

Robust Predictions  
in  
Games with Incomplete Information

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- in games of incomplete information, private information represents information about:  
    payoff environment and strategic environment
- first order beliefs vs. higher order beliefs
- game theoretic predictions are very sensitive to specification of the strategic environment, the "higher order beliefs", or (equivalently) the information structure
- email game, coordination games in game theory
- revenue equivalence, surplus extraction in mechanism design

# The Role of Higher Order Beliefs: An Example

- first price sealed bid auction with independent, private values
- identical, independent common prior over valuations of bidders
- first order beliefs of bidder  $i$  is about:
  - valuation of bidder  $i$  himself; valuations of bidders  $j, k$
- second order beliefs of bidder  $i$  is about:
  - the first order belief of bidder  $j$  about the valuation of bidder  $i$
- in first price sealed bid auction, bid of  $i$  depends on his first-order, second-order and higher-order beliefs
- revenue equivalence - assuming second-order belief of  $i$  puts probability 1 on  $j$ 's first order belief being equal to common prior
- i.e. private information of bidder  $i$  doesn't contain any additional information about bidder  $j$  beyond common prior; revenue equivalence fails without this restrictive assumption
- what predictions can we offer independent of the higher order beliefs?

# Payoff Environment and Strategic Environment

- fix "payoff relevant environment"
  - = action sets, payoff-relevant variables ("states"), payoff functions, distribution over states
  - = incomplete information game without higher order beliefs about states
- there are many strategic environments consistent with fixed payoff environment
- consistent: after integrating over the higher-order beliefs, the marginal over payoff relevant states coincides with common prior over payoff relevant states
- the possible information environments vary widely: from "zero information", where every agent knows nothing beyond common prior to "complete information", where every agents knows realization of payoff relevant state

# Agenda: Robust Predictions

- analyze what could happen for all possible higher order beliefs (maintaining common prior assumption and equilibrium assumptions)
- each specific information environment generates specific predictions regarding equilibrium behavior
- given that the different strategic environments share the same payoff environment, does the predicted behavior display common features?
- can analyst make predictions which are robust to the exact specification of the strategic environment?
- make set valued predictions about joint distribution of actions and states

# Agenda: Prior Information and Prediction

- perhaps analyst doesn't observe all higher order beliefs but is sure of some aspects of the information structure:
  - is sure that bidders know their private values of an object in an auction, but has no idea what their beliefs and higher order beliefs about others' private values are...
  - is sure that oligopolists know their own costs, but has no idea what beliefs and higher order beliefs about demand and others' private values...
- what can you say then?

# Agenda: Robust Identification

- the observable outcomes of the game are the actions and the payoff relevant states
- the chosen action reveals the preference of the agent given his higher-order belief, but typically does not reveal his higher order belief
- having identified mapping from "payoff relevant environment" to action-state distributions, we can analyze its inverse:
- assume payoff relevant environment is observed by the econometrician
- given knowledge of the action-state distribution , or some moments of it, what can be deduced about the payoff relevant environment?
- partial identification / set identification

## ① General Approach

- set valued prediction is set of "Bayes Correlated equilibria"
- epistemic result linking Bayes Nash and Bayes Correlated equilibrium
- partial prior information monotonically reduces the set of "Bayes Correlated equilibria"

## ② Illustration with Continuum Player, Symmetric, Linear Best Response, Normal Distribution Games

- the resulting equilibrium sets are tractable and intuitive
- cannot distinguish between games with strategic substitutes and complements

- players  $i = 1, \dots, I$
- (payoff relevant) states  $\Theta$
- actions  $(A_i)_{i=1}^I$
- utility functions  $(u_i)_{i=1}^I$ , each

$$u_i : A \times \Theta \rightarrow \mathbb{R}$$

- common prior state distribution  $\psi \in \Delta(\Theta)$
- "basic game", "belief-free game"

$$G = \left( (A_i, u_i)_{i=1}^I, \psi \right)$$

- signals (types)  $(T_i)_{i=1}^I$
- signal distribution

$$\pi : \Theta \rightarrow \Delta(T_1 \times T_2 \times \dots \times T_I)$$

- "higher order beliefs", "type space," "signal space"

$$\mathcal{T} = \left( (T_i)_{i=1}^I, \pi \right)$$

- common prior  $\psi(\theta)$  and conditional distribution

$$\pi[t_i, t_{-i}](\theta)$$

allow agent  $i$  to hold private information  $t_i$  in terms of

- posterior beliefs about the payoff relevant state  $\theta$
- posterior beliefs about the beliefs  $t_{-i}$  of the other agents

- a standard Bayesian game is described by  $(G, \mathcal{T})$
- a behavior strategy of player  $i$  is defined by:

$$\sigma_i : T_i \rightarrow \Delta(A_i)$$

## Definition (Bayes Nash Equilibrium (BNE))

A strategy profile  $\sigma$  is a Bayes Nash equilibrium of  $(G, \mathcal{T})$  if

$$\begin{aligned} & \sum_{t_{-i}, \theta} u_i((\sigma_i(t_i), \sigma_{-i}(t_i)), \theta) \pi[t_i, t_{-i}](\theta) \psi(\theta) \\ & \geq \sum_{t_{-i}, \theta} u_i((a_i, \sigma_{-i}(t_{-i})), \theta) \pi[t_i, t_{-i}](\theta) \psi(\theta). \end{aligned}$$

for each  $i$ ,  $t_i$  and  $a_i$ .

