

# Long-term Risk: An Operator Approach <sup>1</sup>

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## Abstract

We create an analytical structure that reveals the long-run risk-return relationship for nonlinear continuous time Markov environments. We do so by studying an eigenvalue problem associated with a positive eigenfunction for a conveniently chosen family of valuation operators. The members of this family are indexed by the elapsed time between payoff and valuation dates, and they are necessarily related via a mathematical structure called a semigroup. We represent the semigroup using a positive process with three components: an exponential term constructed from the eigenvalue, a martingale and a transient eigenfunction term. The eigenvalue encodes the risk adjustment, the martingale alters the probability measure to capture long-run approximation, and the eigenfunction gives the long-run dependence on the Markov state. We discuss sufficient conditions for the existence and uniqueness of the relevant eigenvalue and eigenfunction. By showing how changes in the stochastic growth components of cash flows induce changes in the corresponding eigenvalues and eigenfunctions, we reveal a long-run risk-return tradeoff.

**Keywords:** risk-return tradeoff, long run, semigroups, Perron-Frobenius theory, martingales.

# 1 Introduction

We study long-run notions of a risk-return relationship that feature the pricing of exposure to uncertainty in the growth rates of cash flows. In financial economics risk-return tradeoffs show how expected rates of return over small intervals are altered in response to changes in the exposure to the underlying shocks that impinge on the economy. In continuous-time modeling, the length of interval is driven to zero to deduce a limiting local relationship. In contrast to the local analysis, we focus on what happens as the length of time between valuation and payoff becomes large.

In a dynamic setting asset payoffs depend on both the future state and on the date when payoff will be realized. Risk averse investors require compensation for their risk exposure giving rise to risk premia in equilibrium pricing. There is a term structure of risk premia to consider. There are many recent developments in asset pricing theory featuring the intertemporal composition of risk. The risk premia for different investment horizons are necessarily related, just as long-term interest rates reflect a compounding of short term rates. The risk counterpart to this compounding is most transparent in log-normal environments with linear state dynamics and constant volatility (*e.g.*, see Hansen et al. (2008)). Our aim, however, is to support the analysis of a more general class of models that allow for nonlinearity in the state dynamics. Risk premia depend on both risk exposure and the price of that exposure. The methods we develop in this paper are useful for representing the exposure of cash flows and the price of that exposure in long horizons.

While we are interested in the entire term structure of risk prices, the aim of this paper is to establish limiting behavior as investment horizons become large. There are two reasons for such an investigation. First, they provide a complementary alternative to the local pricing that is familiar from the literature on asset pricing. Comparison of the two allows us to understand the (average) *slope* of the term structure of risk prices. Second, economics is arguably a more reliable source of restrictions over longer time horizons. Thus it is advantageous to have tools that allow us to characterize how equilibrium risk prices are predicted to behave in the long run and how the prices depend on ingredients of the underlying model of the economy.

The continuous time local analysis familiar in financial economics is facilitated by the use of stochastic differential equations driven by a vector Brownian motion and jumps. An equilibrium valuation model gives the prices of the instantaneous

exposure of payoffs to these risks. Values over alternative horizons can be inferred by integrating appropriately these local prices. Such computations are nontrivial when there are nonlinearities in the evolution of state variables or valuations. This leads us to adopt an alternative approach based on an operator formulation of asset pricing. As in previous research, we model asset valuation using operators that assign prices today to payoffs in future dates. Since these operators are defined for each payoff date, we build an indexed family of such pricing operators. This family is referred to as a *semigroup* because of the manner in which the operators are related to one another.<sup>1</sup> We show how to modify valuation operators in a straightforward way to accommodate stochastic cash flow growth. It is the evolution of such operators as the payoff date changes that interests us. Long-run counterparts to risk-return tradeoffs are reflected in how the limiting behavior of the family of operators changes as we alter the stochastic growth of the cash flows.

We study the evolution of the family of valuation operators in a continuous-time framework, although important aspects of our analysis are directly applicable to discrete-time specifications. Our analysis is made tractable by assuming the existence of a Markov state that summarizes the information in the economy pertinent for valuation. The operators we use apply to functions of this Markov state and can be represented as:

$$\mathbb{M}_t\psi(x) = E [M_t\psi(X_t)|X_0 = x]$$

for some process  $M$  appropriately restricted to ensure intertemporal consistency and to guarantee that the Markov structure applies to pricing. The principal restriction we impose is that  $M$  has a “multiplicative” property, resulting in a family of operators indexed by  $t$  that is a semigroup.

