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BY

HERBERT SCARF

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APPLIED MATHEMATICS AND STATISTICS LABORATORY
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

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Summary

In this paper we discuss a theory of consumer's choice, based on the idea of utility maximization, subject to constraints in expenditures. A description is given of those utility functions for which future tastes are independent of past consumption, and an even smaller class is described for which tastes are stationary in time. It is shown that the utility function will have these properties only if it is a discounted sum of functions pertaining to consumption in the separate periods. The discount factor leads to the notion of a personal interest rate, which may be constant, or a function of the consumption level.

The question of whether a decreasing ratio of expenditure to total wealth is compatible with such a utility function is discussed. The case of constant personal interest is examined in detail and it is shown that no stationary utility function permits a decreasing ratio of expenditure to wealth for a sufficiently wide range of market interest rates. On the other hand, an example is given of a utility indicator with a variable personal interest rate, which predicts a decreasing ratio of expenditure to wealth for all market interest rates.

I. Introduction

In this paper we shall discuss several mathematical problems that arise in the construction of a dynamic theory of consumer's choice. During the last several years there have been at least two mathematical approaches to the problem of consumer's choice over time, that of Friedman (3) and that of Modigliani and Brumberg (5). Our discussion will be similar in spirit, though not in detail, to that offered by Modigliani and Brumberg, in the sense that these authors view the problem specifically as one of utility maximization, subject to constraints on expenditures.

The problem faced by a consumer is first of all, that of expressing preferences among alternative streams of consumption bundles. Let $x_1, x_2, x_3, \ldots$ be such a stream of consumption bundles, where x_t is a vector whose j^{th} component represents the number of units of commodity j to be consumed in time period t. We shall begin by assuming that the preferences of the typical consumer are expressed by a utility function

$$U(x_1, x_2, \ldots),$$

such that when alternative consumption streams are presented to the consumer, he will attempt to secure that stream with the highest utility.

Actually we shall assume somewhat more; in the general case, the consumer does not select levels of consumption for all future time periods, but rather he takes a series of actions which lead to consumption levels, perhaps depending randomly upon the actions taken. For this reason the consumer should, ideally at least, be able to express preferences among probability distributions of consumption streams.[1] This consideration leads to the fact that $U(x_1, x_2, \dots)$ is a Bernoulli utility indicator, with preferences being ranked according to expected utility. Of course, if the Bernoulli utility indicator is left in a perfectly general form, there is very little that can be said about the actual behavior of the consumer. There is an enormous variety of behavior consistent with a general utility function, and if we are interested in predicting specific patterns of consumption and saving over time, some restrictions must be made as to the form of the utility function.

Our restriction, which seems quite plausible as a first approximation, may be described by saying that the future tastes are independent of past consumption. As we shall show in Section II, this implies that the utility function U may be written in the following form:

$$(1) \qquad U(x_1, x_2, x_3, \dots) = \varphi_1(x_1) + \beta_1(x_1)\,\varphi_2(x_2) + \beta_1(x_1)\,\beta_2(x_2)\,\varphi_3(x_3) + \dots, \text{ with } \beta_n > 0 \,.$$

At this point we are being deliberately vague as to whether the number of time periods under consideration is finite or infinite. The reasoning

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[1] No distinction will be made in this paper between subjective and objective probability distributions.

which produces (1) is valid for both of these cases. The function β_n may be looked upon as the personal discount factor which is operative in time period n . β_n is a function rather than a constant, and therefore the personal discount factor depends on the level of consumption during the period. The personal interest rate would, of course, be defined by $\frac{1-\beta_n}{\beta_n}$ and is in no way connected with the market rate of interest.

