COMMENT ON "MODELING ASSET RETURNS
WITH ALTERNATIVE STABLE DISTRIBUTIONS"

1. DATA DESCRIPTION AND MODELING

In the search for satisfactory descriptive models of economic data, huge numbers of distributions have been tried and many new distributions have been discovered. Entire classes of distributional types have been constructed and these often serve to direct the search process for a suitable choice. Some of these classes rely on certain limit theories, others rely on solutions to certain functional equations, while others are based solely on a family of descriptive characteristics. In any particular case it is, of course, always possible to find a distribution that fits the data well provided one works within a suitably broad and flexible class of candidates. Systems of distributions are especially useful in this task of data description.

It is one thing to fit given data well through the choice of a good distribution (though this is not always so easy in multivariate cases). But it is an entirely different matter to explain data well through the use of a statistical model that predicts the data's main characteristics and thereby goes some distance toward explaining features of the mechanism of data generation. Capturing the mechanism well enough to be able to explain events that did not occur in the given sample but do occur later on or in other related populations is the essence of statistical modeling. The model with the best fit to the sample data is sometimes far from being the best model of the underlying mechanism in this primitive predictive sense.
General concerns of this type motivated economists like Gibrat (1931), Champernowne (1953) and Sargan (1956) to model economic processes such as income generation and wealth creation over time, and to ascertain the form of statistical distributions for these economic quantities that were consistent with the underlying model. This usually involves finding the solution of certain functional equations that emerge from the transition processes incorporated in the model. The approach has led to a justification for the lognormal (or Gibrat) distribution and the Champernowne distribution for income and wealth, the latter distribution conforming to the (descriptive) Pareto law in the tail.

Purely statistical arguments, such as the experimental conditions that lead to a central limit or extreme value tendency, can also be advanced to justify a certain distributional form. Mandelbrot (1963a, 1963b, 1967) put forward such arguments in advocating the stable Paretoian family as a model for certain long-tailed economic distributions like commodity prices and financial asset returns. Mandelbrot’s arguments relied on the central limit tendency of an outcome determined by a multitude of individual shocks each in the domain of attraction of a certain stable law. The limiting stable law then became the statistical model for the data favored by Mandelbrot and it is predictive in the sense discussed above because the size of the tail exponent parameter (α) determines the magnitude and the regularity of outlier occurrences in the data.

Similar statistical arguments can be put forward to justify the Weibull distribution that is the candidate distribution preferred by the authors of this article on asset return distributions. The Weibull distribution is one of the simple extreme value laws and also has a stability property under random geometric summation. Random summation schemes have been common in central limit theory for several decades (see, e.g. Billingsley, 1968, Chapter 17) and came to the attention of researchers in finance after Clark’s (1973) work on the subordinated process. The idea that information time as distinct from clock time is random is certainly appealing and there are good reasons for believing that it may be a more appropriate way to start modeling financial markets. One advantage of the idea that is not mentioned by Mittnik and Rachev is that it can provide a useful basis for multivariate models (e.g. of asset returns and trading volume). Theoretical and empirical work along these lines is developed in a recent Yale doctoral thesis by Torben Andersen (1992).
2. MIXTURES

Random summation schemes typically lead to mixture limit densities (see e.g. Hall and Heyde, 1980, Theorem 3.2, p. 58). In such cases we can still think of a central tendency law operating because of the effects of the summation of random causes, but additional uncertainties in the process (like random information time or random conditional variances) work to produce a random limit variance. If the two sources of randomness are related (as we must often expect them to be) then we obtain even more complex limit laws that need to be represented in terms of semimartingales or stochastic integrals. As yet, this more general probabilistic approach to modeling financial market data as a stochastic integral process does not seem to have been noticed by researchers interested in the largely descriptive issue of the "best" statistical distribution of asset returns. One advantage of the stochastic integral limit representation is that it captures in a very elegant way the main additional feature of these empirical distributions besides long tails, viz., their conditional heterogeneity over time. I expect we will see some useful research on modeling asset return distributions in this way in the future.

Mixture distributions can also be justified by Bayesian arguments and by randomization. In situations where different members of a given class of distributions fit well for different data sets or regimes, randomization provides a mechanism by which we can determine a plausible model that is of more general validity. This consideration seems to be especially relevant in the present case where the authors' Weibull model works satisfactorily in some regimes such as the pre 1987 crash regime, but less well in others such as those that include the crash. Ideally, one would like to see some "stability" or robustness in the class of distributions under randomization. This does not seem to apply to the Weibull.