A central mathematical result that we establish and exploit is a multiplicative factorization:

$$M_t = \exp(\rho t)\hat{M}_t\frac{\phi(X_0)}{\phi(X_t)} \tag{1}$$

where  $\hat{M}$  is a martingale whose logarithm has stationary increments, and  $\phi$  is a positive function.<sup>2</sup> While this decomposition is typically not unique, we show that there

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<sup>1</sup>See Garman (1984) for an initial contribution featuring the use of semigroups in modeling asset prices.

<sup>2</sup>Alvarez and Jermann (2005) proposed the use of such a decomposition to analyze the long-run behavior of stochastic discount factors and cited an early version of our paper for proposing the link between this decomposition and principal eigenvalues and functions. We develop this connec-

is at most one such decomposition that is of value to study long-term approximation. Intuitively, we may think of the scalar  $\rho$  as a deterministic growth rate, and the ratio of positive random variables  $\frac{\phi(X_0)}{\phi(X_t)}$  as a transitory contribution. We construct this representation using a function  $\phi$  which is a principal (that is positive) eigenfunction of every operator  $\mathbb{M}_t$  in the semigroup, that is,

$$\mathbb{M}_t\phi(x) = \exp(\rho t)\phi(x). \quad (2)$$

In our analysis, we use the martingale  $\hat{M}$  to change the probability measure prior to our study of approximation. The principal eigenfunction  $\phi$  used in constructing this change characterizes the limiting state dependence of the semigroup.

We use the multiplicative factorization (1) to study two alternative long-run counterparts to risk-return tradeoffs. It allows us to isolate enduring components to cash-flows or prices and to explore how these components are valued. For instance, cash flows with different stochastic growth components are valued differently. One notion of a long-run risk-return tradeoff characterizes how the asymptotic rate of return required as compensation depends on the long-run cash flow risk exposure. A second notion cumulates returns that are valued in accordance to a local risk-return tradeoff. A corresponding long-run tradeoff gives the asymptotic growth rates of alternative cumulative returns over long time horizons as a function of the risk exposures used to construct the local returns.

Previous papers have explored particular characterizations of the term structure of risk pricing of cash flows. (For instance, see Hansen et al. (2008) and Lettau and Wachter (2007).) In this regard local pricing characterizes one end of this term structure and our analysis the other end. Hansen et al. (2008) characterize the long-run cash flow risk prices for discrete time log-normal environments. Their characterization shares our goal of pricing the exposure to stochastic growth risk, but in order to obtain analytical results, they exclude potential nonlinearity and temporal variation in volatility. Hansen et al. (2008) examine the extent to which the long-run cash-flow risk prices from a family of recursive utility models can account for the value heterogeneity observed in equity portfolios with alternative ratios of book value to market value. Our paper shows how to produce such pricing characterizations for 

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tion formally and establish existence and uniqueness results for a general class of multiplicative functionals.

more general nonlinear Markov environments.

There is a corresponding equation to (2) that holds locally, obtained essentially by differentiating with respect to  $t$  and evaluating the derivative at  $t = 0$ . More generally, this time derivative gives rise to the *generator* of the semigroup. By working with the generator, we exploit some of the well known local characterizations of continuous time Markov models from stochastic calculus to provide a solution to equation (2). While continuous-time models achieve simplicity by characterizing behavior over small time increments, operator methods have promise for enhancing our understanding of the connection between short-run and long-run behavior.

The remainder of the paper is organized as follows. In sections 3 and 4 we develop some of the mathematical preliminaries pertinent for our analysis. Specifically, in section 3 we describe the underlying Markov process and introduce the reader to the concepts of additive and multiplicative functionals. Both type of functionals are crucial ingredients to what follows. In sections 3.3, 3.4 and 3.5 we consider three alternative multiplicative functionals that are pertinent in intertemporal asset pricing. In section 3.3 we use a multiplicative functional to model a stochastic discount factor process, in section 3.4 we introduce valuation functionals that are used to represent returns over intervals of any horizon, and in section 3.5 we use multiplicative functionals to represent the stochastic growth components to cash flows.

