^{N} \lambda_{1,j''}; \\ \mu_2'(R) &= \frac{2}{K''(K''+2)} \sum_{j''=K'+1}^{N} \lambda_{1,j''}^2 + \frac{K''}{K''+2}[\mu_1'(R)]^2. \end{aligned}$$

If the $\lambda_{1,j''}$ are symmetrical (i.e. for each $\lambda_{1,j''}$, there is a $\lambda_{1,k''} = -\lambda_{1,j''}$), the mean of R is 0. For example, if 1 and $(-1)^t$ are fitted for N even, the mean is 0.

6.2. *Approximate moments of R when $\rho = 0$.* Since R and V are independent [8] when $\rho = 0$, $\mu'(R) = \mu'(C)/\mu'(V)$. V is a sum of squares and its moments are the same as for χ^2 with $N - K' = K''$ degrees of freedom. Using methods similar to those given by Dixon [5], we see that the moment generating function for C is

$$(61) \qquad \phi(t) = \alpha(t) \cdot \beta(t) \cdot \gamma(t),$$

where

$$(62) \qquad \begin{aligned} &\alpha(t) = \left(\frac{2}{A}\right)^{\frac{1}{2}N}, \beta(t) = A^N/[A^N - (2t)^N], \\ &\gamma(t) = \textstyle\prod_{j'} (1 - 2t\,\lambda_{1,j'}), \text{ and } A = 1 + \sqrt{1 - 4t^2}. \end{aligned}$$

In this case, $\lambda_{1,j'} = \cos \frac{2\pi j'}{N}$ includes all K' terms corresponding to those in m_i. Since the first N derivatives of $\beta(t)$ are zero at $t = 0$, we can use

$$\tilde{\phi}(t) = \alpha(t) \cdot \gamma(t) = \frac{2^{\frac{1}{2}N} \prod_{j'} (1 - 2t\,\lambda_{1,j'})^{\frac{1}{2}}}{(1 + \sqrt{1 - 4t^2})^{\frac{1}{2}N}} \tag{63}$$

as an approximation to (61). This expression yields the exact moments of C up to order N.

As a special case, consider $K' = 3$, with $\lambda_{1,1} = 1$ and $\lambda_{1,2} = \lambda_{1,3} = \cos \frac{2\pi j^*}{N}$. In this case

$$\tilde{\phi}_3(t) = \left(1 - 2t \cos \frac{2\pi j^*}{N}\right) \tilde{\phi}_1(t). \tag{64}$$

Successive derivatives of (64) at $t = 0$ show that

$$\mu_h'(R_3) = \left[P\, \mu_h'(R_1) - 2hQ \cos \frac{2\pi j^*}{N}\, \mu_{h-1}'(R_1)\right], \tag{65}$$

where $P = \mu_h'(V_1)/\mu_h'(V_3) = (N - 3 + 2h)/(N - 3)$, $Q = \mu_{h-1}'(V_1)/\mu_h'(V_3) = 2/(N - 3)$, and $h = 1, 2, \cdots, N$.

6.3. *Approximate moment generating function of C and V when $\rho \neq 0$.* To obtain an approximate moment generating function for C and V when $\rho \neq 0$, we utilize an approximation method given by Leipnik [10]. The exact moment generating function (58) with $\sigma^2 = 1$ can be written as

$$\phi(t_0, t) = (1 - \rho^N)\theta \exp.\left\{-\tfrac{1}{2} \sum_{i=1}^{N} \log\left[1 + \rho^2 - 2t_0 - 2(\rho + t) \cos \frac{2\pi i}{N}\right]\right\}, \tag{66}$$

where $\theta = \prod_{j'}[1 + \rho^2 - 2t_0 - 2(\rho + t)\lambda_{1j'}]^{\frac{1}{2}}$, and j' refers to the K' fitted terms in m_i. If the sum in the exponent of (66) is replaced by

$$\int_0^N \log\left[1 + \rho^2 - 2t_0 - 2(\rho + t) \cos \frac{2\pi x}{N}\right] dx, \tag{67}$$

and if $(1 - \rho^N)$ is replaced by 1, we obtain the approximate moment generating function

$$\tilde{\phi} = \frac{\prod_{j'} [1 + \rho^2 - 2t_0 - 2(\rho + t)\, \lambda_{1,j'}]^{\frac{1}{2}}}{[\frac{1}{2}(1 + \rho^2 - 2t_0 + \sqrt{(1 + \rho^2 - 2t_0)^2 - 4(\rho + t)^2}\,)]^{\frac{1}{2}N}}. \tag{68}$$

7. Approximate distributions of R.

7.1. *The Pearson Type* I (*Incomplete Beta*) *distribution.* The significance points of ${}_LR$ can be found exactly from equation (48) for $L = 1$ and by integrating equations (52), (53), and (54) for $N = 2L$, $3L$, and $4L$, respectively. These exact probability integrals for $N = 2L$, $3L$, and $4L$ are simply sums of Incomplete Beta functions, and the significance points can be found in Pearson's *Tables of* *the Incomplete Beta-Function* [14] or in the Thompson tables [16]. However, the computation of the exact significance points for $L = 1$ and $N > 4$ by use of equation (48) is quite tedious and actually impossible for large N with present logarithm tables and readily available computing devices. Hence, approximate distributions are called for.