To be somewhat realistic, it should probably be assumed that for large n, $\beta_n \equiv 0$, that is consumption at a time exceeding the possible life-time of the consumer should be discounted completely. However, it is occasionally useful to introduce as an additional restriction on the Bernoulli indicator U , the assumption that future tastes are not only independent of past consumption, but are also stationary over time. This assumption will be introduced primarily for the purposes of mathematical simplification; it requires that the argument of the Bernoulli indicator be an infinite sequence of consumption vectors. As a consequence of the assumption of stationary tastes it will be shown in Section II, that the utility indicator may be written in the form

$$(2) \qquad U(x_1, x_2, \ldots) = \varphi(x_1) + \beta(x_2)\, \varphi(x_2) + \beta(x_1)\, \beta(x_2)\, \varphi(x_3) + \cdots \quad ,$$

again with $\beta > 0$.

It should be remarked that there are corresponding expressions if the consumption takes place continuously over time, rather than at discrete time intervals. In particular (2) is replaced by an expression of the form

$$(3) \qquad U(x(\cdot)) = \int_0^\infty e^{-\int_0^t i(x(\xi))d\xi} \varphi(x(t))dt \ ,$$

where $x(t)$ is the vector valued rate of consumption, $i(x)$ is the personal interest rate (rather than discount factor) corresponding to a consumption rate x (it is generally > 0, but may actually be negative) and $U(x(\cdot))$ is the utility associated with the consumption plan $x(t)$ $(0 \leq t \leq \infty)$. The function φ appearing underneath the integral sign in (3) is analogous to the function φ appearing on the *right* hand side of (2). We shall find it convenient to work with the continuous version (3), rather than the discrete version (2).

In Section III we shall discuss the behavior of the consuming unit when guided by a utility indicator of the form (3). For simplicity we shall assume that there is a single item of consumption so that the vector function $x(t)$ appearing in (3) will actually be a scalar, and that the price of this item remains constant over time (the price will be taken as 1). This general item of consumption will be assumed to be completely non-durable, so that no inventories are carried. The income rate of the consumer will be assumed to be a known function of the time $I(t)$ (random fluctuations of this income will be examined in a future paper.) Strictly speaking we mean

$I(t)$ to be the income component which is independent of returns on investments and interest payments on loans. Any expenditure plan will result in excesses or deficits of expenditures over assets and any disparity between assets and expenditures is meant to be converted into investments or borrowings at the market rate of interest j. The only constraints that we shall impose on the expenditure plan are

$$
(4) \quad
\begin{aligned}
&\text{a.} \quad x(t) \geq 0 \quad \text{and} \\
&\text{b.} \quad \int_0^\infty e^{-jt}\, x(t)dt \leq M_0 + \int_0^\infty e^{-jt}\, I(t)dt = M\ ,
\end{aligned}
$$

where M_0 represents assets at time $t=0$. (We shall always use M in this sense, and talk of it as the assets, meaning the discounted value of current plus future assets.) The latter constraint is a very weak one; it is equivalent to assuming that the present value of indebtedness at time t tends to zero as $t \to \infty$. Other constraints, such as a limit on the actual indebtedness at any time are also possible, but we shall not discuss them in this paper.

In Section III we determine the expenditure plan which maximizes (3) under the constraints (4), when the personal interest rate i is a constant, and φ is an increasing concave function. The solution is as follows

Theorem I.

1. If $i < j$ then the optimal consumption plan $x(t;j,M)$ consists of (possibly) an interval $(0, t_0)$ in which $x=0$, and in (t_0, ∞) $x(t)$

satisfies the differential equation

$$\frac{dx}{dt} = (i-j)\,\frac{\varphi'(x)}{\varphi''(x)}. \tag{5}$$

t_o is taken as the smallest possible value such that the solution satisfies (4).

2. If $i > j$, then $x(t)$ consists of (possibly) an interval (t_o, ∞) in which $x = 0$, and in $(0, t_o)$ $x(t)$ satisfies (5). t_o is taken as the largest possible value such that the solution satisfies (4).

3. If $i = j$, then $x(t) = jM$.

The general features of the solution are clear: if $i < j$, then expenditures are increasing steadily over time. If M is thought of as the capital of the consumer at $t = 0$, then the capital at time t, $M(t)$, will be an increasing function of the time,[2] and expenditures are proceeding at a rate which is less than the interest on the capital. It is probably fair to call this type of consumer a saver.