Take, for instance, the case of the Weibull density

\[(1) \quad p(x) = (c \lambda)(\lambda x)^{c-1}e^{-\lambda x^c} ; \quad c > 0 , \quad \lambda > 0 \]

on the half line \([0, \infty)\). The authors find empirical estimates of \(c\) that are slightly in excess of unity for both the positive and negative half lines. Suppose we randomize \(\lambda\) in (1), taking \(\lambda = \chi^2_1\) with density

\[(2) \quad g(\lambda) = 2^{-1/2}\Gamma(1/2)\lambda e^{-\lambda/2} \lambda^{-1/2} ; \quad \lambda > 0 . \]

Mixing (1) and (2) we have the density
\[ p(x) = 2^{-1/2} \Gamma(1/2)^{-1} x^{-1/2} \int_{\lambda > 0} e^{-\lambda x} \exp\left\{ -\lambda^2 \right\} d\lambda \]

which is an integral that is related to the parabolic cylinder function. It is particularly interesting to characterize the tail behavior of (3). A simple change of variable in the integrand shows that this is given by

\[ p(x) = 2^{-1/2} \Gamma(1/2)^{-1} x^{-3/2}, \quad \text{as} \quad x \to \infty. \]

which is very different from the exponential tail of the Weibull density (1). Mixing the Weibull with a \( \chi^2 \) density for its scale parameter therefore leads to a heavy tailed distribution that is of the stable Paretoian form (4). The mixed density therefore belongs to the domain of attraction of a stable variate with tail exponent parameter \( \alpha = 1/2 \).

Consider the same exercise conducted with a stable variable whose characteristic function is given by

\[ \phi_x(t) = e^{-|t|^\alpha}, \quad \lambda > 0, \quad \alpha > 0. \]

Mixing (5) and (2) we obtain after a simple calculation

\[ \phi(f) = \int_{\lambda > 0} \phi_x(\lambda) \phi(\lambda) d\lambda = \left( 1 + 2|s|^{\alpha} \right)^{-1/2}. \]

Noting that

\[ \phi(f) = \exp\left\{ -(1/2) \ln[1 + 2|s|^{\alpha}] \right\} - \exp\left\{ -|s|^{\alpha} [1 + O(1)] \right\}, \quad \text{as} \quad s \to 0 \]

we deduce that the mixed process is in the normal domain of attraction of a stable law with exponent \( \alpha \). Thus, for this mixing variate, the family of distributions described by the characteristic function (5) display "stability" in the sense that their tail exponent remains unchanged.

Now suppose the mixing variable is the reciprocal of a \( \chi^2_1 \). (It is popular to choose such mixing variates in Bayesian analysis for scale parameters and it is well known that mixing a normal distribution in this way leads to a Cauchy distribution or more generally a \( t \)-distribution when we take a \( 1/\chi^2_1 \) mixing variate.) Then \( f = 1/\lambda = \chi^2_1 \) and we have

\[ \phi(f) = \int_{f > 0} \phi_x(\lambda) \phi(f) df \]

\[ = (2\pi)^{-1/2} \int_{f > 0} e^{-|s|^{\alpha} f^2} f^{-1/2} df \]

\[ = \pi^{-1/2} \int_{h > 0} \exp\left\{ -|s|^{\alpha} h^2 - h \right\} h^{-1/2} dh \]

\[ = (2\pi)^{-1/2} (\pi/2)^{1/2} K_{-1/2}(z), \quad z = 2^{1/2} |s|^{\alpha/2}. \]
where

\[ K_\alpha(z) = (1/2)(z/2)^{-\alpha} \int_0^\infty \exp[-t - (z^2/4)(1/t)]t^{-\alpha-1}dt \]

is the Macdonald function (see e.g. Lebedev, 1972, pp. 109, 119). Noting that

\[ K_{-\alpha}(z) = K_\alpha(z) = (\pi/2)z^{1/2}e^{-z} \]

(op.cit., p. 110) we find that the characteristic function of the mixed variate is

\[ c(f) = \exp\left(-2^{1/2} |\lambda|^{\alpha/2}\right). \]

Thus, the mixed variate lies in the domain of attraction of a stable variate with tail exponent \( \alpha/2 \) rather than \( \alpha \). Hence, mixing a stable variate with the reciprocal of a \( \chi_1^2 \) variate also keeps the stable variate in the stable family but changes the tail exponent so that the tails of the new variate are thicker than those of the original variate. I will not give a demonstration here, but if the mixing variate is \( f = 1/\lambda \cdot \chi_n^2 \) then the mixed stable variate is in the domain of attraction of the same stable law, i.e. a stable law with exponent \( \alpha/2 \), irrespective of the degrees of freedom \( n \).

These brief results should serve to indicate that the stable family has more "stability" properties under mixing than the Weibull and that when mixed the Weibull belongs to the domain of attraction of certain stable laws. The latter property seems rather interesting because it indicates that the long tailed feature of asset returns that is not well captured by the Weibull may indeed be well accommodated by use of a simple mixing process.

