theory related to these variables can be obtained as an application of general theory concerned with the shot effect, developed by Takács [6], [7]. Our contribution in this case is to highlight the relevance of these ideas to the study of inventory models. This is the principal subject matter of Section 2. In particular, if $d\Psi = \mu e^{-\mu t}\,dt$, which means that orders in time constitute a Poisson process, the distribution of N_t is Poisson with parameter

$$\mu\int_0^t [1 - \Omega(s)]\,ds\ .$$

In Section 3 we shall introduce a new stochastic quantity which in the language of queueing theory represents the time necessary for all of the servers to finish their current work. A functional equation is obtained for the distribution of this random variable, which is solved completely when customers (orders) arrive according to a Poisson process, and which may be analyzed quite readily for other special cases (Ψ = gamma distribution of integral order). For the Poisson process, the stationary distribution function of this maximum is given by

$$P\{\omega \leqq x\} = e^{-\mu\int_x^\infty [1-\Omega(\xi)]d\xi}\ .$$

In Section 4 this stochastic quantity is generalized so that instead of time necessary for service on all outstanding orders to be completed, we determine the distribution of the time necessary for all outstanding orders but one or two, etc., to be filled.

Section 5 is devoted to a study of a modified version of the model discussed in preceding sections of this chapter. We assume that only limited facilities for service are available, so that whenever N orders are being served, all additional orders are refused. This modification of the model examined in Section 2 cannot be treated by the methods of Takács. The problem was solved for an arbitrary interarrival distribution and a negative exponential service rate by Scarf (see also [8]). In this chapter we investigate the case where the input process is Poisson and the service distribution is general. The infinitesimal character of a Poisson process is exploited in order to derive partial differential equations satisfied by the distribution of random variables analogous to those studied in Sections 3 and 4. It is then shown (assuming a continuous service distribution) that the limiting distribution of N^* is a truncated Poisson. A special case of this result was treated in [3] (see also [8]).

Some results are appended at the close of the chapter discussing the circumstances in which the parameters of the Poisson process depend on t.

2. The Distribution of the Number of Outstanding Orders for General Input and Service Distributions

In this section we shall apply the method developed by Takács in [6] to a generalization of one of the inventory models discussed by Scarf

[5]. The reader is referred to [5] for a full discussion of this model, which may be summarized as follows. Demands for an item arrive singly with an interarrival distribution given by $\Theta(t)$. Every time that D items are sold out of current inventory, a shipment of D more is requested, and this shipment is delivered with a time lag whose distribution is $\Omega(\lambda)$. If we assume that the process begins with S items in current inventory and 0 items which have been requested but not delivered, then at any future date the sum of the items in current inventory plus the items requested will lie between $S - D + 1$ and S.

Two different models arise, depending on how we treat the situation, of a demand arriving when no current inventory is available. In this section we shall assume that such demands may be satisfied by subsequent deliveries. This permits the number of items in current inventory to be negative.

Let $w(t)$ represent the number of shipments requested but not yet delivered at time t. This may be represented as follows. Define

$$f(u, v) = \begin{cases} 1 & 0 \leqq v - u \\ 0 & \text{otherwise} . \end{cases} \tag{1}$$

Then if $\tau_1, \tau_2, \cdots$ represent successive sums of independent observations from $\Psi(\tau)$ (the D-fold convolution of $\Theta(\tau)$), and $\lambda_1, \lambda_2, \cdots$ represent independent observations from $\Omega(\lambda)$:

$$w(t) = \sum_{\tau_n \leq t} f(t - \tau_n, \lambda_n) . \tag{2}$$

For any y, define $\pi(y ; t) = E(y^{w(t)})$. This is, of course, the generating function for the number of undelivered orders at time t. We shall obtain an integral equation for this function. Let us examine $w(t)$ under the condition that $\tau_1 = \tau$. Then under this condition

$$w(t) = \begin{cases} f(t - \tau, \lambda_1) + \overline{w}(t - \tau) & \tau \leqq t \\ 0 & \tau > t , \end{cases} \tag{3}$$

where $\overline{w}$ is independent of $f(t - \tau; \lambda_1)$ and has the same distribution as w. Therefore

$$E(y^{w(t)} | \tau_1 = \tau) = \begin{cases} \pi(y ; t - \tau) E_\Omega y^{f(t-\tau, \lambda_1)} & \tau \leqq t \\ 1 & \tau > t . \end{cases} \tag{4}$$