On the other hand if $i > j$, the situation is reversed, and the consumer is dissaving.

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[2] $M(t)$ satisfies the equation

$$\frac{dM}{dt} = jM(t) - x(t).$$

We should remark that it is possible for there to be no optimal consumption plan. It is possible to construct examples in which any consumption plan can be improved. Theorem 1 only applys to the case in which an optimal consumption plan exists.

Actual data on consumption patterns seem to indicate that the ratio of consumption to income falls as income rises. This would seem to be reflected in our theory by the statement that $\frac{x(0;j,M)}{M}$ should be strictly decreasing in M. In Section IV we shall take a look at this topic. It will be shown that as long as the personal interest rate i is constant, there is no utility function φ which has the property that $\frac{x(0;j,M)}{M}$ is decreasing in M for all j. The only utility indicators which are in any way close to fulfilling this condition are $\varphi(x) = \log x$ or $\varphi(x) = x^a$,[3] and for these utility functions $\frac{x}{M}$ is actually independent of M, for each fixed j. This seems to indicate, in no uncertain terms, that the personal interest rate should depend on the rate of consumption. In Section V an example of a utility indicator with a variable personal interest rate is given, for which $\frac{x(0;j,M)}{M}$ is strictly decreasing for all j.

We should like to thank K.J. Arrow, L. Hurwicz, and S. Karlin for a number of stimulating conversations on these topics.

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[3] Linear transformations are possible.

II. The Independence Assumption.

There are several excellent discussions in the literature of Bernoulli utility indicators, and we shall not repeat these discussions here (6,1). The main points are the following: Consider a set consisting of a number of alternatives, and with a preference relationship which permits one to compare an arbitrary pair of probability distributions for these alternatives. If the preference relationship is assumed to have a number of simple and intuitively plausible properties, then it may be shown that there is a utility indicator U, defined on the set of alternatives, with the property that one probability distribution is preferred to another if its expected utility is larger. Moreover, the function U is unique up to a linear transformation, in the following sense: if $U' = a+bU$ with $b > 0$, then preferences ranked according to the expectation of U are the same as preferences ranked according to the expectation of U', and if U' and U are not related in this linear fashion, then they express different preferences for probability distributions over the alternatives. We shall take it for granted, in the remainder of this paper, that all preference relationships will be such as to imply the existence and uniqueness of a Bernoulli utility indicator in this sense.

Now let us turn our attention to the independence assumption described in Section I. We consider, as the space of possible alternatives, all sequences of consumption vectors, $x_1, x_2, \ldots$ (These may be finite sequences, if we wish, or else infinite.), and along with this a Bernoulli utility indicator $U_1(x_1, x_2, \dots)$. Preferences over probability distributions are obtained by ranking the expectations of this function. In particular, all probability distributions for which consumption in the first period is not random, but actually equal to $\bar{x}_1$, may be compared. This implies that if we consider all consumption streams which are equal to $\bar{x}_1$ in the first period, then preferences are determined by ranking the expectations of $U_1(\bar{x}_1, x_2, x_3, \dots)$, and therefore $U_1(\bar{x}_1, x_2, x_3, \dots)$ as a function of $x_2, x_3, \dots$ is a Bernoulli utility indicator for consumption from period two onward, given that the consumption in period 1 is $\bar{x}_1$. But according to our assumption about future tastes being independent of past consumption, we see that as $\bar{x}_1$ ranges over all possible values, the family of functions of $x_2, x_3, \dots, U_1(\bar{x}_1, x_2, x_3, \dots)$ should be equivalent Bernoulli indicators, and therefore should be linked by a linear relationship. This implies that

$$U_1(x_1, x_2, \dots) \equiv \varphi_1(x_1) + \beta_1(x_1)\, U_2(x_2, \dots)\,, \tag{6}$$

for some $\varphi_1, \beta_1 > 0$ and U_2. In order to obtain (1), we apply the same argument to the function U_2, etc.