# Bayes Nash Equilibrium Distribution

- given a Bayesian game  $(G, \mathcal{T})$ , a BNE  $\sigma$  generates a joint probability distribution  $\mu_\sigma$  over outcomes and states  $A \times \Theta$ ,

$$\mu_\sigma(a, \theta) = \psi(\theta) \sum_t \pi[t](\theta) \left( \prod_{i=1}^I \sigma_i(a_i | t_i) \right)$$

- equilibrium distribution  $\mu_\sigma(a, \theta)$  is specified without reference to type space  $\mathcal{T}$  which gives rise to  $\mu_\sigma(a, \theta)$

## Definition (Bayes Nash Equilibrium Distribution)

A probability distribution  $\mu \in \Delta(A \times \Theta)$  is a Bayes Nash equilibrium distribution (over action and states) of  $(G, \mathcal{T})$  if there exists a BNE  $\sigma$  of  $(G, \mathcal{T})$  such that

$$\mu = \mu_\sigma.$$

- recall the original equilibrium conditions on  $(G, \mathcal{T})$ :

$$\begin{aligned} & \sum_{t_{-i}, \theta} u_i((\sigma_i(t_i), \sigma_{-i}(t_i)), \theta) \psi(\theta) \pi[t_i, t_{-i}](\theta) \\ & \geq \sum_{t_{-i}, \theta} u_i((a_i, \sigma_{-i}(t_{-i})), \theta) \psi(\theta) \pi[t_i, t_{-i}](\theta). \end{aligned}$$

- with the equilibrium distribution

$$\mu_\sigma(a, \theta) = \psi(\theta) \sum_t \pi[t](\theta) \left( \prod_{i=1}^I \sigma_i(a_i | t_i) \right)$$

- an implication of BNE of  $(G, \mathcal{T})$  : for all  $i, a_i \in \text{supp } \mu_\sigma(a, \theta)$  :

$$\sum_{a_{-i}, \theta} u_i((a_i, a_{-i}), \theta) \mu_\sigma(a, \theta) \geq \sum_{a_{-i}, \theta} u_i((a'_i, a_{-i}), \theta) \mu_\sigma(a, \theta);$$

# Bayes Correlated Equilibrium

- joint distribution over action-state  $\mu(a, \theta)$  describes choices, but is silent about reason (information) for choice

## Definition (Bayes Correlated Equilibrium (BCE))

An action state distribution  $\mu \in \Delta(A \times \Theta)$  is a Bayes Correlated Equilibrium (BCE) of  $G$  if is *obedient*, i.e., for each  $i$ ,  $a_i$  and  $a'_i$ ,

$$\sum_{a_{-i} \in A_{-i}, \theta \in \Theta} (u_i((a_i, a_{-i}), \theta) - u_i((a'_i, a_{-i}), \theta)) \mu((a_i, a_{-i}), \theta) \geq 0$$

and *consistent*, i.e., for each  $\theta$

$$\sum_{a \in A} \mu(a, \theta) = \psi(\theta).$$

- no restrictions on private information beyond  $\psi(\theta)$ ; zero information and complete information of  $\theta$  are possible

# Bayes Correlated Equilibrium

- BCE is defined in terms of the payoff environment and without reference to type space:

$$\mu(a, \theta)$$

- without uncertainty,  $\Theta = \{\theta\}$ , Bayes correlated equilibrium reduces to correlated equilibrium (Aumann (1974))
- earlier definitions of correlated equilibrium for games of incomplete information, see Forges (1993), (2006), define solution concepts for  $(G, \mathcal{T})$ , integrate out payoff relevant states  $\theta$ , and describe correlated equilibrium as action type distributions  $\Delta(A \times T)$
- we work with a basic game  $G$ , describe correlated equilibrium as action state distributions  $\Delta(A \times \Theta)$
- in companion paper, “Correlated Equilibrium in Games with Incomplete Information” we relate definitions and establish comparative results wrt information environments

- now given  $(G)$ , what is the set of equilibrium distributions  $\mu$  across all possible information structures  $\mathcal{T}$

## Theorem (Equivalence )

*A probability distribution  $\mu \in \Delta(A \times \Theta)$  is a Bayes correlated equilibrium of  $G$  if and only if it is a Bayes Nash Equilibrium distribution of  $(G, \mathcal{T})$  for some information system  $\mathcal{T}$ .*

- $BCE \Rightarrow BNE$  uses the richness of the possible information structure to complete the equivalence result:

$$\forall i, \forall a_i : t_i = \mu(\theta, a_{-i} | a_i), \sigma_i(t_i) = a_i$$

- Aumann (1987) established the above characterization result for complete information games

# Payoff Environment: Quadratic Payoffs

- utility of each agent  $i$  is given by quadratic payoff function:
- determined by individual action  $a_i \in \mathbb{R}$ , state of the world  $\theta \in \mathbb{R}$ , and average action  $A \in \mathbb{R}$ :

$$A = \int_0^1 a_i di$$

and thus  $u_i(a_i, A, \theta) =$

$$\begin{pmatrix} \lambda_a \\ \lambda_A \\ \lambda_\theta \end{pmatrix}' \begin{pmatrix} a_i \\ A \\ \theta \end{pmatrix} + \begin{pmatrix} a_i \\ A \\ \theta \end{pmatrix}' \begin{pmatrix} \gamma_a & \gamma_{aA} & \gamma_{a\theta} \\ \gamma_{aA} & \gamma_A & \gamma_{A\theta} \\ \gamma_{a\theta} & \gamma_{A\theta} & \gamma_\theta \end{pmatrix} \begin{pmatrix} a_i \\ A \\ \theta \end{pmatrix}$$

