CHAPTER V

DEMAND FOR CAPITAL GOODS

1. Definitions. So far as the author knows, no one has yet published a demand study for capital goods.* Various attempts have been made to obtain such laws but all attempts have ended in failure. One of the difficulties has been lack of adequate data, but there are other fundamental reasons. For example, another important reason why such studies have been difficult to make is that a certain fairly long period of time has to elapse before an incentive to demand is consummated in actual demand. The length of this period of time varies with the particular kind of capital goods demanded. There is, of course, a time interval for consumer goods also, but it is not nearly so long and by using yearly data the difficulty can be surmounted, as will be made apparent at the end of this chapter. In fact, there are some consumer goods for which the time interval is very short, a few minutes, an hour, a day, or a week. For capital goods the time interval is quite long; for example, two or three years on the average.

Some economists have attempted to differentiate between various kinds of economic goods on the basis of the time required to consume the product, i.e., the life of the product. Such differentiation into durable and non-durable goods is inadequate for demand studies. Another differentiation sometimes made is on the basis of expected income, i.e., capital goods are goods that are to be used to produce income. This is a much better definition for studies of demand, but, of course, it should be recognized that no matter what definition is used there will be some goods that will fall in more than one class and be difficult to classify. Thus, according to the latter differentiation an automobile used for pleasure would be a consumer good, whereas the same automobile used as a taxicab would be a capital good. Furthermore, some capital goods will be durable and others will be non-durable.

*R. H. Whitman presented a paper on the demand for pig iron before the Econometric Society in Syracuse in June, 1932, but this has not been published. The paper uses the method of trend analysis. After elimination of trends the method is essentially that of determining integral weights $K(x_i, t)$, t fixed, for prices three months in advance of demand by the method of multiple correlation. The correlations are unfortunately far from convincing, since the signs of terms are theoretically correct only three times out of five.

The price of a capital good (using the income definition) would, of course, depend upon the expected income from the good and the cost of production of the good. In some instances the expected income is the most important factor (there are usually physical factors of at least as much importance as price), whereas in other instances the price plays a role of some importance. In the work that follows p will be used to represent either the price of the good or the expected income. As far as the analysis is concerned, it is unnecessary to differentiate between the two and the careful reader will be able to differentiate and also to interpret the results in terms of either or both according to the interpretation which is applicable to the particular problem in hand. Furthermore, even though p be used only in linear relations, it does not follow that the theory applies only to linear demand laws, for if p is expected income, this might in the case of residential building be defined to be $(R - T)/C$ where R is rent, T is taxes and C is cost (a price).

2. Demand Incentives and Consummations. In Chapter III it was assumed that demand depended upon past prices and other quantities and present prices and other quantities. On this assumption an equation of demand was obtained in the form

$$y(t) = \varphi(t) + K(t,t)F(p) + \int_{-\infty}^{t} K(x,t)p(x)dx. \tag{2.1}$$

It was then assumed that for many products the integral term could be replaced by a constant factor, so that the equation of demand became

$$y(t) = \varphi(t) + K(t,t)F(p). \tag{2.2}$$

No theoretical justification for this assumption was given, but in Chapters III and IV it was shown that this equation is sufficiently general to apply to the demand for such consumer goods as gasoline, pork, wheat and cotton.

As a further problem it is now proposed to investigate the function $K(x,t)$. This study of the questions of why demand depends upon past prices and how individuals build up wants or desires throws considerable light on the nature of the function $K(x,t)$ and makes it possible to give theoretical justification to the assumption pointed out above.

To investigate the nature of $K(x, t)$ suppose that, at the time $t_i < t$, a group consisting of G_i individuals comes into contact with an incentive to purchase, $p(t_i)$. Suppose further that this group purchases M_t units and decides to purchase a total of N_i additional units of the goods as soon as money or credit becomes available to them. Some of the group, perhaps, decide not to purchase any units while others may decide to purchase more than one unit. Some, perhaps, purchase at the time t_i and decide to purchase other units in the future. The number of units $N(t_i)$ that would be purchased per unit time in the future from such decisions (without inhibitions which can be discussed separately) may be assumed to be proportional to the incentive $p(t_i)$; that is,

$$N(t_i) = ap(t_i) + b \ ,$$

where a and b are constants (or functions of time in the sense explained in Chapter II).