This may be simplified to

$$\begin{cases} \pi(y ; t - \tau)\{\Omega(t - \tau) + y - y\Omega(t - \tau)\} & \tau \leqq t \\ 1 & \tau > t . \end{cases} \tag{5}$$

We obtain $\pi(y, t)$ by integrating τ with respect to its distribution $\Psi(\tau)$. Therefore

$$\pi(y, t) = 1 - \Psi(t) + \int_0^t \pi(y ; t - \tau)\{(y - 1)[1 - \Omega(t - \tau)] + 1\} \, d\Psi(\tau) . \tag{6}$$

If we expand $\pi(y, t)$ as a power series in y about the point 1,

$$\pi(y, t) = \sum_0^\infty a_n(t)(y-1)^n \,, \tag{7}$$

then the factorial moments $a_n(t)$ satisfy the equations

$$\begin{aligned} &a_0(t) = 1, \\ &a_n(t) = \int_0^t a_n(t-\tau)\, d\Psi(\tau) + \int_0^t a_{n-1}(t-\tau)[1-\Omega(t-\tau)]\, d\Psi(\tau) \,. \end{aligned} \tag{8}$$

This set of equations may be expressed very simply in terms of the renewal quantity $M(\tau)$ based on the distribution $\Psi(\tau)$, i.e.,

$$M(\tau) = \sum_1^\infty \Psi^{(n)}(\tau) \,, \tag{9}$$

where $\Psi^{(n)}(\tau)$ represents the n-fold convolution of the distribution $\Psi(\tau)$. We have, solving the Volterra equation (8) by the standard method,

$$\begin{aligned} &a_0(t) = 1, \\ &a_n(t) = \int_0^t a_{n-1}(t-\tau)[1-\Omega(t-\tau)]\, dM(\tau) \,. \end{aligned} \tag{10}$$

These equations furnish a recursive procedure for computing the factorial moments, and hence the distribution of the number of shipments requested but not yet delivered at time t.

Let us, as a first example, apply these equations to the case where $\Psi(\tau) = 1 - e^{-u\tau}$, i.e., requests for shipments are a Poisson process. Then $M(\tau) = u\tau$, and

$$\begin{aligned} a_n(t) &= u\int_0^t a_{n-1}(t-\tau)[1-\Omega(t-\tau)]\, d\tau \\ &= u\int_0^t a_{n-1}(\xi)[1-\Omega(\xi)]\, d\xi, \end{aligned} \tag{11}$$

and hence

$$a_n'(t) = u a_{n-1}(t)[1-\Omega(t)] \,.$$

Therefore

$$\frac{\partial \pi(y, t)}{\partial t} = u(y-1)[1-\Omega(t)]\pi(y, t) \,, \tag{12}$$

and

$$\pi(y, t) = \exp\left\{ u(y-1)\int_0^t [1-\Omega(\xi)]\, d\xi \right\} \,, \tag{13}$$

which is the generating function of a Poisson distribution with mean

$$u\int_0^t [1-\Omega(\xi)]\, d\xi \,.$$

$\pi(y, \infty)$ is the generating function of a Poisson distribution with mean $uE_\Omega(\lambda)$.

As a second example, let us apply the above equations to the case in which $\Omega(\lambda) = 1 - e^{-\lambda/m}$, i.e., a negative exponential distribution for delivery time. This is the untruncated version of the problem treated by Scarf [5]. The equations become

$$a_n(t) = 1,$$

$$a_n(t) = \int_0^t a_{n-1}(t-\tau)e^{-t/m}e^{\tau/m}\,dM(\tau)\ .$$

If we denote the Laplace transform of the function $a_n(t)$ by $\tilde{a}_n(s)$, and the Laplace transform of the measure $M(t)$ by $\tilde{M}(s)$, then these equations become

$$\tilde{a}_0(s) = \frac{1}{s}\ ,$$

$$\tilde{a}_n(s) = \tilde{a}_{n-1}\left(s + \frac{1}{m}\right)\tilde{M}(s)\ ,$$

and therefore

$$\tilde{a}_n(s) = \frac{1}{s + \frac{n}{m}}\tilde{M}(s)\cdots\tilde{M}\left(s + \frac{n-1}{m}\right)\ . \tag{14}$$

