

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS  
AT YALE UNIVERSITY

Box 2125, Yale Station  
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 887

Note: Cowles Foundation Discussion Papers are preliminary materials circulated to stimulate discussion and critical comment. Requests for single copies of a Paper will be filled by the Cowles Foundation within the limits of the supply. References in publications to Discussion Papers (other than mere acknowledgment by a writer that he has access to such unpublished material) should be cleared with the author to protect the tentative character of these papers.

A NEW PROOF OF KNIGHT'S THEOREM  
ON THE CAUCHY DISTRIBUTION

Peter C. B. Phillips

October 1988

A NEW PROOF OF KNIGHT'S THEOREM  
ON THE CAUCHY DISTRIBUTION

by

P. C. B. Phillips

*Cowles Foundation for Research in Economics  
Yale University*

0. ABSTRACT

We offer a new and straightforward proof of F. B. Knight's [3] theorem that the Cauchy type is characterized by the fact that it has no atom and is invariant under the involution  $i : x \rightarrow -1/x$ . Our approach uses the representation  $X = \tan \theta$  where  $\theta$  is uniform on  $(-\pi/2, \pi/2)$  when  $X$  is standard Cauchy. A matrix generalization of this characterization theorem is also given.

*Key Words:* Cauchy distribution; Involution; Matrix variate; Uniform distribution.

September 1988

---

\*My thanks go to Glenna Ames for her work in keyboarding the manuscript and to the NSF for research support.

## 1. INTRODUCTION

E. J. Williams [4] showed that  $X$  is standard Cauchy with density  $f(x) = [\pi(1 + x^2)]^{-1}$  (we shall write  $X \equiv C(0,1)$ ) iff

$$X \equiv \frac{1 + bX}{b - X} \tag{1}$$

where  $b$  is some constant which is not the tangent of a rational multiple of  $2\pi$ . (In (1) and hereafter we use the symbol " $\equiv$ " to signify equivalence in distribution.) F. B. Knight [3] sharpened this result considerably by proving that  $X$  is of the Cauchy type (i.e. belongs to the equivalence class of distributions of  $a + pX$ ,  $p \neq 0$ , where  $X$  is standard Cauchy) iff

$$\frac{aX + b}{cX + d} \equiv aX + \beta$$

for some constants  $a \neq 0$  and  $\beta$  whenever  $ad - bc \neq 0$ . C. Hassenforder [2] pointed out that this is equivalent to the statement that elements of the type have no atoms and the type is invariant under the involution  $x \rightarrow -1/x$ . This leads to the central result:

### 1.1. THEOREM

$$X \equiv C(0,1) \text{ i.e. } X \text{ is standard Cauchy}$$

*iff*

$$X \equiv -1/X$$

*and the distribution has no atoms.*

## 2. A NEW PROOF OF THEOREM 1.1

We offer a new and straightforward proof of Theorem 1.1. Our approach is to use the representation of  $C(0,1)$  in terms of  $X = \tan \theta$ , where  $\theta$  is uniform on  $(-\pi/2, \pi/2)$ . We have:

2.1. PROPOSITION. *The following statements are equivalent:*

- (i)  $X \equiv -1/X$  and has no atoms;
- (ii)  $\theta$  is uniform on  $(-\pi/2, \pi/2)$  ;
- (iii)  $X \equiv C(0,1)$  .

PROOF. Direct calculation shows that (iii)  $\Rightarrow$  (i) and (ii)  $\Rightarrow$  (iii). It remains to prove that (i)  $\Rightarrow$  (ii). Observe that (i) implies that  $X$  is symmetric so that (i) also implies that  $X \equiv 1/X$ . The transformation  $X = \tan \theta$  induces a unique (and, under (i), continuous) probability distribution for  $\theta$  on  $(-\pi/2, \pi/2)$ . By replicating this distribution over  $(\pi/2, 3\pi/2)$  and assigning probability mass 1/2 to each interval we obtain an induced distribution on the complete unit circle. Observe that since  $X \equiv 1/X$  we have