- game is completely described by linear returns  $\lambda$  and interaction matrix  $\Gamma = \{\gamma_{ij}\}$

# Payoff Environment: Normal Payoffs

- the state of the world  $\theta$  is normally distributed

$$\theta \sim N(\mu_\theta, \sigma_\theta^2)$$

with mean  $\mu_\theta \in \mathbb{R}$  and variance  $\sigma_\theta^2 \in \mathbb{R}_+$

- the distribution of the state of the world,  $\theta$ , is commonly known common prior

- the matrix  $\Gamma$  defines the nature of the interaction:

$$\Gamma = \begin{pmatrix} \gamma_a & \gamma_{aA} & \gamma_{a\theta} \\ \gamma_{aA} & \gamma_A & \gamma_{A\theta} \\ \gamma_{a\theta} & \gamma_{A\theta} & \gamma_\theta \end{pmatrix}$$

- diagonal entries:  $\gamma_a, \gamma_A, \gamma_\theta$  describe “own effects”
- off-diagonal entries:  $\gamma_{a\theta}, \gamma_{A\theta}, \gamma_{aA}$  “interaction effects”
- best response of agent  $i$  depends on  $\gamma_a, \gamma_{aA}, \gamma_{a\theta}$ :
  - cost of adjustment:  $\gamma_a$
  - informational externality:  $\gamma_{a\theta}$
  - strategic externality:  $\gamma_{aA}$ ; strategic complements and strategic substitutes,  $\gamma_{aA} > 0$  vs.  $\gamma_{aA} < 0$

# Complete Information Game

- suppose  $\theta$  is commonly known
- best response of agent  $i$ :

$$a = - \frac{\lambda_a + \gamma_{a\theta} \cdot \theta + \gamma_{aA} \cdot A}{\gamma_a}$$

- equilibrium response of agent  $i$ :

$$a^*(\theta) = - \frac{\overbrace{\lambda_a}^{\text{intercept}}}{\gamma_a + \gamma_{aA}} - \frac{\overbrace{\gamma_{a\theta}}^{\text{slope}}}{\gamma_a + \gamma_{aA}} \cdot \theta$$

- maintained assumptions:
  - concavity at individual level (well-defined best response):  
 $\gamma_a < 0$
  - concavity at aggregate level (existence of interior equilibrium):  
 $\gamma_a + \gamma_{aA} < 0$
- concave payoffs imply that complete information game has unique Nash **and** correlated equilibrium (Neyman (1997))

## Example 1: Competitive Market

- action (= quantity):  $a_i \in \mathbb{R}$
- cost of production  $c(a_i) = \frac{1}{2}\gamma_a (a_i)^2$
- state of the world (= demand intercept):  $\theta \in \mathbb{R}$
- inverse demand (= price):

$$p(A) = \gamma_{a\theta}\theta - \gamma_{aA}A$$

where  $A$  is average supply:

$$A = \int_0^1 a_i di$$

- see Guesnerie (1992) and Vives (2008)

## Example 2: Beauty Contest, Macroeconomic Coordination Games

- continuum of agents:  $i \in [0, 1]$
- action (= message):  $a \in \mathbb{R}$
- state of the world:  $\theta \in \mathbb{R}$
- payoff function

$$u_i = -(1-r)(a_i - \theta)^2 - r(a_i - A)^2$$

with  $r \in (0, 1)$

- see Morris and Shin (2002), Angeletos and Pavan (2007)

- fix an arbitrary information system:
- ① every agent  $i$  observes a public signal  $y$  about  $\theta$  :

$$y \sim N(\theta, \sigma_y^2)$$

- ② every agent  $i$  observes a private signal  $x_i$  about  $\theta$  :

$$x_i \sim N(\theta, \sigma_x^2)$$

- binary information structure with a private and a public component
- every pair  $(\sigma_x^2, \sigma_y^2)$  generates a different information system, type space

- the best response of each agent is:

$$a = -\frac{1}{\gamma_a} (\gamma_{a\theta} \mathbb{E}[\theta | x, y] + \gamma_{Aa} \mathbb{E}[A | x, y])$$

- suppose the equilibrium strategy is given by a linear function:

$$a(x, y) = \alpha_0 + \alpha_x x + \alpha_y y,$$

- denote the sum of the precisions:

$$\sigma^{-2} = \sigma_{\theta}^{-2} + \sigma_x^{-2} + \sigma_y^{-2}$$

## Theorem

The unique Bayesian Nash equilibrium (given the bivariate information structure) is a linear equilibrium:

$$s^*(x, y) = \alpha_0^* + \alpha_x^* x + \alpha_y^* y$$

with

$$\alpha_x^* = -\frac{\gamma_{a\theta}\sigma_x^{-2}}{\gamma_{Aa}\sigma_x^{-2} + \gamma_a\sigma^{-2}},$$

and

$$\alpha_y^* = -\frac{\gamma_a}{\gamma_a + \gamma_{aA}} \frac{\gamma_{a\theta}\sigma_y^{-2}}{\gamma_{Aa}\sigma_x^{-2} + \gamma_a\sigma^{-2}}.$$

- the linear coefficients  $\alpha_x^*$  and  $\alpha_y^*$  satisfy the relationship:

$$\frac{\alpha_y^*}{\alpha_x^*} = \frac{\sigma_y^{-2}}{\sigma_x^{-2}} \frac{\gamma_a}{\gamma_a + \gamma_{aA}}, \quad \text{w/o strategic interaction } \gamma_{aA} = 0 : \frac{\alpha_y^*}{\alpha_x^*} = \frac{\sigma_y^{-2}}{\sigma_x^{-2}}$$