There will be many factors affecting the length of time required for individuals to act on an incentive. Some of these factors will be of the same relative importance, but a few, such as credit, will have important special influences. For the many small factors it can be assumed that the time frequency distribution of those acting on the incentive will obey the normal law.* The peak of the distribution will occur M units of time after the stimulus or incentive; i.e., M units of time after t_i, so that the number reacting at time $t > t_i$ to the stimulus at t_i may be taken to be approximately

$$\psi(t_i,t) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(t-t_i-M)^2/2\sigma^2} [ap(t_i) + b] \ .$$

The parameters σ and M depend on many economic and psychological factors and in the sense already explained may be assumed to be constants, at least for some periods of time.

The total number reacting at any particular time to all stimuli prior to t, i.e., to stimuli at $t_0, t_1, t_2, \cdots, t_i, t_{i+1}, \cdots, t_{n+1} = t$, is

$$\psi(t) = \sum_{i=0}^{n} \psi(t_i,t) = \sum_{i=0}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(t-t_i-M)^2/2\sigma^2} [ap(t_i) + b] \ ,$$

where $i = 0$ refers to the first time at which a stimulus or incentive was given. More exactly,

*Whittaker and Robinson, *Calculus of Observations*, pp. 168-175.

$$\psi(t) = \int_{-\infty}^{t} \frac{ap(x) + b}{\sqrt{2\pi}\,\sigma} e^{-(t-x-M)^2/2\sigma^2} dx \;,$$

where x has been used to designate time $t_0, t_1, \cdots, t_{n+1}$.*

A simple transformation $(x-t+M)/\sqrt{2}\,\sigma = S$ leads to a form which, in some respects, is more suggestive of lags.

The quantity M_t consists of units purchased at t due to important factors operating at that particular time. For example, some buyers coming into contact with an article for the first time might decide to purchase immediately. An individual in a group G_i who had made his decision to purchase four units at $1,000 each might purchase five units at $850 each, the latter number being better suited to his needs. There would also be individuals who had not quite decided to purchase, but whose desires had been growing ever since some previous time τ. The amount purchased by this group would depend upon past prices as well as upon present price, depending less and less on past prices as the time is more and more remote. Thus, M_t might be given by a formula of the type

$$M_t = F[p(t), t] + \int_{\tau}^{t} h(t, x)\, e^{-w_1(t-x)}\; p(x)dx$$

where M_t is used to denote a function of p and t, $h(t) > 0$ and $w_1 > 0$. Here t is used to indicate that there will be present factors other than price, $p(t)$. A function of the type $\varphi(t)$ is included in F.

The demand $y(t)$ now takes the form

$$(2.3) \qquad y(t) = F[p(t), t] + \int_{\tau}^{t} h(t, x) e^{-w_1(t-x)} p(x)dx$$

$$+ \int_{-\infty}^{t} \frac{ap(x) + b}{\sqrt{2\pi}\,\sigma} e^{-(t-x-M)/2\sigma^2} dx.$$

As already pointed out, M and σ depend on many economic factors, the effects of any one of which are assumed to be small. For impor-

*Irving Fisher seems to have been the first to use the idea of a distributed lag. See Irving Fisher, "The Business Cycle Largely a Dance of the Dollar," *Journal of the American Statistical Association*, December, 1923, p. 5. For further references see Irving Fisher, *Theory of Interest*, New York, 1930, pp. 419-420.

tant special factors in the past, as, for example, credit in the case of residential building, corrections can be made by adding (or subtracting) other integral terms (corresponding to the Gram-Charlier method of representing frequencies*) or by using weighting functions other than the normal probability function (corresponding to the Pearsonian method of representing frequency distributions†). In the study of residential building presented in the next chapter the first method will be explained and used. A short discussion of the other alternative is given here.

