94.3.4. **Spurious Regression in Forecast-Encompassing Tests**, proposed by Peter C.B. Phillips. A time series $y_t$ is generated by the unit root model

$$y_t = y_{t-1} + u_t, \quad t = 1, \ldots, n,$$

where $y_0$ is a random variable and the shocks $u_t$ are driven by the linear process

$$u_t = \sum_{j=0}^{\infty} c_j \epsilon_t, \quad \sum_{j=0}^{\infty} j^{1/2} |c_j| < \infty,$$

with $\epsilon_t$ i.i.d. $(0, \sigma^2)$.

Two models are used for producing $h$-period-ahead forecasts of $y_{n+h} (h = 1, \ldots, N)$:

Model 1: $\hat{y}_{n+h} = y_n$.

Model 2: $\hat{y}_{n+h} = b(n + h)$, 

where $b$ in model 2 is the least-squares coefficient from the fitted regression

$$y_t = b t + \text{error}$$

that uses data generated by (1).

It is proposed to test model (1) by a “forecast-encompassing” test in which the prediction error from model (1), i.e., $y_{n+h} - \hat{y}_{n+h}$, is regressed on the predictions $\hat{y}_{n+h}$ from model (2) to determine whether the latter have any explanatory power. This is achieved by the regression

$$y_{n+h} - \hat{y}_{n+h} = \gamma \hat{y}_{n+h} + \text{error} \quad (h = 1, \ldots, N)$$

(3)
where $\gamma^*$ is the fitted coefficient and its $t$-ratio is $t$, Suppose $N = n\tau$ for some fixed $\tau \in (0, 1)$.

Find the limit behavior of $t$, as $n \to \infty$. Discuss the implications of your result for this forecast-encompassing test of model (1) against model (2).