# Equilibrium Response and Information

- linear coefficients  $\alpha_x^*$  and  $\alpha_y^*$  respond to signals  $x_i$  and  $y$ :

$$x_i = \theta + \varepsilon_i \text{ and } y = \theta + \varepsilon,$$

and hence to state of the world  $\theta$

- sum of linear coefficients  $\alpha_x^*$  and  $\alpha_y^*$  display “attenuation”:

$$\alpha_x^* + \alpha_y^* = \frac{\gamma_a \theta}{\gamma_a + \gamma_{aA}} \cdot \left( 1 - \frac{\gamma_a \sigma_\theta^{-2}}{\gamma_a \sigma^{-2} + \gamma_{aA} \sigma_x^{-2}} \right) < -\frac{\gamma_a \theta}{\gamma_a + \gamma_{aA}},$$

yet mean action:

$$\alpha_0^* + \alpha_x^* \mu_\theta + \alpha_y^* \mu_\theta = -\mu_\theta \frac{\gamma_a \theta}{\gamma_a + \gamma_{aA}},$$

for all  $\sigma_x^2, \sigma_y^2$

- first equilibrium moment is constant across information structures
- second equilibrium moments vary across information structures.

# Joint Action State Distribution

- given normally distributed random variables  $(\sigma_\theta^2, \sigma_x^2, \sigma_y^2)$  and linear strategy, the joint distribution of action and state  $\mu(a_i, a_j, \theta)$  is multivariate normal as well:

$$\Sigma_{a_i, a_j, \theta} = \begin{pmatrix} \sigma_a^2 & \rho_a \sigma_a^2 & \rho_{a\theta} \sigma_a \sigma_\theta \\ \rho_a \sigma_a^2 & \sigma_a^2 & \rho_{a\theta} \sigma_a \sigma_\theta \\ \rho_{a\theta} \sigma_a \sigma_\theta & \rho_{a\theta} \sigma_a \sigma_\theta & \sigma_\theta^2 \end{pmatrix}$$

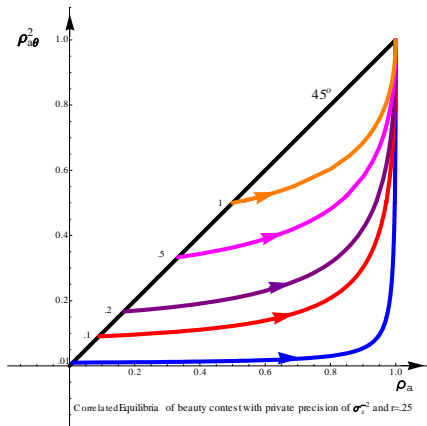
- and in terms of equilibrium coefficients:

$$\begin{pmatrix} \sigma_a^2 \\ \rho_a \sigma_a^2 \\ \rho_{a\theta} \sigma_a \sigma_\theta \end{pmatrix} = \begin{pmatrix} \alpha_x^2 \sigma_x^2 + \alpha_y^2 \sigma_y^2 + \sigma_\theta^2 (\alpha_x + \alpha_y)^2 \\ \alpha_y^2 \sigma_y^2 + \sigma_\theta^2 (\alpha_x + \alpha_y)^2 \\ \sigma_\theta^2 (\alpha_x + \alpha_y) \end{pmatrix}$$

- represent equilibrium in terms of correlation coefficients  $\rho_a, \rho_{a\theta}$

# Equilibrium and Private Information

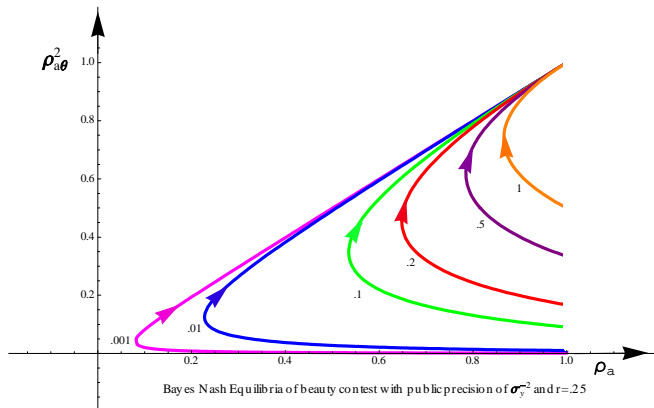
- equilibrium covariances via correlation coefficients  $\rho_a, \rho_{a\theta}$
- consider given precision  $\sigma_x^{-2}$  of private information



- increase in precision of public information  $\sigma_y^{-2}$  is upward movement along level curve

# Equilibrium and Public Information

- equilibrium covariances via correlation coefficients  $\rho_a, \rho_{a\theta}$
- consider given precision  $\sigma_y^{-2}$  of public information



- increase in precision of private information  $\sigma_x^{-2}$  is upward movement along level curve