For definiteness suppose that the factors affecting the periods of time required for the desires to purchase to materialize into actual purchases are such that decisions to purchase are translated into actual purchases according to a Pearsonian Type III frequency function. For this type of function the amount $v(t, t_i)$ purchased by the group G at the time $t > t_i$ is given by the equation

$$v(t, t_i) = v_0 e^{-\gamma(t-t_i-\mu)} \left[\frac{t-t_i}{\mu}\right]^{\gamma\mu}, \tag{2.4}$$

where v_0, the mode of purchases, is proportional to N, $v_0 = \lambda N$; where N is the total number of proposed purchases and γ and μ are parameters not involving the time t.

The curve (2.4), (t_i, γ and μ constants) has a zero at t_i, reaches a maximum μ units of time after t_i and then approaches the t-axis asymptotically. Thus, (2.4) requires an assumption that many of the group will be able to make their purchases within a relatively short time (for some products one week, for others a month and for some perhaps a year) after t_i; others will require more and more time and some will never be able to make their proposed purchases. The parameters γ and μ quite evidently depend upon monetary and credit conditions, upon the prices of competing or substitute goods and upon various other quantities. For example, convincing advertising of the product presented to the group at the time t_i may cause certain of the group to do without other products sooner than they would if the advertising were not so alluring.

As pointed out above, v_0 is proportional to N; that is,

$$v_0 = \lambda N, \text{ where } \lambda = (\gamma\,\mu)^{\gamma\,\mu+1} \quad \mu\, e^{\gamma\,\mu}; \Gamma(\gamma\,\mu + 1) \text{ and } \Gamma(\gamma\,\mu + 1)$$

*See Arne Fisher, *Mathematical Theory of Probabilities*, New York, 1930, pp. 202 et seq.

†See H. L. Reitz, *Mathematical Statistics*, Chicago 1927, pp. 54 et seq.

is the gamma function for $(\gamma\,\mu + 1)$.* The number of proposed purchases N, will certainly be a function of the price $p(t_i)$. As a first approximation suppose that $N = a(t)p(t) + b(t)$. Then by the process of summation (integration) already described, the demand y can be taken to be

$$(2.5) \qquad y(t) = F(p(t), t) + \int_{\tau}^{t} h(t, x) e^{-w_1(t, x)} p(x) dx$$

$$+ \int_{\tau_1}^{t} \lambda(a(x)p(x) + b(x)) e^{-\gamma(t-x-\mu)} \left[(t-x)/\mu \right]^{\gamma\,\mu} dx \,.$$

Formula (2.5) is simply a special case of formula (2.1) of Chapter II obtained in a slightly different way. Quite obviously different functions $K(x, t)$ can be obtained by assuming different distributions for the consummation of purchases at the time t. Thus, the distribution

$$v = v_0 \left[1 + \frac{(t - t_i - \mu_1)}{\mu_1} \right]^{m_1} \left[1 + \frac{(t - t_i - \mu_2)}{\mu_2} \right]^{m_2};$$

$$m_1/\mu_1 = m_2/\mu_2 \,,$$

and the same linear law of demand decision would lead to

$$K(x, t) = \lambda_1 a(x) \left[1 + (t - x - \mu_1)/\mu_1 \right]^{m_1}$$

$$\left[1 - (t - x - \mu_2)/\mu_2 \right]^{m_2}$$

$$\text{where } \lambda_1 = \frac{(m_1)^{m_1} (m_2)^{m_2}}{(m_1 + m_2)^{m_1 + m_2}} \, \frac{\Gamma(m_1 + m_2 + 2)}{\Gamma(m_1 + 1)\Gamma(m_2 + 1)} \quad \frac{1}{\mu_1 + \mu_2} .$$

Different kernels, $K(x, t)$, can also be obtained by assuming laws of demand decision other than the linear one which has been assumed here, that is, by replacing p by a function of p. There are, therefore, a variety of forms $K(x, t)$ that are theoretically possible.

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*See, for example, D. C. Jones, *A First Course in Statistics*, London 1921, p. 221.