Now we know that

$$a_n(\infty) = \lim_{s\to 0} s\tilde{a}_n(s)$$

(in this case the conditions for the application of the appropriate Abelian theorem are easily justified), so that

$$a_n(\infty) = \frac{m}{n}\prod_{j=1}^{n-1}\tilde{M}\left(\frac{j}{m}\right)\lim_{s\to 0} s\int_0^\infty e^{-st}\,dM(t)\ .$$

But

$$\lim_{s\to 0} s\int_0^\infty e^{-st}\,dM(t) = \lim_{s\to 0}\frac{s\int_0^\infty e^{-st}\,d\Psi(t)}{1 - \int_0^\infty e^{-st}\,d\Psi(t)} = \frac{1}{\int_0^\infty t\,d\Psi(t)} = \frac{u}{D}\ ,$$

where u is the average number of demands per unit time.

Therefore

$$a_n(\infty) = \frac{mu}{nD}\prod_{j=1}^{n-1}\tilde{M}\left(\frac{j}{n}\right)\ .$$

As a third example let us apply these techniques to the case in which the time lag is a constant λ. We shall show that if Ψ is not a lattice distribution, then

$$\lim_{t\to\infty}\pi_n(t) = \begin{cases} 1 - \frac{1}{\mu}\int_0^\lambda [1-\Psi]\,d\xi & n = 0 \\ \frac{1}{\mu}\int_0^\lambda [\Psi^{(n-1)} - 2\Psi^{(n)} + \Psi^{(n+1)}]\,d\xi & n > 0\ , \end{cases}$$

where $\pi_n(t)$ represents the probability of n undelivered orders at time t.

Let us begin by computing the factorial moments $a_n(t)$, by means of equations (10). If $t < \lambda$, we see that

$$a_n(t) = \int_0^t a_{n-1}(t-\tau)\, dM(\tau) ,$$

so that

$$a_n(t) = M^{(n)}(t) ,$$

the n-fold convolution of $M(t)$. On the other hand, for $t > \lambda$, we have

$$\begin{aligned} a_n(t) &= \int_{t-\lambda}^t a_{n-1}(t-\tau)\, dM(\tau) \\ &= \int_{t-\lambda}^t M^{(n-1)}(t-\tau)\, dM(\tau) = \int_\lambda^0 M^{(n-1)}(\xi)\, dM_\xi(t-\xi) . \end{aligned}$$

If we integrate this by parts, we obtain, for $t > \lambda$,

$$a_n(t) = M^{(n-1)}(\lambda)\{M(t) - M(t-\lambda)\} + \int_0^\lambda \{M(t-\xi) - M(t)\}\, dM^{(n-1)}(\xi) .$$

We now apply a theorem of Blackwell and Doob ([1], [2]) which implies that if Ψ is not a lattice distribution, then

$$\lim_{t\to\infty} \{M(t) - M(t-\xi)\} = \frac{\xi}{\mu} .$$

Therefore, if Ψ is not a lattice distribution,

$$\begin{aligned} \lim_{t\to\infty} a_n(t) &= M^{(n-1)}(\lambda)\frac{\lambda}{\mu} - \frac{1}{\mu}\int_0^\lambda \xi\, dM^{(n-1)}(\xi) \\ &= \frac{1}{\mu}\int_0^\lambda M^{(n-1)}\, d\xi = \frac{1}{\mu}\int_0^\lambda a_{n-1}(\xi)\, d\xi . \end{aligned}$$

In order to obtain $\lim_{t\to\infty} \pi_n(t)$, we examine the generating function $\pi(y, t)$. We have

$$\begin{aligned} \lim_{t\to\infty} \pi(y, t) &= 1 + \frac{(y-1)}{\mu}\sum_0^\infty (y-1)^n \int_0^\lambda a_n(\xi)\, d\xi \\ &= 1 + \frac{(y-1)}{\mu}\int_0^\lambda \pi(y, \xi)\, d\xi . \end{aligned}$$

Therefore

$$\pi_0(\infty) = 1 - \frac{1}{\mu}\int_0^\lambda \pi_0(\xi)\, d\xi ,$$

and

$$\pi_n(\infty) = \frac{1}{\mu}\int_0^\lambda [\pi_{n-1}(\xi) - \pi_n(\xi)]\, d\xi \quad \text{for } n > 0 .$$

Since it is clear that $\pi_n(\xi) = \Psi^{(n+1)}(\xi) - \Psi^{(n)}(\xi)$ for $\xi < \lambda$, the above result follows.