# Multitude of Information Environments

- every type  $t_i$  of agent  $i$  could contain many pieces of information

$$t_i = (s, s_i, s_{ij}, s_{ijk}, \dots)$$

every agent  $i$  may observe a public (common) signal  $s$  centered around the state of the world  $\theta$ :

$$s \sim N(\theta, \sigma_s^2)$$

- every agent  $i$  may observe a private signal  $s_i$  centered around the state of the world  $\theta$ :

$$s_i \sim N(\theta, \sigma_i^2)$$

- every agent  $i$  may observe a private signal  $s_{i,j}$  about the signal of agent  $j$ :

$$s_{i,j} \sim N(s_j, \sigma_{i,j}^2)$$

- every agent  $i$  may observe a private signal  $s_{i,j,k}$  about ....:

$$s_{i,j,k} \sim N(s_{j,k}, \sigma_{i,j,k}^2)$$

- object of analysis: joint distribution over actions and states

$$\mu(a, \theta)$$

- for exposition, focus on symmetric Bayes correlated equilibria:

$$\mu(a_i, A, \theta)$$

which are normally distributed:

$$\begin{pmatrix} a_i \\ A \\ \theta \end{pmatrix} \sim N \left( \begin{pmatrix} \mu_a \\ \mu_A \\ \mu_\theta \end{pmatrix}, \begin{pmatrix} \sigma_a^2 & \rho_{aA}\sigma_a\sigma_A & \rho_{a\theta}\sigma_a\sigma_\theta \\ \rho_{aA}\sigma_a\sigma_A & \sigma_A^2 & \rho_{A\theta}\sigma_A\sigma_\theta \\ \rho_{a\theta}\sigma_a\sigma_\theta & \rho_{A\theta}\sigma_A\sigma_\theta & \sigma_\theta^2 \end{pmatrix} \right)$$

- $\sigma_A^2$  is the aggregate volatility (common variation)
- $\sigma_a^2 - \sigma_A^2$  is the cross-section dispersion (idiosyncratic variation)
- statistical representation of equilibrium in terms of first and second order moments

# Symmetric Bayes Correlated Equilibria

- with focus on symmetric equilibria:

$$\mu_A = \mu_a, \quad \sigma_A^2 = \rho_a \sigma_a^2, \quad \rho_{aA} \sigma_a \sigma_A = \rho_a \sigma_a^2, \quad \rho_{a\theta} \sigma_a \sigma_\theta = \rho_{A\theta} \sigma_A \sigma_\theta$$

- the first and second moments of the correlated equilibria are:

$$\begin{pmatrix} a_i \\ A \\ \theta \end{pmatrix} \sim N \left( \begin{pmatrix} \mu_a \\ \mu_a \\ \mu_\theta \end{pmatrix}, \begin{pmatrix} \sigma_a^2 & \rho_a \sigma_a^2 & \rho_{a\theta} \sigma_a \sigma_\theta \\ \rho_a \sigma_a^2 & \rho_a \sigma_a^2 & \rho_{a\theta} \sigma_a \sigma_\theta \\ \rho_{a\theta} \sigma_a \sigma_\theta & \rho_{a\theta} \sigma_a \sigma_\theta & \sigma_\theta^2 \end{pmatrix} \right)$$

- correlated equilibria characterized by first and second moments:

$$\{\mu_a, \sigma_a, \rho_a, \rho_{a\theta}\}$$

- in the complete information game, the best response is:

$$a = -\theta \frac{\gamma_{a\theta}}{\gamma_a} - A \frac{\gamma_{Aa}}{\gamma_a}$$

- best response is weighted linear combination of fundamental  $\theta$  and average action  $A$  relative to the cost of action:

$$\gamma_{a\theta}/\gamma_a, \gamma_{Aa}/\gamma_a$$

- in the incomplete information game,  $\theta$  and  $A$  are uncertain:

$$\mathbb{E}[\theta], \quad \mathbb{E}[A]$$

- given the correlated equilibrium distribution  $\mu(a, \theta)$  we can use the conditional expectations:

$$\mathbb{E}_\mu[\theta | a], \quad \mathbb{E}_\mu[A | a]$$

- in the incomplete information game, the best response is:

$$a = -\mathbb{E}_\mu [\theta | a] \frac{\gamma_{a\theta}}{\gamma_a} - \mathbb{E}_\mu [A | a] \frac{\gamma_{Aa}}{\gamma_a}$$

- best response property has to hold for all  $a \in \text{supp } \mu(a, \theta)$
- a fortiori, the best response property has to hold in expectations over all  $a$  :

$$\mathbb{E}_\mu [a] = \mathbb{E}_\mu \left[ - \left( \mathbb{E}_\mu [\theta | a] \frac{\gamma_{a\theta}}{\gamma_a} + \mathbb{E}_\mu [A | a] \frac{\gamma_{Aa}}{\gamma_a} \right) \right]$$

- by the law of iterated expectation, or “law of total expectation”:

$$\mathbb{E}_\mu [\mathbb{E}_\mu [\theta | a]] = \mu_\theta, \quad \mathbb{E}_\mu [\mathbb{E}_\mu [A | a]] = \mathbb{E}_\mu [A] = \mathbb{E}_\mu [a],$$

- the best response property implies that for all  $\mu(a, \theta)$  :

$$\mathbb{E}_\mu [a] = \mathbb{E}_\mu \left[ - \left( \mathbb{E}_\mu [\theta | a] \frac{\gamma_{a\theta}}{\gamma_a} + \mathbb{E}_\mu [A | a] \frac{\gamma_{Aa}}{\gamma_a} \right) \right]$$

or by the law of iterated expectation:

$$\mu_a = -\mu_\theta \frac{\gamma_{a\theta}}{\gamma_a} - \mu_a \frac{\gamma_{Aa}}{\gamma_a}$$

## Theorem (First Moment)

*In all Bayes correlated equilibria, the mean action is given by:*

$$\mathbb{E}[a] = -\mu_\theta \frac{\gamma_{a\theta}}{\gamma_a + \gamma_{aA}}.$$

- result about “mean action” is independent of symmetry or normal distribution

## Equilibrium Moments: Variance

- in any correlated equilibrium  $\mu(a, \theta)$ , best response demands

$$a = - \left( \mathbb{E}[\theta | a] \frac{\gamma_{a\theta}}{\gamma_a} + \mathbb{E}[A | a] \frac{\gamma_{Aa}}{\gamma_a} \right), \quad \forall a \in \text{supp } \mu(a, \theta)$$