3. Approximations to Past Effects—Time Lags. It is now proposed to investigate conditions under which the integral terms of demand equations such as (2.3) and (2.5) can be neglected, or approximated by other expressions which for some purposes are simpler. Consider first the expression

$$\psi(t) = \int_{-\infty}^{t} \frac{ap(x) + b}{\sqrt{2\pi}\,\sigma} e^{-(t-x-M)^2/2\sigma^2} dx \;.$$

Such an expression for ψ may be mathematically correct, but it is, nevertheless, impracticable from the points of view of economic reality and of statistical analysis. Obviously, observations on p do not extend in time to $t = -\infty$ and, furthermore, if the observations did extend that far in the past all the individuals who had such early incentives would be dead. In other words it is necessary to apply certain boundary conditions of the problem.

For this purpose, ψ can be broken down into an integral from $-\infty$ to $t-t_0$, and one from $t-t_0$ to t, where t_0 is a constant sufficiently large to make the deviation of the first integral from its mean value over the infinite period of $-\infty$ to $t-t_0$ negligible; that is,

$$\psi(t) = A_0 + A_1 p_0 + \int_{t-t_0}^{t} \frac{ap(x)}{\sqrt{2\pi}\,\sigma_1} e^{(t-x-M)^2/2\sigma_1^2} dx \;, \tag{3.1}$$

where A_0 and A_1 are constants (or functions of time in the sense used in this book) and p_0 is the normal (average) value of p on the infinite range $-\infty$ to $t-t_0$.

This is the type of formula required to represent ψ if it is assumed that once an incentive is offered it will be acted upon some time in the future, the particular time at which the action occurs being determined by a great many economic and physical forces, any one of which has only a small effect.

It is especially interesting to derive the formula (3.1) since the method of proof gives important information regarding the nature of time lags. To transform from the infinite range on which ψ is defined to a finite one, write

$$\psi = \int_{-\infty}^{t} \frac{ap(x) + b}{\sqrt{2\pi}\,\sigma} e^{-(t-x-M)^2/2\sigma^2} dx$$

$$= \int_{-\infty}^{t} \frac{b}{\sqrt{2\pi}\,\sigma} e^{-(t-x-M)^2/2\sigma^2} dx$$

$$+ \int_{-\infty}^{t-t_0} \frac{ap(x)}{\sqrt{2\pi}\,\sigma} e^{-(t-x-M)^2/2\sigma^2} dx$$

$$+ \int_{t-t_0}^{t} \frac{ap(x)}{\sqrt{2\pi}\,\sigma} e^{-(t-x-M)^2/2\sigma^2} dx\ .$$

By putting $(x + M - t)/\sqrt{2}\,\sigma = z$ in the first and second integral, ψ may be written in the form

$$\psi = \int_{-\infty}^{M/\sqrt{2}\,\sigma} \left(\frac{b}{\sqrt{\pi}}\right) e^{-z^2} dz$$

$$+ \int_{-\infty}^{\alpha} \left(\frac{a}{\sqrt{\pi}}\right) p(\sqrt{2}\,\sigma z + t - M) e^{-z^2} dz + F[p]$$

where

$$F[p] = + \int_{t-t_0}^{t} \frac{ap(x)}{\sqrt{2\pi}\,\sigma} e^{-(t-x-M)^2/2\sigma^2} dx,$$

and

$$\alpha = (M - t_0)/\sqrt{2}\,\sigma\ .$$

Now, the first integral in ψ is a constant which may be called A_0. In the second integral let p_0 be the average value of I in the interval $-\infty \leq z \leq (M - t_0)/\sqrt{2}\,\sigma$ and let $p_\Delta\,(\sqrt{2}\,\sigma z + t - M)$ be a function such that

$$p(\sqrt{2}\,\sigma z + t - M) = p_0 + p_\Delta(\sqrt{2}\,\sigma z + t - M).$$

Then by an application of the law of the mean for integrals it follows that

$$\psi = A_0 + A_1 p_0 + \int_{-\infty}^{\alpha} \frac{a}{\sqrt{\pi}} p_\Delta(\sqrt{2}\,\sigma z + t - M)\ e^{-z^2} dz + F[p]$$