If in addition to the independence of future tastes and past consumption we also require that tastes be stationary over time, the preceding argument implies that $U_1(\bar{x}_1, x_2, x_3, \dots)$ and $U_1(x_2, x_3, \dots)$ are equivalent Bernoulli indicators for each fixed $\bar{x}_1$, and therefore,

$$U_1(x_1, x_2, \dots) \equiv \varphi(x_1) + \beta(x_1)\, U_1(x_2, \dots) \ . \tag{7}$$

We may therefore drop the calendar date from the utility indicator and by the iteration of (7) we obtain (2).

It is interesting to note that unless the personal discount factor β is constant, the function φ should not be interpreted as a utility indicator, in the sense that if φ is changed by a linear transformation the over-all tastes of the consumer are actually modified.

If the consumer is depicted as consuming continuously over time, at a rate $x(t)$, then the Bernoulli utility indicator would associate a number with each such function. We may arrive at such a number, by a limiting operation on (2), by letting the periods between successive consumption choices approach zero. If the personal interest rate is considered to be proportional to the time between successive consumption decisions, then (2) becomes

$$U(x(\cdot)) \sim \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} \frac{1}{1+i(x(k\Delta t))\Delta t} \right) \varphi(x(n\Delta t)) \ .$$

If we multiply (normalizing) U by Δt, and pass to the limit, we obtain

$$U(x(\cdot)) = \int_0^{\infty} e^{-\int_0^t i(x(\xi))d\xi}\, \varphi(x(t))dt \ , \tag{8}$$

where $i(x)$ may be interpreted as the personal interest rate corresponding consumption rate x. This formula, for the case of stationary independent tastes, may also be derived directly by considerations similar to those of the discrete time period model. We shall find it convenient to work with the continuous time model rather than the discrete model.

III. The Optimal Consumption Plan for Constant Personal Interest Rate.

In this section we shall discuss the optimal consumption plan when the consumer is guided by a utility function of the form (8), and with a constant personal interest rate. In this case φ may be considered as a proper utility function, and we shall assume that as a function of the rate of consumption it is both increasing and concave.

The only constraints that we shall impose on the possible consumption plans $x(t)$ are

$$(9) \qquad \begin{aligned} &a; \quad x(t) \geq 0 \qquad \text{and} \\ &b; \quad \int_0^\infty e^{-jt}\, x(t)dt \leq M \ . \end{aligned}$$

(j is the market rate of interest.)

The problem is to determine the function $x(t)$ which maximizes (8), subject to the constraints (9). The mathematical technique is similar to that used by Karlin in (4), and Karlin and Arrow in (3).

Let us begin by assuming the existence of an optimal consumption plan $x^*(t)$, and deduce several conditions that must by satisfied by this plan.

As was remarked in the introduction, it is possible for there to be no optimal consumption plan. For the moment, let us consider the case where the personal interest rate i is smaller than the market rate j. If $x(t)$ is any other consumption plan satisfying (9), then for any θ between zero and one $\theta x^*(t) + (1-\theta)x(t)$ is a consumption plan which satisfies the constraints. The utility of this latter plan is given by

$$(10) \qquad J(\theta) = \int_0^\infty e^{-it}\, U(\theta x^*(t) + (1-\theta)x(t))dt\ .$$

$J(\theta)$ is concave in θ, and therefore, a necessary and sufficient condition that it assumes its maximum at $\theta=1$ is that $J'(1) \geq 0$. But

$$(11) \qquad J'(1) = \int_0^\infty e^{-it}\, U'(x^*(t))\ (x^* - x)dt$$

$$= \int_0^\infty e^{(j-i)t}\, U'(x^*(t))\ (e^{-jt}\, x^*(t) - e^{-jt}\, x(t))dt.$$

