91.4.3. *Testing for Stationarity in the Components Representation of a Time Series*—Solution, proposed by D. Kwiatkowski, P.C.B. Phillips, and P. Schmidt. (a) Note that $r_i = \Sigma_i u_i$ and set $w_i = r_i + u_i$ so that the model can be written as

$$y_i = x_i' \gamma + w_i; \quad \gamma' = (\gamma_0, \gamma_1), \quad x_i' = (1, t)$$
or in observation format as

\[ y = X\gamma + w. \]

Now \( E(w) = 0 \) and

\[
\text{var}(w) = \text{var}(u) + \text{var}(r) = \sigma_u^2 I_n + \sigma_r^2 LL' = \sigma_u^2 I_n + \sigma_r^2 A
\]

where

\[
L = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 0 \\
1 & 1 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 1 & 1 & \cdots & 1
\end{pmatrix}
\quad \text{and} \quad
A = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & 2 & 2 & \cdots & 2 \\
1 & 2 & 3 & \cdots & 3 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 2 & 3 & \cdots & n
\end{pmatrix}
\]

\[ = \Omega(\sigma_u^2, \sigma_r^2), \text{ say}. \]

The log likelihood is then

\[
L(\gamma, \sigma_u^2, \sigma_r^2; y) = -\frac{n}{2} \ln 2\pi - \frac{1}{2} \ln |\Omega| - \frac{1}{2} (y - X\gamma)'\Omega^{-1}(y - X\gamma).
\]

(b) \( \partial L/\partial \sigma_r^2 = -\frac{1}{2} \text{tr}(\Omega^{-1}A) + \frac{1}{2} (y - X\gamma)'\Omega^{-1}A\Omega^{-1}(y - X\gamma) \) and then

\[
\tilde{\lambda} = \partial L(\tilde{\gamma}, \tilde{\sigma}_u^2, \tilde{\sigma}_r^2 = 0)/\partial \sigma_r^2 = (1/2\tilde{\sigma}_r^2) \text{tr}(A) + (1/2\tilde{\sigma}_u^2) (y - X\tilde{\gamma})'A(y - X\tilde{\gamma})
\]

where \( \tilde{\gamma}, \tilde{\sigma}_u^2 \) are the restricted ML estimates, that is, the OLS estimates of \( y = X\gamma + w \). Write \( \tilde{u} = y - X\tilde{\gamma} \) and then we have

\[
\tilde{\lambda} = -\frac{1}{2\tilde{\sigma}_r^2} \text{tr}(A) + \frac{1}{2\tilde{\sigma}_u^2} \tilde{u}'A\tilde{u}.
\]

The LM test of

\[ H_0: \sigma_r^2 = 0 \]

is based on \( \tilde{\lambda} \). We can construct a “studentized test” based on \( \tilde{\lambda} \) and an estimate of its standard error. Note that

\[
\text{var}
\left(
\frac{1}{2\tilde{\sigma}_u^2} u' Au
\right)
= \left(\frac{1}{4\tilde{\sigma}_u^4}\right) 2\tilde{\sigma}_u^2 \text{tr}(A^2), \quad \text{under normality.} \quad (1)
\]

We set

\[
LM_1 = \frac{\tilde{\lambda}}{[(1/2\tilde{\sigma}_r^4)\text{tr}(A^2)]^{1/2}} = \frac{\tilde{u}'A\tilde{u}}{2^{1/2}\tilde{\sigma}_u^2(\text{tr}(A^2))^{1/2}} - \frac{\text{tr}(A)}{2^{1/2}(\text{tr}(A^2))^{1/2}}. \quad (2)
\]
Equivalently, we may work with

\[ LM_2 = \frac{\tilde{u}'A\tilde{u}}{\tilde{u}'\tilde{u}} \]

(removing the fixed term and scale coefficient of (2)).

Next note that

\[ \tilde{u}'A\tilde{u} = \tilde{u}'LL'\tilde{u} = \sum_{i=1}^{n} S_{i-1}^2 \]

where \( S_i = \sum_{j=1}^{n} \tilde{u}_j \). Hence we have the representation

\[ LM_2 = \frac{\sum_{i=1}^{n} S_{i-1}^2}{\tilde{u}'\tilde{u}}. \]  \hspace{1cm} (3)