- or varying in  $a$

$$1 = - \left( \frac{\partial \mathbb{E}[\theta | a]}{\partial a} \frac{\gamma_{a\theta}}{\gamma_a} + \frac{\partial \mathbb{E}[A | a]}{\partial a} \frac{\gamma_{Aa}}{\gamma_a} \right),$$

- the change in the conditional expectation

$$\frac{\partial \mathbb{E}[\theta | a]}{\partial a}, \quad \frac{\partial \mathbb{E}[A | a]}{\partial a}$$

is a statement about the correlation between  $a, A, \theta$

- “law of total variance”

# Equilibrium Moment Restrictions

- the best response condition **and** the condition that  $\Sigma_{a,A,\theta}$  forms a multivariate distribution, meaning that the variance-covariance matrix has to be positive definite
- we need to determine:

$$\{\sigma_a, \rho_a, \rho_{a\theta}\}$$

## Theorem (Second Moment)

*The triple  $(\sigma_a, \rho_a, \rho_{a\theta})$  forms a Bayes correlated equilibrium iff:*

$$\rho_a - \rho_{a\theta}^2 \geq 0,$$

*and*

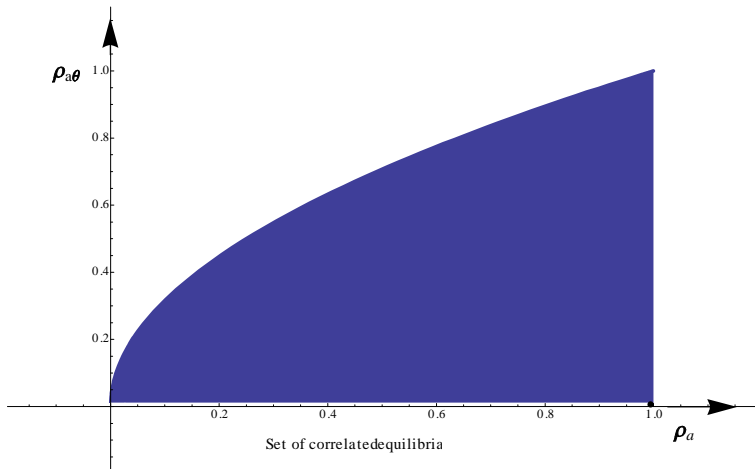
$$\sigma_a = -\frac{\sigma_\theta \gamma_{a\theta} \rho_{a\theta}}{\rho_a \gamma_{Aa} + \gamma_a}.$$

- correlation of actions across agents:  $\rho_a$
- correlation of actions and fundamentals:  $\rho_{a\theta}$

# Moment Restrictions: Correlation Coefficients

- the equilibrium set is completely characterized by inequality:

$$\rho_a - \rho_{a\theta}^2 \geq 0$$



# Equivalence between BCE and BNE

- bivariate information structure which generates volatility (common signal) and dispersion (idiosyncratic signal)

## Theorem

*There is BCE with  $(\rho_a, \rho_{a\theta})$  if and only if there is a BNE with  $(\sigma_x^2, \sigma_y^2)$ .*

- a public and a private signal are sufficient to generate the entire set of correlated equilibria...
- but a given BCE does not uniquely identify the information environment of a BNE

- the analyst may not know how much private information the agents have, yet he may have a lower bound on how much information the agents have
- how does the set of BCE change in the lower bound
- assume that all agents observe a public signal  $y$  :

$$y = \theta + \varepsilon$$

and a private signal  $x_i$

$$x_i = \theta + \varepsilon_i$$

with

$$\begin{pmatrix} \varepsilon \\ \varepsilon_i \end{pmatrix} \sim N \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_y^2 & 0 \\ 0 & \sigma_x^2 \end{pmatrix} \right)$$

# Information Bounds and Correlated Equilibrium

- the equilibrium conditions are now augmented from for all “ $a$ ” to “for all  $a, x, y$ ” as additional “incentive constraints”

$$a = -\frac{1}{\gamma_a} (\gamma_{a\theta} \mathbb{E}[\theta | a, x, y] + \gamma_{Aa} \mathbb{E}[A | a, x, y]), \quad \forall a, x, y.$$

- we determine  $\sigma_a, \rho_{ax}, \rho_{ay}$  in terms of  $\rho_a, \rho_{a\theta}$ , e.g.:

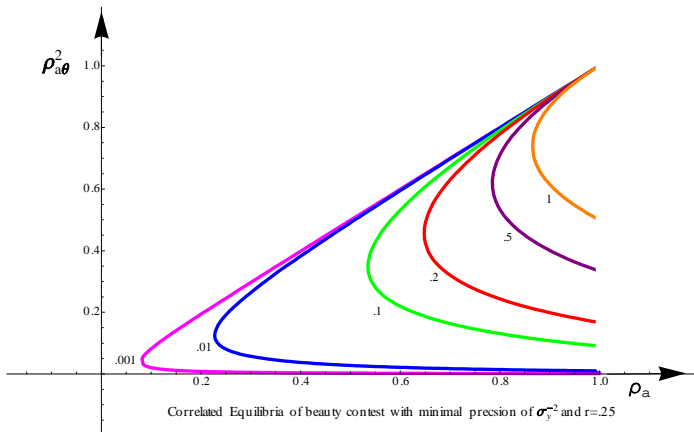
$$\rho_{ay} = \frac{\sigma_\theta}{\sigma_y \rho_{a\theta}} \left( \frac{\gamma_a + \rho_a \gamma_{Aa}}{\gamma_a + \gamma_{Aa}} - \rho_{a\theta}^2 \right)$$