Let us integrate this last expression by parts, defining $C(t)$ by the expression

$$(12) \qquad C(t) = \int_0^t e^{-j\xi}\, x(\xi)d\xi\ ,$$

and $C^*(t)$ in a corresponding way. Since $C(\infty) = C^*(\infty) = M$, (we are comparing the optimal plan with other plans that utilize all resources.) the integration by parts yields

$$0 \le \int_0^\infty \frac{d}{dt}\left\{ e^{(j-i)t}\, U'(x^*(t))\right\} \left\{ C^*(t) - C(t)\right\} dt \tag{13}$$

and therefore, $C^*(t)$ maximizes the integral

$$\int_0^\infty \frac{d}{dt}\left\{ e^{(j-i)t}\, U'(x^*(t))\right\}\, C(t)dt \quad , \tag{14}$$

for all choices of $C(t)$ arising from a feasible consumption plan $x(t)$.

This remark permits us to deduce some simple properties satisfied by the optimal consumption plan $x^*(t)$. First of all, let us suppose that there is some interval of t-values such that

$$\frac{d}{dt}\left\{ e^{(j-i)t}\, U'(x^*(t))\right\} > 0 \ , \qquad \text{say,} \tag{15}$$

t in (a,b). We shall show that this is impossible, unless $C^*(t)$ is actually constant in this interval. For, assume that it is not constant, so that $C^*(b) > C^*(a)$. Then we can construct a consumption plan $x(t)$ such that $C(a) = C^*(a)$, and $C(b) = C^*(b)$, and $C(t) > C^*(t)$ for t between a and b . (A simple graph will show that this is possible.) This clearly improves the integral (14) so that if (15) holds, we must have $x^*(t) = 0$ in (a,b). But since $j < i$ this contradicts (15) and therefore (15) can never hold.

We have shown that $\frac{d}{dt}\, e^{(j-i)t}\, U'(x^*(t))$ must be everywhere ≤ 0. An argument similar to that of the preceding paragraph shows that if this function is ever strictly less than zero in an interval, then $x^*(t) = 0$

in this interval. These remarks may be summarized in the following lemma.

Lemma 1: The optimal consumption plan $x^*(t)$ is composed of pieces, in which $x^*(t)$ is either identically zero or given by the solution of the differential equation

$$\frac{d}{dt}\left\{ e^{(j-i)t}\, U'(x^*(t)) \right\} = 0 . \tag{16}$$

The next step is to show that when $j < i$ the only place where $x^*(t)$ can be identically zero is (possibly) in an interval connected to the origin, and that if this interval is given by $(0, t_o)$ with $t_o > 0$, then $x^*(t_o + 0) = 0$.

Let us begin by assuming that the optimal strategy contains an interval of zero consumption which is not connected to the origin. The function $C^*(t)$ will then have the appearance of the solid line in Figure 1.

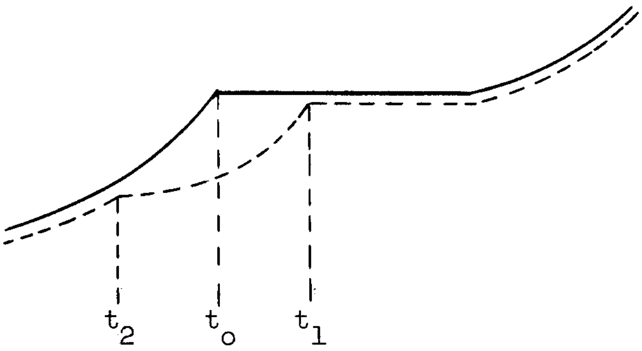

FIGURE 1.

We mean to compare the optimal policy with one whose $C(t)$ function has the form of the dashed line in Figure 1. This comparison policy may be obtained in the following way: pick a point t_2 slightly to the left of t_o. t_2 is in an interval in which $x^*(t)$ satisfies the differential equation (16). It may be shown directly from this equation, since $j < i$ and φ is increasing and concave, that $x^*(t)$ is increasing, and therefore, if t_2 is close to t_o, $x^*(t_2) > 0$. We change the x^* policy by beginning at t_2 with a slightly lower value of $x(t_2 + 0)$, and continuing by means of the differential equation until the two C functions intersect, as is shown in Figure 1. There will always be such an intersection point if $x(t_2 + 0)$ is taken sufficiently close to $x^*(t_2)$.

