92.2.7. *Simultaneous Equations Bias in Level VAR Estimation*, proposed by Peter C.B. Phillips. Consider the bivariate cointegrated system

\[
\begin{bmatrix}
  y_{1t} \\
  y_{2t}
\end{bmatrix}
= \begin{bmatrix}
  0 & a_{12} \\
  0 & 1
\end{bmatrix}
\begin{bmatrix}
  y_{1t-1} \\
  y_{2t-1}
\end{bmatrix}
+ \begin{bmatrix}
  u_{1t} \\
  u_{2t}
\end{bmatrix}
\]

where \( u_t = (u_{1t}, u_{2t})' \) is i.i.d. \((0, \Sigma = (\sigma_{ij}))\). Data generated from this system is estimated using a levels vector autoregression with one lag, that is, a VAR(1). Denote the estimates of the coefficients in the first equation obtained in this way by \( \hat{a}_{11} \) and \( \hat{a}_{12} \).

(a) Find the asymptotic distribution of \((\hat{a}_{11}, \hat{a}_{12})\).
(b) Show that the asymptotic distribution of \( \hat{a}_{12} \) suffers from “simultaneous equations bias.”
(c) Show that the asymptotic distribution of \( \hat{a}_{12} \) is identical to that of the OLS estimate of \( a_{12} \) in the model

\[ y_{1t} = a_{12} y_{2t-1} + u_{1t}. \]

Use this to explain the presence of “simultaneous equations bias” in the limit distribution.