- set of correlated equilibria is given by the inequalities:

$$\begin{aligned} \rho_a - \rho_{a\theta}^2 - \rho_{ay}^2 &\geq 0, \\ 1 - \rho_a - \rho_{ax} &\geq 0, \end{aligned}$$

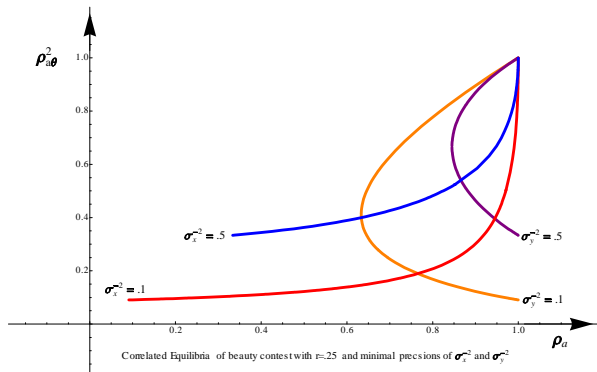
# Lower Bound on Public Information

- movements along level curve are variations in  $\sigma_x^{-2}$  given  $\sigma_y^{-2}$
- the interior of each level curve describes the correlated equilibria for a given lower bound on public information



# Lower Bounds on Private and Public Information

- interior of intersection of level curves is the set of Bayes correlated equilibrium subject to lower bounds on private and public information



- more information reduces set of possible distributions, as it adds incentive constraints but does not remove correlation possibilities

## 1 Predictions:

What restrictions are imposed by the structural model  $(u, \psi)$  on the observable endogenous statistics  $(\mu_a, \sigma_a, \rho_a, \rho_{a\theta})$ ?

## 2 Identification:

What restrictions can be imposed/inferred on the structural model  $(u)$  by the observations of the outcome variables  $(\mu_a, \mu_\theta, \sigma_a, \sigma_\theta, \rho_a, \rho_{a\theta})$ ?

- can we identify sign and size of interaction?
- can we identify the nature of the informational externality  $\gamma_{a\theta}$  and the strategic externality  $\gamma_{aA}$

# Identification and Information: Single Agent

- zero interaction:  $\gamma_{aA} = 0$ ,
- normalize cost of adjustment:  $\gamma_a = -1$
- complete information: the state  $\theta$  is observable to the agent
- the best response of the agent is given by:

$$a(\theta) = \gamma_{a\theta} \cdot \theta + u,$$

where  $u$  is an error, observable to the agent, but unobservable to the econometrician

- unobservable error  $u$ ,  $\mathbb{E}[u] = 0$ ,  $u \perp \theta$
- the econometrician observes  $a$  and  $\theta$  but not  $u$
- the covariance between  $a$  and  $\theta$  point identifies  $\gamma_{a\theta}$

# Identification and Noisy Information

- noisy information: the state  $\theta$  is unobservable, the agent only observes a noisy signal

$$s = \theta + \varepsilon$$

with  $\varepsilon \sim N(0, \sigma_\varepsilon^2)$

- the agent chooses  $a$  upon observing  $s$  and  $u$ ,

$$a(s) = \gamma_{a\theta} \cdot \frac{s\sigma_\varepsilon^{-2} + \mu_\theta\sigma_\theta^{-2}}{\sigma_\varepsilon^{-2} + \sigma_\theta^{-2}} + u$$

- the econometrician observes  $a$  and  $\theta$  but neither  $s$  nor  $u$
- the best response of the agent has the slope:

$$\gamma_{a\theta} \cdot \frac{\sigma_\varepsilon^{-2}}{\sigma_\varepsilon^{-2} + \sigma_\theta^{-2}} < \gamma_{a\theta},$$

- the best response is attenuated by the signal to noise ratio

- the covariance between  $a$  and  $\theta$  point identifies:

$$\hat{\gamma}_{a\theta} = \gamma_{a\theta} \cdot \frac{\sigma_{\varepsilon}^{-2}}{\sigma_{\varepsilon}^{-2} + \sigma_{\theta}^{-2}},$$

- how to identify  $\gamma_{a\theta}$  without knowing the information of the agent, i.e. without knowing the variance  $\sigma_{\varepsilon}^2$  of  $\varepsilon$  :
- without knowing  $\sigma_{\varepsilon}^2$ , there is only set identification:

$$\gamma_{a\theta} = \hat{\gamma}_{a\theta} \cdot \frac{\sigma_{\varepsilon}^{-2} + \sigma_{\theta}^{-2}}{\sigma_{\varepsilon}^{-2}} \Rightarrow \gamma_{a\theta} \in [\hat{\gamma}_{a\theta}, \infty)$$

- noise in the predictor variable induces a bias: “attenuation bias”, “regression dilution”
- in the presence of interaction, “attenuation bias” impacts identification of other structural parameters, here the strategic externality  $\gamma_{a\theta}$

# Identification and Information: Many Agents

- non-zero interaction:  $\gamma_{aA} \neq 0$
- information structure  $(\sigma_x^2, \sigma_y^2)$  of Bayesian game is assumed to be known
- the identification, given the hypothesis of BNE, uses variance-covariance matrix of actions and states:

$$\Sigma_{A,\theta} = \begin{bmatrix} \alpha_y^2 \sigma_y^2 + \sigma_\theta^2 (\alpha_x + \alpha_y)^2 & \sigma_\theta^2 (\alpha_x + \alpha_y) \\ \sigma_\theta^2 (\alpha_x + \alpha_y) & \sigma_\theta^2 \end{bmatrix}$$