If $x^*(t)$ is, indeed, optimal then the integral appearing in (11) must be ≥ 0, when x is the comparison policy described above. Let us show that this is false; x^* and x agree everywhere, except in the interval from t_2 to t_1, and we may therefore write (11) as

$$(17) \quad \int_{t_2}^{t_o} e^{-it} \varphi'(x^*(t))(\bar{x}^* - x)dt + \int_{t_o}^{t_1} e^{-it} \varphi'(x^*(t))(x^* - x)dt \ .$$

Consider the first integral in (17). In the interval (t_2, t_o) $x^*(t)$ satisfies the differential equation (16). This implies that

$$(18) \qquad e^{(j-i)t}\varphi'(x^*(t)) = e^{(j-i)t_o} \varphi'(x^*(t_o - 0)),$$

and therefore the first integral is equal to

(19) $$e^{(j-i)t_o} \varphi'(x^*(t_o-0)) \int_{t_2}^{t_o} e^{-jt}(x^*-x)\,dt.$$

Now consider the second integral. For this range of t values, $x^*(t) = 0$, and therefore the second integral is

$$\begin{aligned} &-\varphi'(0) \int_{t_o}^{t_1} e^{-it}\, x(t)dt \\ &= -\varphi'(0) \int_{t_o}^{t_1} e^{-jt} e^{(j-i)t}\, x(t)dt \\ &< -\varphi'(0) e^{(j-i)t_o} \int_{t_o}^{t_1} e^{-jt}\, x(t)dt \\ &< -\varphi'(x^*(t_o-0)) e^{(j-i)t_o} \int_{t_o}^{t_1} e^{-jt}\, x(t)dt, \end{aligned}$$

since $x^*(t_o-0) > 0$ and φ' is a strictly decreasing function of its argument.

Therefore (17), which is meant to be ≥ 0 if x^* is optimal, is actually

$$\begin{aligned} &< \varphi'(x^*(t_o-0))\, e^{(j-i)t_o} \left\{ \int_{t_2}^{t_o} e^{-jt}(x^*-x)\,dt - \int_{t_o}^{t_1} e^{-jt}\, x\,dt \right\} \\ &= \varphi'(x^*(t_o-0))\, e^{(j-i)t_o} \left\{ \int_{t_2}^{t_1} e^{-jt}(x^*-x)\,dt \right\} = 0, \end{aligned}$$

which shows that x^* is not optimal if it contains an interval of zero consumption which is not connected to the origin.

On the other hand let us assume that $x^*(t)$ indicates zero consumption for the interval $(0,t_o)$. We shall show that this policy is only optimal if $x^*(t_o+0) = 0$. For suppose that $x^*(t_o+0) > 0$. In this case let us compare, by means of (11), the x^* policy with one whose $C(t)$ function is given by the dashed line in Fig. 2. The solid line is meant to be $C^*(t)$.

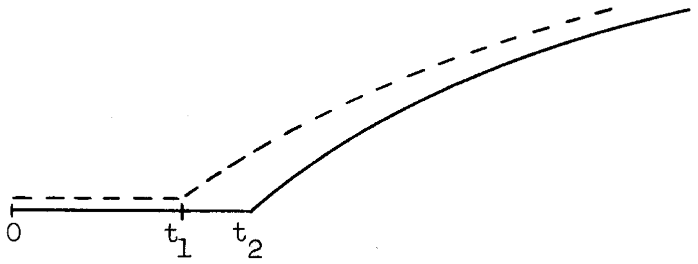

FIGURE 2.