3. EMPIRICALLY DETERMINED TAIL BEHAVIOR

The analysis of the previous section suggests that it may be interesting to work in the much wider family of distributions constituting the domain of attraction of a stable law. As we have seen this includes certain mixed Weibull variates. This family is much larger than the stable Pareto family itself and is characterized by the fact that its members have asymptotically Paretoan tails. One example is the aforementioned Champernowne distribution, which has been found to model income distributions well, capturing the long tailed behavior of these distributions yet at the same time doing better in the body of the distribution than the Pareto. Similar remarks might well apply to the distribution of asset returns. Typically, the stable Pareto distributions do very well in the tails but less well in the main body of the distribution.
This feature is manifest in the empirical results given to us by Mittnik and Rachev on stock price returns.

One way of finding out whether a distribution is likely to be adequate in the tails is to estimate the tail slope directly. Let \( \alpha \) be the maximal moment exponent of the variate \( X \), i.e.,
\[
\alpha = \sup\{ s > 0 : E|X|^s < \infty \}.
\]
For distributions with asymptotically Pareto tails of the form
\[
\rho(x) = kx^{-\alpha-1}[1 + \beta(x)], \ x \to \infty
\]
for some constant \( k > 0 \) and where \( \beta(x) = 0 \) as \( x \to \infty \), the maximal moment exponent is the tail exponent parameter \( \alpha \). Let \( (X_i)_{i \geq 1} \) be a sample of independent draws from a population whose distribution has a tail shape of the form (9). If \( X_{n1} \leq X_{n2} \leq \ldots \leq X_{nm} \) are the order statistics from this sample then the estimators
\[
\hat{\alpha}_s = \left( \sum_{j=1}^{s} X_{n-j+1}^{-s} \ln X_{n,n-j+1} - \ln X_{n,n-s} \right)^{-1}
\]
\[
\hat{k}_s = s n^{-1} X_{n,n-s}^{-s}
\]
are consistent for \( \alpha \) and \( k \) in (9) when \( s = s(n) \to \infty \) as \( n \to \infty \). If \( \beta(x) = O(x^{-\gamma}) \) in (9) for some \( \gamma > 0 \), then for \( s = o(n^{2/(\gamma+\epsilon)}) \) as \( n \to \infty \) we have
\[
l_{1/2}(\hat{\alpha}_s - \alpha) \rightarrow_d N(0, \sigma^2),
\]
and
\[
l_{1/2}(\ln(n/s))^{-1}(\hat{k}_s - k) \rightarrow_d N(0, \kappa^2).
\]
This limit theory is due to Hall (1982) and can be used to examine the tail shape of distributions that belong to the general family (9).

This suggestion was put forward in recent work by the author and M. Lorentza (1990) and some extensive empirical implementations of the method are provided in a recent Yale doctoral dissertation by Lorentza (1991). A paper by Lorentza-Phillipss (1992) contains an overview of this methodology and applications to several financial data sets, including the daily S&P 500 index of stock returns over the period 1962-1987. The latter series is a longer version of the series considered in the Mittnik-Rachev application, where data over the period 1982-1987 is used. The Lorentza-Phillips results give estimates of \( \alpha \) in the range 2.5-3.2 for monthly stock return data and estimates in the range 3.1-3.8 for daily data. These estimates are (statistically) significantly below 4.0, so that fourth moments of the population distribution would
appear to be infinite. While this rules out the Weibull distribution as a potential contender, it also rules out infinite variance stable laws. In such cases, it seems likely that some finite variance mixture density that has Pareto tails and infinite fourth moments is a good candidate.

Some work that is related to the above has recently been done by Jansen and de Vries (1991). Using a maintained hypothesis that the tails are exactly Paretin these authors estimate tail shape for stock return series for several US companies and find results that are broadly similar to those discussed above, viz. finite variances but possibly infinite fourth moments for stock return data.

A final issue that is of importance in the empirical context of financial data sets is the question of the constancy of the marginal distributions over time. All methods that are used for fitting distributions to the empirical data presume constancy of the marginal distributions over time. Recent work has shown that this presumption may be very unrealistic. Pagan and Schwert (1990a, 1990b) presented some persuasive evidence that US stock market return data cannot be presumed to be covariance stationary over long historical periods. Some formal tests of constant unconditional variances are developed in the Phillips-Loretan (1990) paper and their empirical results support those of Pagan-Schwert. Mittnik-Rachev also find that the empirical support for both the Weibull and the stable Paretin distributions deteriorates when their data includes the 1987 stock market crash. When data on trading volume is taken into account the evidence of nonstationarity in the series is overwhelming. Clearly, there have been major institutional changes even over the last decade that belie the assumption of stationarity. Dealing with and modeling these changes and the induced nonstationarity of asset return and trading volume data seems to be every bit as important as finding a good distributional model for the data over shorter periods where constancy of the marginal distributions can be more safely assumed.

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4. REFERENCES