$$= A_0 + A_1 p_0 + p_\Delta \; (t - M - \lambda) \frac{\bar{a}}{\sqrt{\pi}} \int_{-\infty}^{a} e^{-z^2} dz + F[p] \;,$$

where

$$A_1 = \int_{-\infty}^{a} \frac{a}{\sqrt{\pi}} e^{-z^2} dz$$

and λ is a positive quantity such that $t - M - \lambda$ represents a time previous to the present time and $\bar{a}$ is an average value of a. Thus, $p_\Delta(t - M - \lambda)$ represents an average value of past fluctuations in incentive from an assumed normal value I_0.*

In all cases it is possible to find a t_0 such that

$$p_\Delta(t - M - \lambda) \frac{a}{\sqrt{\pi}} \int_{-\infty}^{a} e^{-z^2} dz$$

is negligible, since the integral can be made as small as desired by proper choice of t_0. It is, therefore, always possible to write

$$\psi = A + \int_{t-t_0}^{t} \frac{ap(x)}{\sqrt{2\pi}\,\sigma} e^{-(t-x-M)^2/2\sigma^2} dx$$

where A is a constant or function of time defined by the equation $A = A_0 + A_1 p_0$.

If t_0 is a year or less, the effect of the integral term is completely lost when yearly data are used, since the integral gives a weighted average of p for the year. Thus, for the consumer goods, gasoline, wheat, pork and cotton, there is no need to use the integral term. For these goods a formula of the type

$$y = \varphi(t) + F[p(t), t]$$

is sufficiently general.

*J. A. Hobson, *Theory of Functions of a Real Variable*, Vol. 1, page 617. The theorem referred to here does not apply to an integral with an infinite range, but the transformation $1/z = y$ reduces the infinite range to a finite one, the function $p(z)$ is bounded for all values of z and $(e^{-1/y})/y^2$ is also bounded on the transformed finite range. The law of the mean referred to here can thus be applied to the transformed integral and then a transformation can be made back to z.

When t_0 is greater than a year and yearly data are used, the law of the mean can be applied to the integral term. In this case the demand is given by an equation of the form

$$y(t) = \varphi(t) + F[p(t), t] + F_1[p(t-\vartheta), t],$$

or by further averaging by a formula of the type

$$y(t) = \varphi(t) + \psi[p(t-\vartheta), t].$$

This is an equation involving a fixed average time lag. In statistical work of a preliminary nature results are more likely to be obtained by the use of a fixed average lag than by the use of an integral lag. After such a preliminary study a formula involving distributed lag can be obtained. Methodology for accomplishing this is given in Appendix V.

4. Price Forecasting and Speculation. An analysis similar to the preceding can be used to determine the form a demand law should take when speculation is taken into account. Speculative demand depends upon future prices, (price used here in the general sense explained in Section 1) so that demand for goods that lend themselves to speculation depends upon past prices, present price and expected future prices; that is, the amount demanded at the time t is

$$y(t) = f[p(t_1), \cdots, p(t_n)p(t_{n+1}), \cdots, p(t_{n+\sigma}), t, p_1 \cdots, p_m],$$

where t_n stands for time t when purchases are made and $p(t_{n+1})$, $\cdots, p(t_{n+\sigma})$ are expected future prices. The analysis already developed shows that, for hypotheses similar to those made for obtaining equation (3.1) of Chapter II, this expression can be replaced by an integral,

$$y(t) = \varphi(t) + \int_{t-t_0}^{T} K(x, t)p(x)dx \ ,$$

where $T = t_{n+\sigma}$. More generally, since special weight may be attached to the present price $p(t)$,

$$y(t) = \varphi(t) + \int_{t-t_0}^{T} K(x, t)p(x)dx + a_1(t)p(t).$$

This expression can be written as

$$(4.1) \quad y(t) = \varphi(t) + a_1(t)p(t) + \int_{t-t_0}^{t} K(x,t)p(x)dx + \int_t^T \psi(x,t)p(x)dx,$$

where for purposes of clarity $K(x,t)$ for $x > t$ has been named $\psi(x,t)$. The first three terms of (4.1) can be taken to be the same as the expression (3.1) of Chapter II. Thus

$$S = \int_t^T \psi(x,t)p(x)dx$$

represents demand due to speculation.