3. The Maximum Time for Service to be Completed

In this section we shall discuss another quantity associated with the process described in Section 2. We define the random variable $m(t)$ to be the time necessary for all of the shipments outstanding at time t to be delivered where initially there were no unfilled orders. In the notation of Section 2, we may write

$$m(t) = \max_{\tau_n \leq t} \{\lambda_n - (t - \tau_n), 0\} . \tag{15}$$

Let us define, for any t, $F(y, t)$ to be the distribution function corresponding to $m(t)$. This is in contrast to Section 2, in which a generating function rather than a distribution function was discussed. We shall obtain an integral equation for this function. Let us examine $m(t)$ under the condition that $\tau_1 = \tau$. Then under this condition

$$m(t) = \begin{cases} \max \{\overline{m}(t - \tau), \lambda_1 - (t - \tau)\} & \tau \leqq t \\ 0 & \tau > t , \end{cases} \tag{16}$$

where $\overline{m}$ is independent of λ_1 and has the same distribution as m. Therefore

$$\text{prob}\,\{m(t) \leqq y \mid \tau_1 = \tau\} = \begin{cases} F(y, t - \tau)\Omega(y + t - \tau) & \tau \leqq t \\ 1 & \tau > t , \end{cases}$$

and if we integrate out the condition, we obtain

$$F(y, t) = 1 - \Psi(t) + \int_0^t F(y, t - \tau)\Omega(y + t - \tau)\, d\Psi(\tau) . \tag{17}$$

It is to be noticed that generally there will be a non-zero probability that no shipments are outstanding at time t. This reflects itself in the fact that $F(0, t) > 0$, in most cases.

With relatively general assumptions on the functions Ω and Ψ, this equation may be solved by substituting a power series in t for $F(y, t)$ and equating coefficients. We shall not discuss this method in detail. Successive approximations may also be employed in order to obtain numerical evaluations of (17). Let us instead examine the above equation in the special case in which the time between successive requests for shipments is given by a negative exponential distribution. In other words, $\Psi(\tau) = 1 - e^{-u\tau}$. Then

$$F(y, t) = e^{-ut} + u\int_0^t F(y, t - \tau)\Omega(y + t - \tau)e^{-u\tau}\, d\tau . \tag{18}$$

Making the substitution $t - \tau = \xi$, this becomes

$$F(y, t) = e^{-ut} + u\int_0^t F(y, \xi)\Omega(y + \xi)e^{-ut}e^{u\xi}\, d\xi .$$

This shows us that $F(y, t)$ is differentiable in t, and we may write

$$\frac{\partial F(y,t)}{\partial t} = -u[1-\Omega(y+t)]F(y,t)\,. \tag{19}$$

Using the initial condition $F(y, 0) = 1$, we obtain

$$F(y,t) = \exp\left(-u\int_y^{t+y}[1-\Omega(\xi)]\,d\xi\right). \tag{20}$$

If the distribution of the time between successive requests for shipments is a member of the Γ family of distributions, then a procedure similar to the one above will give us an ordinary linear differential equation for $F(y, t)$, but of higher order than the first. Indeed, suppose

$$d\Psi(t) = \frac{u^k t^{k-1} e^{-ut}}{\Gamma(k)}\,dt \quad (k \geqq 1)\,.$$

The integral equation (17) becomes

$$e^{ut}F(y,t) = P_k(t) + \frac{u^k}{\Gamma(k)}\int_0^t F(y,\xi)\Omega(y+\xi)(t-\xi)^{k-1}e^{u\xi}\,d\xi\,, \tag{21}$$

where

$$P_k(t) = e^{ut}\int_t^\infty \frac{u^k \xi^{k-1} e^{-u\xi}}{\Gamma(k)}\,d\xi$$

is a polynomial of degree $k-1$. Differentiation of (21) k times and then cancellation of the common factor e^{ut} yields

$$\frac{\partial^k F}{\partial t^k} + \binom{k}{1}u\frac{\partial^{k-1}F}{\partial t^{k-1}} + \binom{k}{2}u^2\frac{\partial^{k-2}F}{\partial t^{k-2}} + \cdots + \binom{k}{i}u^i\frac{\partial^{k-i}F}{\partial t^{k-i}} + \cdots + \binom{k}{k}u^k F = \Omega(y+t)F\,. \tag{22}$$

