94.2.6. *Convergence of a Nonlinear Time Series Model*, proposed by Peter C.B. Phillips. In the model

\[ X_t = \left( \frac{1}{2} + e_t \right) X_{t-1}, \quad t = 1, 2, \ldots \]

the shocks \( e_t \) are independent and identically distributed with mean zero and variance \( \frac{1}{4} \), and \( X_0 \) is a random variable with zero mean and finite variance \( \sigma^2 > 0 \).

Show that \( Z_t = 2^{t/2} X_t \) converges almost surely as \( t \to \infty \). (Hint: Use the martingale convergence theorem for a suitable function of \( Z_t \).) Hence, show that \( X_t \to_{a.s.} 0 \) as \( t \to \infty \).