Long range economic forecasting has never been consistently successful. In fact, in view of a recent study by Alfred Cowles, III* of the frequency of successes of stock forecasting houses, it is somewhat doubtful that any great measure of success, perhaps only success that might be expected by chance, can, at present, be obtained for even short time forecasting of such things as security price changes. There are, of course, other phenomena for which there is some hope of short time forecasting.

Many individuals seem to be engaged in the science (or guessing game) of attempting to predict what will happen to prices the next minute, the next hour, the next day, or the next week. An important speculative problem is, therefore, the one for which $T = t + 1$. For this problem the extrapolation curve for $p(x)$ may be most conveniently taken as a straight line

$$p(x) = p(t) + (x-t)dp/dt \;,\; x > t,$$

where dp/dt is the derivative of price with respect to time; that is, dp/dt is the slope of the price-time curve at the time t. The assumption made here is that estimates of future quantities are based on present evidence. A substitution of the above value of $p(x)$ in S yields

$$S = \int_t^T \psi(x,t)p(x)dx = \int_t^{t+1} \psi(x,t)\;[p(t) + (x-t)dp/dt]\,dx.$$

*Alfred Cowles, III, "Can Stock Market Forecasters Forecast?", *Econometrica,* Vol. I, 1933, pp. 309-324.

An application of the first law of the mean for integrals* followed by an integration gives

$$S = \psi(t+\vartheta, t) \int_t^{t+1} [p(t) + (x-t)dp/dt]\,dx$$

$$= \psi(t+\vartheta, t)\ [p(t) + \tfrac{1}{2}\,dp/dt]\ , \qquad 0 < \vartheta < 1.$$

It follows that equation (7) can be replaced by the simpler form

$$(4.2) \quad y(t) = \varphi(t) + A(t)p\ (t) + H(t)dp/dt + \int_{t-t_0}^{t} K(x,t)p(x)dx\ ,$$

where $A(t) = a_1(t) + \psi(t+\vartheta, t)$ and $H(t) = \psi(t+0, t)/2$.

It will be recalled that the integral equation (4.2), or rather the equation for which the lower limit $t - t_0$ is fixed, is invertible, so that it is possible to talk about forecasting either prices or consumption without stopping to reformulate the problem.

In general, when prices are rising and there is reason to believe that they will continue to rise, speculative demand is positive. When prices are falling and there is reason to believe that they will continue to fall, speculative demand is negative; that is, there is a tendency to dump. To satisfy the above conditions $H(t)$ must be a positive function.

Many times prices rise for a period, drop back slightly, rise a bit and again drop back to the preceding recession level. In such instances there is uncertainty regarding trends, that is, there is uncertainty whether prices will rise or fall. In such instances some speculative traders sell to avoid losses or sell short expecting to buy back at lower prices, some traders buy in the expectation of gain and some simply stay out of the market. This situation is characterized by the fact that $dp/dt = 0$; that is, demand is neither increased nor decreased by speculation.

The introduction of a derivative of price into the demand equation marks one of the important contributions of mathematics to economics in the past decade. This is due to G. C. Evans, who in 1925 proposed a demand equation $y(t) = a\ p(t) + b + h\ dp/dt$,

*E. W. Hobson, *Theory of Functions of a Real Variable*, Cambridge, 1927, Vol. I, p. 617. Continuity of $\psi(x, t)$ in x for all t is assumed.

where a, b and h are constants, in order to approximate the phenomenon that "when prices are going up the demand (for lumber) is insatiable, but when prices are going down it is nil until the price movement stops."* Although it is possible that the rise in price of lumber mentioned by Evans was brought about by an expected increase in price, it is probable that the price of lumber increases because of a rising demand brought about by other factors (see the study on residential building in the next chapter). Nevertheless, there is probably some speculative influence at work on lumber and there are certainly such forces at work on other products.