The initial data are obtained after each successive differentiation of (21) by setting $t = 0$. We obtain $F(y, 0) \equiv 1$ and

$$\frac{\partial F}{\partial t}(y,0) = \frac{\partial^2 F}{\partial t^2}(y,0) = \cdots = \frac{\partial^{k-1}F}{\partial t^{k-1}}(y,0) = 0\,. \tag{23}$$

For general Ω, it is very difficult to explicitly solve (22) for F. If $k = 2$ and $\Omega(y + t) = 1 - e^{-\lambda(y+t)}$ (lag is an exponential distribution), then the solution of (22) may be represented as a combination of hypergeometric functions. For an arbitrary k, the representation of the solution can be expressed in terms of generalized hypergeometric functions. One particular case of importance is where

$$\Omega(u) = \begin{cases} 1 & u > a \\ 0 & u < a\,. \end{cases}$$

This is the physical circumstance where the lag in delivery is of fixed length a. Of course, for this case $F(y, t) = 1$ for $y \geqq a$ and consequently $F(y, t)$ need be evaluated only for the range $y < a$. In this situation, (22) reduces to the pair of equations

$$(24) \qquad \frac{\partial^k F}{\partial t^k} + \binom{k}{1} u \frac{\partial^{k-1} F}{\partial t^{k-1}} + \cdots + u^{k-1} \binom{k}{k-1} \frac{\partial F}{\partial t} + u^k F = 0 \qquad y + t < a$$

$$(25) \qquad \frac{\partial^k F}{\partial t} + \binom{k}{1} u \frac{\partial^{k-1} F}{\partial t^{k-1}} + \cdots + u^{k-1} \binom{k}{k-1} \frac{\partial F}{\partial t} = 0 \qquad y + t \geqq a \, .$$

Inspection of (21) will uncover the fact that

$$F(y, t), \cdots, \frac{\partial^{k-1} F}{\partial t^{k-1}}$$

are all continuous for $y + t = a$. From (21), we deduce directly that

$$(26) \qquad F(y, t) = \int_t^\infty \frac{u^k \xi^{k-1} e^{-u\xi}}{\Gamma(k)} d\xi \quad \text{for } y + t < a \, .$$

Turning to the range $y + t \geqq a$, we observe that the algebraic characteristic equation associated with (25) is

$$(27) \qquad (\alpha + u)^k - u^k = 0 \, .$$

If ω denotes a primitive kth root of unity, then the solutions of (27) are all distinct and are indeed

$$(28) \qquad \alpha_r = -u + u\omega^r \qquad r = 0, 1, 2, \cdots, k-1 \, .$$

The general solution of (25) may now be explicitly constructed from knowledge of the roots α_r. In fact,

$$(29) \qquad F(y, t) = \sum_{r=0}^{k-1} A_r e^{\alpha_r t} \quad \text{for } y + t \geqq a \, ,$$

where A_r represents arbitrary constants. These constants are to be determined so that

$$F, \frac{\partial F}{\partial t}, \cdots, \frac{\partial^{k-1} F}{\partial t^{k-1}}$$

are all continuous for $t = a - y$ ($y \leqq a$). These conditions translate into the system of linear equations

$$(30) \qquad \sum_{r=0}^{k-1} A_r \alpha_r^l e^{\alpha_r (a-y)} = \frac{d^l}{dt^l} \left(\int_t^\infty \frac{u^k \xi^{k-1} e^{-u\xi}}{\Gamma(k)} d\xi \right) \Bigg|_{a-y}$$

where $l = 0, 1, 2, \cdots, k-1$, from which the constants A_r are uniquely determined since the matrix of the coefficients is the familiar Vandermonde matrix with α_r distinct. For the special case $k = 2$, we have in particular

$$(31) \quad F(y, t) = \begin{cases} (1 + ut)e^{-ut} & y + t < a \\ \left[1 + \left(u - \frac{u^2}{2}\right)(a - y)\right] e^{-u(a-y)} + \frac{u^2 (a-y)}{2} e^{(2-u)(a-y)} e^{-2t} & y + t \geqq a \text{ and } y < a \\ 1 & y \geqq a \, . \end{cases}$$