$$\tan \theta \equiv \tan(\pi/2 - \theta) \tag{2}$$

Next, for any constant  $b$  we find by direct use of (i) that

$$Z = \frac{X + b}{1 - bX} \equiv -\frac{1 - bX}{X + b} = -\frac{1}{Z} .$$

This implies symmetry of  $Z$  and we have

$$Z \equiv 1/Z . \tag{3}$$

Let  $b = \tan \beta$  for some  $\beta \in (-\pi/2, \pi/2)$ . Then (3) may be written in the equivalent form

$$\tan(\theta+\beta) \equiv \tan(\pi/2 - \theta - \beta) .$$

Since

$$\tan(\pi/2 - \theta - \beta) = \frac{\tan(\pi/2 - \theta) - \tan \beta}{1 + \tan(\pi/2) \tan \beta} \equiv \frac{\tan \theta - \tan \beta}{1 + \tan \theta \tan \beta} = \tan(\theta-\beta)$$

we deduce that

$$\tan(\theta+\beta) \equiv \tan(\theta-\beta) .$$

By reversing the transformation this implies that the distribution on the unit circle that is induced by  $\theta$  is invariant under a rotation of  $2\beta$  radians. Since  $\beta$  is arbitrary we deduce that the distribution on the unit circle is invariant to rotation and therefore  $\theta$  is uniform on  $(-\pi/2, \pi/2)$ . Hence, (i)  $\Rightarrow$  (ii) and the proposition follows  $\square$ .

## 2.2. REMARK

As observed in the proof, (i) implies that  $X$  is symmetrically distributed i.e.  $X \equiv -X$ . We may therefore write (i) in the equivalent form:

$$(i)' \quad X \equiv Xk \equiv 1/X, \quad k \in O(1) .$$

Here,  $O(n)$  represents the orthogonal group of order  $n$  so that since  $n = 1$ ,  $k = \pm 1$ . This form is useful because it helps to suggest a multivariate generalization of the proposition.

## 3. A MULTIVARIATE EXTENSION

Let  $X = (x_{ij})_{n \times m}$  be multivariate (matrix) Cauchy with density

$$f(X) = \left[ \pi^{nm/2} \Gamma_n(n/2) \right]^{-1} \Gamma_n((n+m)/2) |I + XX'|^{-(n+m)/2} . \quad (5)$$

We shall write  $X \equiv C_{n,m}(0,I)$ . It is known that all submatrices of  $X$

are distributed as matrix Cauchy (e.g. see Dawid (1)) and that  $X$  is spherically symmetric. These properties help us to characterize  $C_{n,m}(0,I)$ . We have:

### 3.1. PROPOSITION

$$X \equiv C_{n,m}(0,I) \text{ iff}$$

$$(i) X \equiv HXK ; H \in O(n) , K \in O(m)$$

$$(ii) x_{ij} \equiv 1/x_{ij} \text{ and has no atoms } (i = 1, \dots, n; j = 1, \dots, m) .$$

PROOF. If  $X \equiv C_{n,m}(0,I)$  then (i) follows directly from the form of the density (5). Moreover, since all univariate marginals of  $X$  are  $C(0,1)$ , (ii) is a consequence of Theorem 1. To prove sufficiency, note that (i) implies  $X$  is spherically symmetric. This means that all of its marginal distributions are of the same type. This includes the univariate marginals. But, in view of (ii), we have

$$x_{ij} \equiv 1/x_{ij} \equiv -1/x_{ij} ,$$

the second distributional identity following from (i). Theorem 1 now implies that  $x_{ij} \equiv C(0,1)$  (all  $i, j$ ). It follows that  $X$  is of the Cauchy type and necessarily  $X \equiv C_{n,m}(0,I)$  with density (5).  $\square$

#### 4. REFERENCES

1. Dawid, A. P. (1981). "Some matrix variate distribution theory: Notational considerations and a Bayesian application," *Biometrika*, 68, 265-274.
2. Hassenforder, C. (1988). "An extension of Knight's theorem on Cauchy distribution," *Journal of Theoretical Probability*, 1, 205-209.
3. Knight, F. B. (1976). "A characterization of the Cauchy type," *Proc. Am. Math. Soc.*, 55, 130-135.
4. Williams, E. J. (1969). "Cauchy distributed functions and a characterization of the Cauchy distribution," *Annals of Mathematical Statistics*, 40, 1083-1085.