One difficulty about forecasting security prices is that there is always the possibility of introducing a new element that has not previously been taken into account. In general, security prices are determined by so many causes that series of security prices appear to be random series, but the series are, of course, subject to special factors, such as abandonment of gold, favorable Government pronouncements, etc.

There are some consumer goods for which it is possible to forecast demand and prices two or three months in the future. As will be seen in the subsequent chapter, it is now possible to forecast residential construction for a year or so in advance of the present time with the probability of a reasonable degree of accuracy.

5. *Demand for Competing Products.* It will be recalled that in Section 2 the prices $p_1, \cdots, p_m$ of m goods competing with the goods whose price is p were assumed to be constant with respect to time. Since the theory just developed is a dynamic one, that is, one in which forces are allowed to modify situations as time progresses and hence one in which the time element plays an important role but not necessarily *per se,* it was unnecessary to carry these parameters along in the equations. It should be remembered now, however, that, in accordance with the hypotheses made in Section 2, each of the quantities φ, A, H and K depends upon the prices of competing goods. For example, suppose that in a study of the relation of the demand for pig iron to the price of pig iron the co-

*G. C. Evans, "The Dynamics of Monoply," *Amer. Math. Monthly,* Vol. XXXI, February, 1924, p. 77. See also Evans, *Mathematical Introduction to Economics,* New York, 1930. For the first use of an integral equation of demand, see C. F. Roos, "A Mathematical Theory of Competition," *Amer. Jour. of Math.,* Vol. 47, 1925, p. 173. See also, C. F. Roos, "Theoretical Studies of Demand," *Econometrica,* Vol. 2, 1934, pp. 73-90.

efficient $A(t)$ is determined statistically as a number a. This number a on further analysis will be found to be made up of prices of competing goods such as copper, aluminum, lumber and so forth. Thus a might be defined by a formula such as $a = a\ p_a + c\ p_c + l\ p_l$, where a, c and l are constants and p_a, p_c and p_l are the prices of aluminum, copper and lumber respectively.* More generally a, c and l may be assumed to vary with the time t so that a will be a function of time. Thus as far as the whole of the preceding analysis is concerned, the quantities φ, A, H and K may be considered to be functions of present prices of competing or substitute goods and the time t either implicitly, as already explained, or explicitly, since for seasonal goods φ and possibly other quantities may contain periodic functions of the time as, for example, cosines and sines.

The statements made above in regard to changes in φ, A, H and K with respect to time do not mean, however, that the quantities are so continuously changing that statistical laws of demand cannot be determined. On the contrary, as indicated previously, due to compensation of prices of competing goods, these quantities may remain fairly constant for periods of time that may be much in excess of five years for some commodities. In the example considered above, it might be possible for the price of aluminum to increase, the prices of copper and lumber to decrease and yet a might remain constantly equal to some given value, but such a situation would probably not prevail for long, especially if there were noticeable price movements. The question of what must be regarded as "long" may, of course, be answered differently for each product.

If the prices $p_j(t_i)$, $j = 1, 2, \cdots, m$, $i = 0, \cdots, n$, $t_0 = 0$, $t_n = t$, are not assumed to be all equal to $p_j(t)$ respectively, the theory of demand becomes much more complicated, but, nevertheless, it can be readily formulated in terms of systems of integral equations or functional equations. Thus, the simplest case of the non-speculative problem leads to a system of $m + 1$ integral equations,

$$y_k = \sum_{j=1}^{m+1} \left[\varphi_{kj}(t)p_j(t) + \int_0^t K_{kj}(x, t)p_j(x)dx \right], \quad k = 1, \cdots, m + 1.$$

There is in general no difficulty in inverting this system of Volterra integral equations to determine the $p_j(t)$ in terms of the