- the relationship between equilibrium coefficients

$$\frac{\alpha_y^*}{\alpha_x^*} = \frac{\sigma_x^2}{\sigma_y^2} \frac{\gamma_a}{\gamma_a + \gamma_{aA}}$$

lends information about the sign of  $\gamma_{aA}$

# Identification with Incomplete Information:

- information structure  $(\sigma_x^2, \sigma_y^2)$  of Bayesian game is assumed to be known

## Proposition (**Sign and Point Identification**)

If  $0 < \sigma_x^2, \sigma_y^2 < \infty$ , then:

- ① BNE identifies the sign of  $\gamma_{a\theta}$  **and** of  $\gamma_{aA}$ .
- ② BNE point identifies the slope of the equilibrium response:

$$\frac{\gamma_{a\theta}}{\gamma_a + \gamma_{aA}}.$$

# Robust Identification with Incomplete Information

- information structure of Bayesian game is assumed to be unknown
- we can use observed mean, variance, and covariance of  $(a, \theta)$  and equilibrium conditions:

$$\mu_a = \frac{\lambda_a + \gamma_{a\theta}\mu_\theta}{\gamma_a + \gamma_{aA}}$$

and

$$\sigma_a = -\frac{\gamma_{a\theta}\rho_{a\theta}\sigma_\theta}{\rho_a\gamma_{aA} + \gamma_a}$$

## Proposition (Partial Identification)

- ① *The sign of  $\gamma_{a\theta}$  is, but the sign of  $\gamma_{aA}$  is **not** identified.*
- ② *If  $\rho_a < 1$ , the slope of the equilibrium response is partially identified:*

$$-\frac{\gamma_{a\theta}}{\gamma_a + \gamma_{aA}} \in \left( \frac{\rho_a \sigma_a}{\rho_{a\theta} \sigma_\theta}, \infty \right).$$

- failure to identify the strategic nature of the game, the sign of  $\gamma_{aA}$  describes whether the game is one of strategic complements or strategic substitutes

# Prior Information and Identification

- earlier, we analyzed how information bounds in terms of private and public information, represented by  $\sigma_x^2$  and  $\sigma_y^2$ , sharpen the prediction
- similarly, the information bounds sharpen the identification of the interaction ratios

## Proposition ( **Prior Information and Identification** )

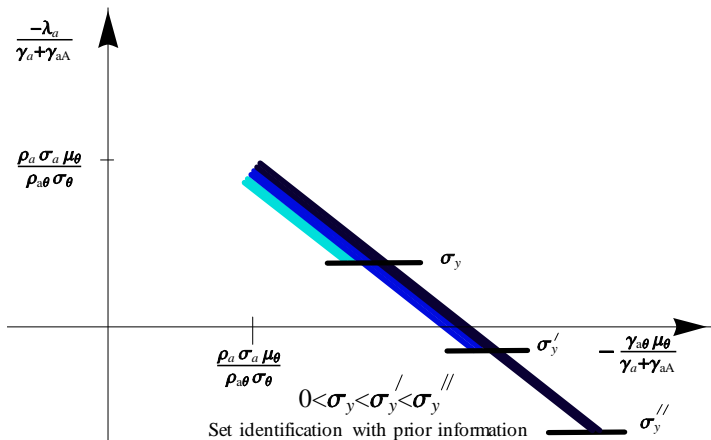
*With prior information, the interaction ratios are sharper identified with:*

$$-\frac{\gamma_{a\theta}}{\gamma_a + \gamma_{aA}} \in (r(\sigma_x, \sigma_y), R(\sigma_x, \sigma_y)) \subset \left( \frac{\rho_a \sigma_a}{\rho_{a\theta} \sigma_\theta}, \infty \right),$$

and

$$\frac{\partial r(\sigma_x, \sigma_y)}{\partial \sigma_x} < 0, \quad \frac{\partial R(\sigma_x, \sigma_y)}{\partial \sigma_y} > 0.$$

# Prior Information and Set Identification



- the private information set provides the lower bound on  $\gamma_{aA}$
- the public information set provides the upper bound on  $\gamma_{aA}$
- as  $\sigma_x^2, \sigma_y^2 \rightarrow 0$ , sign of strategic interaction is identified

# Demand and Supply Identification

- linear inverse demand is given by:

$$P_d = \lambda_d + \gamma_{aA}Q + \gamma_{a\theta_d}\theta_d$$

- linear inverse supply is given by:

$$P_s = \lambda_s + \gamma_a Q + \gamma_{a\theta_s}\theta_s$$

- two-dimensional uncertainty:  $\theta_d$  and  $\theta_s$  are demand and supply shocks ("demand, supply shifters")
- firm  $i$  makes supply decision with noisy information about  $(\theta_d, \theta_s)$
- only aggregate data is observed: aggregate quantity and price
- in Bayes Nash equilibrium  $(\gamma_a, \gamma_{aA}, \gamma_{a\theta_d}, \gamma_{a\theta_s})$  are point identified
- in Bayes Correlated equilibrium  $\gamma_a$  and  $\gamma_{a\theta_s}$  are only **set identified**

- Bayes correlated equilibrium encodes concern for robustness to strategic information environment
- next items on the agenda: robust comparative statics, robust policy analysis
- what can say about the impact of changes in parameters of the payoff relevant environment (for example, policy choices) for the set of possible outcomes?
- robust information policy
- robust taxation policy