(c) Under the null

\[ n^{-1/2} S_{[\nu]} = n^{-1/2} \sum_{i=1}^{[\nu]} u_i = B(r) = B M (a_u^2) \]

whereas

\[ n^{-1/2} \tilde{S}_{[\nu]} = n^{-1/2} \sum_{i=1}^{[\nu]} \tilde{u}_i = n^{-1/2} \sum_{i=1}^{[\nu]} u_i - \left( n^{-1/2} \sum_{i=1}^{[\nu]} x_i' \right) (X'X)^{-1} X'u. \]

Now \( x_i' = (1, t) \) and setting

\[ D_n = \begin{pmatrix} n^{1/2} & 0 \\ 0 & n^{3/2} \end{pmatrix} \]

we have

\[ D_n^{-1}X'XD_n^{-1} = \left[ \begin{array}{c} 1 \\ \sum_{i=1}^{n} t/n^2 \end{array} \right] \rightarrow \left[ \begin{array}{c} 1 \\ \frac{1}{2} \end{array} \right]. \]

\[ D_n^{-1}X'u = \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} u_i \right) \rightarrow \left( \int_{0}^{1} dB \right) \]

\[ \left( \frac{1}{n^{3/2}} \sum_{i=1}^{n} t u_i \right) \rightarrow \left( \int_{0}^{1} r dB \right). \]
Hence,
\[ n^{-1/2} \tilde{S}_{(nr)} = n^{-1/2} S_{(nr)} - \left( \frac{[nr]}{n}, \frac{1}{n^2} \right) \left( D_n^{-1} X' X D_n^{-1} \right)^{-1} D_n^{-1} X' u \]
\[ = B(r) - \left[ r, \frac{r^2}{2} \right] \left[ \begin{array}{cc} 1 & 1 \\ \frac{1}{2} & \frac{1}{3} \end{array} \right]^{-1} \left( \int_0^r dB \right) \]
\[ = \tilde{B}_2(r), \text{ say.} \]  \hspace{1cm} (4)

We also obtain
\[ \frac{1}{n^2} \sum_{i=1}^n \tilde{S}_{i-1}^2 = \frac{1}{n} \sum_{i=1}^n \left( \frac{S_{i-1}^2}{\sqrt{n}} \right) = \int_0^1 \tilde{B}_2(r)^2 \, dr. \]

Hence
\[ n^{-2} LM_2 = n^{-2} \sum_{i=1}^n \tilde{S}_{i-1}^2 / \tilde{c}_u^2 \]
\[ = \int_0^1 \tilde{B}_2(r)^2 \, dr / \tilde{c}_u^2 \]
\[ = \int_0^1 \tilde{W}_2(r)^2 \, dr \]

since \( \tilde{B}_2(r) = \tilde{c}_u \tilde{W}_2(r) \). Note that \( \tilde{W}_2(r) \), which is defined in the same way as (4), is free of nuisance parameters.

Remark. Observe that in the \( LM_1 \) form we have \( (2 \text{ tr} A^2)^{1/2} \) in the denominator. Now
\[ \text{tr}(A^2) = \text{tr} \left( \begin{array}{cccc}
1 & 1 & 1 & \cdots & 1 \\
1 & 2 & 2 & \cdots & 2 \\
1 & 2 & 3 & \cdots & 3 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 2 & 3 & \cdots & n
\end{array} \right)^2 \]
\[ = n \sum_{i=1}^n i^2 - \left( \sum_{i=1}^n i \right)^2 \]
\[ = n \left( \frac{n+1}{2} \right)^2 - n^2 \left( \frac{n+1}{2} \right)^2 \]
\[ = n \sum_{i=1}^n i^2 + (n-1) \sum_{i=1}^n i^2 + (n-2)^2 + \cdots + (n-1)^2 \]
\[ + (n-2)^2 \cdots + 1(n-1)^2 \]  \hspace{1cm} (continued)
\[
\frac{1}{n^2} \sum_{i=1}^{n} S_i^2 \frac{\hat{a}}{\sum_{i=1}^{n} S_i^2} \cdot \frac{\hat{a}}{\sum_{i=1}^{n} S_i^2} = \frac{1}{n^2} \sum_{i=1}^{n} S_i^2 \frac{\hat{a}}{\sum_{i=1}^{n} S_i^2} \cdot \frac{\hat{a}}{\sum_{i=1}^{n} S_i^2} = \int_{0}^{1} \tilde{W}_2^2(r) \, dr
\]

is invariant to the normality assumption so that the statistic \( n^{-2} L M_2 \) based on the calculation (1) is in this sense robust.
Remark. The reader is referred to Kwiatkowski, Phillips, and Schmidt [1] for a theoretical development and empirical application of the $LM$ test derived herein.

REFERENCES