The Type I approximation to the distribution of R is

$$f_1(R) = \frac{(1 + R)^{p-1}(1 - R)^{q-1}}{2^{p+q-1}\beta(p, q)}, \quad -1 \leq R \leq 1, \tag{69}$$

where p and q are chosen so that the first two moments of this approximate distribution agree with the first two moments of the exact distribution. It can be shown that each moment of the approximate distribution approaches the corresponding exact moment quite rapidly as N increases. On the basis of the approximation, the probability α of the significance point R' being exceeded can be found from the Incomplete Beta function. Thus

$$\alpha = Pr\{R > R'\} = 1 - I_x(p, q) = I_{x'}(p', q'), \tag{70}$$

where

$$I_x(p, q) = \frac{1}{\beta(p, q)} \int_0^x y^{p-1}(1 - y)^{q-1}\, dy, \tag{71}$$

and $x = (1 + R')/2$, $x' = (1 - x)$, $p' = q$, and $q' = p$. Hence, $R' = 2x - 1 = 1 - 2x'$.

The parameters in (69) are taken to be

$$2p = (1 + \mu_1')(1 - \mu_2')/\mu_2, \qquad 2q = (1 - \mu_1')(1 - \mu_2')/\mu_2, \tag{72}$$

where $\mu_2 = \mu_2' - (\mu_1')^2$ and $\mu_i' - \mu_i'(R)$ given in (60). Hence, when the distribution of R is symmetric, $\mu_1' = 0$ and $2p = 2q = (1 - \mu_2')/\mu_2'$.

In Section 3.1, we set up significance points for four special trends for which $\mu_1' = 0$:

(b) $P = 2$; (c) $P = 2, 4$; (d) $P = 2, 3, 6$; (e) $P = 2, 12/5, 3, 4, 6, 12$. The values of μ_2' for these four trends are:

(b) $(N - 4)/[N(N - 2)]$, (c) $1/(N - 2)$, (d) $1/(N - 4)$, (e) $1/(N - 10)$. Naturally the third moments for these symmetric distributions are 0. The fourth moments are as follows:

Trend	(b)	(c)	(d)	(e)
Exact	$\frac{3(N^2 - 2N - 16)}{(N + 4)(N + 2)N(N - 2)}$	$\frac{3(N^2 - 2N - 16)}{(N + 2)(N)(N - 2)(N - 4)}$	$\frac{3}{(N - 2)(N - 4)}$	$\frac{3}{(N - 8)(N - 10)}$
Incomplete Beta	$\frac{3(N - 4)^2}{N(N - 2)(N^2 - 8)}$	$\frac{3}{N(N - 2)}$	$\frac{3}{(N - 2)(N - 4)}$	$\frac{3}{(N - 8)(N - 10)}$

We note that for (d) and (e), the fourth moments for the Incomplete Beta are exact and for (b) and (c), they approach the exact values quite rapidly as N increases.

In Section 3.2, we considered some significance points for the following single-period trends: $P = 3, 4, 6$, and 12. The values of $2p$ and $2q$ for these asymmetrical cases are

$$2p = \frac{(N - 4 - 2\lambda)E}{D}; \qquad 2q = \frac{(N - 2 + 2\lambda)E}{D}, \tag{73}$$

where $\lambda = \cos \frac{2\pi}{P}$, $E = (N - 1)(N - 4) - 4\lambda$ and $D = (N - 3)(N - 1 + 4\lambda) - (N - 1)(1 + 2\lambda)^2$.

Equation (69) has the drawback of using the range $(-1, +1)$ instead of the true range of R, which varies between the last (smallest) λ_k'' to the first (largest) λ_k''. For example, if $N = 12$ and we fit the constant, $\cos \frac{2\pi i}{12}$, and $\sin \frac{2\pi i}{12}$, then $\lambda_{1.1} = 1$, $\lambda_{1.2} = \lambda_{1.3} = \cos \frac{2\pi}{12} = \frac{\sqrt{3}}{2}$, and the range of R is $\left(-1, \cos \frac{4\pi}{12} = \frac{1}{2}\right)$. However, if we fit the constant and $\cos \pi i = (-1)^i$, then $\lambda_{1.1} = 1$ and $\lambda_{1.2} = -1$, the true range would be $\left(-\frac{\sqrt{3}}{2}, +\frac{\sqrt{3}}{2}\right)$. From these examples we see that the error in using the approximate range $(-1, +1)$ varies according to the fitted terms in m_i, and that the error is worse on one tail than on the other, unless symmetric terms are fitted. A more accurate approximation could be obtained by use of the exact curtailed range, but it was not thought desirable because the exact range rapidly approaches the approximate range as N increases.

We might add that the significance point, R', can also be calculated from the Inverted Beta (F) distribution, for which tables are given by Merrington and Thompson [13], Snedecor [15], and Fisher and Yates [6]. Cochran [4] has provided an approximate formula for $z = \frac{1}{2} \log_e F$ when n_1 and n_2 are not given in the F-tables.

7.2. *The normal approximation.* It should be noted that R is asymptotically normally distributed for $\rho = 0$, as shown by the form of the characteristic function. We have considered the normal approximation with mean $\mu_1'(R)$ and variance $\mu_2(R)$. The variance of R was given in the previous section for the four special trends. For all single period trends, except $P = 2$, $\mu_1' = -(1 + 2\lambda)/(N - 3)$ and the variance is

$$\mu_2 = \frac{(N - 1 + 4\lambda)}{(N-1)(N-3)} - (\mu_1')^2, \tag{74}$$

where, as before, $\lambda = \cos(2\pi/P)$. Further terms in an asymptotic expansion of the distribution would take account of higher moments of R as Hsu has done for the case of fitting only the mean (m_i = a constant) [7].

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