The dashed policy indicates no consumption in $(0,t_1)$ and consumption according to the differential equation in (t_1,∞), with $\int_o^\infty e^{-jt}x(t)dt = M$. Let us show that such a policy can be constructed when $x^*(t_o+0) > 0$ and t_1 is close to t_o. For consider the consumption curve which follows the differential equation in (t_1,∞) and for which $x(t_1+0) = 0$. For t_1 close to t_o, $C(\infty)$ for this policy, will be close to the $C(\infty)$ for the policy which follows the differential equation from t_o onwards and for which $x(t_o+0) = 0$. But for such a policy $C(\infty) < M$, since $\int e^{-jt}x(t)dt$ is monotone in the initial conditions (because of the properties of φ' and φ''.) Therefore for t_1 close to t_o, discounted expenditure will be less than M if we start out with zero expenditure at

t_1. Raise the expenditure gradually at t_1, until discounted expenditure is actually equal to M and we get the dashed policy.

Now let us compute (11) where x is the dashed policy. This integral may be written as

$$(20) \qquad \int_{t_1}^{t_o} e^{-it}\varphi'(x^*(t))(x^*-x)dt + \int_{t_o}^{\infty} e^{-it}\varphi'(x^*(t))(x^*-x)dt.$$

In the first integral $x^* = 0$ and therefore it may be written as

$$-\varphi'(0)\int_{t_1}^{t_o} e^{-it}xdt$$

$$= -\varphi'(0)\int_{t_1}^{t_o} e^{-jt}e^{(j-i)t}xdt$$

$$< -\varphi'(0)\, e^{(j-i)t_1}\int_{t_1}^{t_o} e^{-jt}xdt\,.$$

Consider the second integral in (20) In this integral x^* satisfies the differential equation and therefore

$$e^{(j-i)t}\varphi'(x^*(t)) = e^{(j-i)t_o}\varphi'(x^*(t_o)),$$

so that the integral may be written as

$$e^{(j-i)t_o}\varphi'(x^*(t_o))\int_{t_o}^{\infty} e^{-jt}(x^*-x)dt.$$

Now let us pick t_1 so close to t_o such that

$$\varphi'(0)e^{(j-i)t_1} > \varphi'(x^*(t_o))\, e^{(j-i)t_o}\,,$$

which we can do since $x^*(t_o) > 0$. Then (20) is less then

$$e^{(j-i)t_o}\varphi'(x^*(t_o))\left\{\int_{t_o}^{\infty} e^{-jt}(x^*-x)\,dt - \int_{t_1}^{t_o} e^{-jt}x\,dt\right\} = 0,$$

and this shows that x^* is not optimal if $t_o > 0$ and $x^*(t_o + 0) > 0$.

If these remarks are combined, they indicate the procedure for constructing the optimal consumption plan when $i < j$. Pick an initial consumption level $x^*(0) \geq 0$, and solve the differential equation (5) with this initial condition. If there is an initial condition such that the resulting solution gives $\int_0^{\infty} e^{-jt}x^*(t)dt = M$, then x^* is the optimal plan. On the other hand if all solutions give rise to an $x^*(t)$ with $\int_0^{\infty} e^{-jt}x^*(t)dt > M$, then we must have recourse to an initial interval of zero consumption, say $(0, t_o)$. But the point t_o is quite simple to determine. Solve the differential equation (5) with the initial condition $x^*(0) = 0$. Then

$$e^{jt_o}M = \int_0^{\infty} e^{-jt}x^*(t)\,dt, \tag{21}$$

and the actual optimal plan is given by $x^*(t-t_o)$ for $t \geq t_o$.

This disposes of the case when the personal rate of interest is less than the market interest rate. A similar set of remarks may be made when the personal interest rate i is larger than the market rate. By reasoning similar to that used above it is possible to show that the optimal consumption plan consists of at most two parts: a part governed by the solution of the differential equation (5) and a part of zero consumption. In this case however the zero consumption level, if it exists, must be connected to infinity, rather than to the origin. It is also possible to show that if such an interval (t_o, ∞) exists, then $x^*(t_o - 0) = 0$.