*For a statistical study of the effects of prices of competing goods, see Mordecai Ezekiel, "Statistical Examination of Factors Related to Lamb Prices," *Journal of Political Economy*, Vol. 35, April, 1927, p. 254.

y_k.* In fact, entirely similar equations result with the y_k replacing the p_k and resolvent kernels replacing the K_{kj}. When derivatives are introduced and a more general functional relation is assumed, a system of $m + 1$ functional-differential equations are obtained.†

6. The Elasticity of Demand. For a great while economists have been classifying demand as elastic or inelastic depending upon whether the value of the good demanded increased or decreased with a fall in price and vice versa. In 1838 Augustin Cournot proposed to separate articles into two categories depending upon whether $(\Delta y/\Delta p)p/y$ is less than or greater than one. Alfred Marshall defined the quantity $(dy/dp)/(p/y)$ to be the *coefficient of elasticity of demand.* He and H. L. Moore popularized the concept and Moore applied the concept extensively to agricultural data and used it to obtain various laws of demand.‡

Whenever the coefficient of elasticity of demand is *numerically* greater than one, the demand is said to be elastic. Whenever it is numerically less than one, the demand is inelastic and if it is numerically equal to one, the demand is neither elastic nor inelastic.

For equation (4.2) the elasticity of demand is simply $A(t)p(t)/y$. For a functional demand equation of this type a functional coefficient of elasticity of demand might be more useful.

Let δy be the variation of y corresponding to a variation, δp, in p in the sense of the calculus of variations. Then, the *functional coefficient of elasticity of demand* may be defined to be $(\delta y/\delta p)p/y$. Thus for (4.2) with $H = 0$, the variation in y would be given by

$\delta y = \int_0^t K(x, t)\, \delta p dx + A(t)\delta p$. In particular, if each price of the

goods from the time 0 to t were increased by a constant amount δp, the functional coefficient of elasticity of demand would be

$$\left[\int_t^0 K(x, t)dx + A(t) \right] p(t)/y(t) \ .$$

If H were not zero, there would be added a term $Hd(\delta p)/dt$ to δy.

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*Vito Volterra, *loc. cit.*, pp. 71-74.

†C. F. Roos, "A Dynamical Theory of Economics," *Journal of Political Economy*, Vol. XXXV, 1927, pp. 648-650.

‡ See, for example, Alfred Marshall, *Principles of Economics*, London, 1920, H. L. Moore, *Synthetic Economics*, New York, 1930, and Augustin Cournot, loc. cit., pp. 52-54.

It is important to notice that no attempt has been made to determine why a demand decision curve should take the form postulated, or why the number making a demand decision should depend upon the price, or why N or M_t should take any of the forms postulated. It would, of course, be possible to pursue these questions as far as one liked by analyzing human behavior, utility and so forth, but these studies properly belong to the sciences of psychology and sociology rather than to economics.

The size of a market can be increased by advertising. This could be taken into account by postulating that the number introduced to a product at a certain time depends upon the advertising expenditures. It would then be possible to determine how demand varies with advertising.

Again, if the product is one for which there will be a repeat demand, the number of prospective purchases will certainly depend upon the life of the product. Thus, one could study the effects of obsolescence, risk of damage, seasonal changes and so forth. Also, the parameters of the functions $K(x, t)$, $\varphi(t)$, etc., depend upon such things as monetary conditions, the psychology of the buying public and so forth. However, all these factors deserve consideration in their own right and cannot be adequately treated here. It may be hoped that it will be possible to push the analysis of demand further and further back to fundamental conceptions of behavior, but it must not be forgotten that empirical formulae will have to be introduced at some stage.

One might say that the theory of demand has progressed to a point where it is questionable that further theoretical work should be done until many statistical studies have been made to verify or disprove the hypotheses and conclusions so far reached. The statistical investigation is, in itself, a tremendous task. A new type of mathematical statistician is undoubtedly required to make the studies. Nevertheless, as has been indicated in Chapters III and IV and will be demonstrated in the next chapter, there are reasons to be optimistic regarding the possibilities of the discovery of statistical laws of demand both for consumer goods and for capital goods.