86.3.4. An Integral Over a Matrix Space, proposed by Peter C. B. Phillips.

Prove that

\[
\int_{S>0} \text{etr} (-S)(\det S)^{-\alpha+1/2}(a'S^{-1}a)^{\alpha}(a'S^{-2}a)^{\beta/2} \, dS
\]

\[
= (a'a)^{\beta/2} \frac{\Gamma_n(q)\Gamma(q + \frac{1}{2})\Gamma(q + p/2 + 1 - n/2)}{\Gamma(q + 1 - n/2)\Gamma(q + p/2 + 1/2)}
\]

where the integral is over the domain of all positive definite \( n \times n \) matrices \( S > 0 \), \( \Gamma_n(q) = \pi^{n(n-1)/4} \prod_{i=1}^{n} \Gamma(q - (i-1)/2) \) is the multivariate gamma
function, $a$ is a constant $n \times 1$ vector and $\text{Re}(q) > (n - 1)/2$, $\text{Re}(p) \geq -1$. 

The operator $\exp\{\text{trace}(\ )\}$.

The integral arose recently in some exact distribution theory for the seemingly unrelated regression model [1].

**REFERENCE**