In section 4 we study semigroups. Semigroups are used to evaluate contingent claims written on the Markov state indexed by the elapsed time between trading date and the payoff date. In section 5 we define an extended generator associated with a multiplicative functional, and in section 6 we introduce principal eigenvalues and functions and use these objects to establish a representation of the form (1). We present approximation results that justify formally the long-run role of a principal eigenfunction and eigenvalue and show that there is at most one appropriate eigenfunction in section 7. Applications to financial economics are given in section 8. Among other things, we derive long-term counterparts to risk-return tradeoffs by making the valuation horizon arbitrary large. Finally, in section 9 we discuss sufficient conditions for the existence of the principal eigenvectors needed to support our analysis.

## 2 Stochastic discount factors and pricing

Consider a continuous time Markov process  $\{X_t : t \geq 0\}$ , and the (completed) filtration  $\mathcal{F}_t$  generated by its histories. A strictly positive stochastic discount factor process  $\{S_t : t \geq 0\}$  is an adapted ( $S_t$  is  $\mathcal{F}_t$  measurable) positive process that is used to discount payoffs. The date zero price of a claim to the payoff  $\Pi_t$  at  $t$  is

$$E[S_t \Pi_t | \mathcal{F}_0].$$

For each date  $t$ , this representation follows from the representation of positive linear functionals on appropriately constructed payoff spaces. The stochastic discount factor process is positive with probability one and satisfies  $S_0 = 1$  because of the presumed absence of arbitrage.

Let  $\tau \leq t$  be an intermediate trading date between date zero and date  $t$ . The time  $t + u$  payoff could be purchased at date zero or alternatively it could be purchased at date  $\tau$  with a prior date zero purchase of a claim to the date  $\tau$  purchase price. The *Law of One Price* guarantees that these two ways of acquire the payoff  $\Pi_t$  must have the same initial cost. The same must be true if we scale the  $\Pi_t$  by a bounded random variable available to the investor at date  $\tau$ . This argument allows us to infer the date  $\tau$  prices from the date zero prices. Specifically, for  $\tau \leq t$ :

$$E\left[\frac{S_t}{S_\tau} \Pi_t | \mathcal{F}_\tau\right] \tag{3}$$

is the price at time  $\tau$  of a claim to the payoff  $\Pi_t$  at  $t$ . Thus once we have a date zero representation of prices at alternative investment horizons via a stochastic discount factor process, we may use that same process to represent prices at subsequent dates by forming the appropriate ratios of the date zero stochastic discount factors. This representation imposes temporal consistency in valuation enforced by the possibility of trading at intermediate dates.<sup>3</sup>

We add a Markov restriction to this well known depiction of asset prices. Expression (3) can then be used to define a *pricing operator*  $\mathbb{S}_t$ . In particular if  $\psi(X_t)$  is a random payoff at  $t$  that depends only on the current Markov state, its time zero price

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<sup>3</sup>This temporal consistence property follows formally from a “consistency axiom” in Rogers (1998).

is:

$$\mathbb{S}_t\psi(x) = E[S_t\psi(X_t)|x_0 = x] \quad (4)$$

expressed as a function of the initial Markov state. The fact that the price depends only on the current state restricts the stochastic discount factor process. While  $S_t$  can depend on the Markov process between dates 0 and  $t$ , we do not allow it to depend on previous history of the Markov state; otherwise this history would be reflected in the date zero prices. As first remarked by Garman (1984), the temporal consistency in valuation insures the that family of such linear pricing operators  $\{\mathbb{S}_t : t \geq 0\}$  satisfies a *semigroup* property:  $\mathbb{S}_0 = \mathbb{I}$  and  $\mathbb{S}_{t+u}\psi(x) = \mathbb{S}_t\mathbb{S}_u\psi(x)$ . In our asset pricing setting, the semigroup property is thus an iterated value property that connects pricing over different time horizons.<sup>4</sup>

Consider again trading at the intermediate date  $\tau$ . Then Markov valuation between dates  $\tau$  and  $t$  can be depicted using the operator  $\mathbb{S}_{t-\tau}$ , or it can be depicted using (3). As a consequence  $\frac{S_t}{S_\tau}$  should depend only on the Markov process between dates  $t$  and  $\tau$ . Moreover,

$$\frac{S_t}{S_\tau} = S_{t-\tau}(\theta_\tau) \quad (5)$$

where  $\theta_\tau$  is a *shift operator* that moves forward the time subscript of the Markov process used in the construction of  $S_{t-\tau}$  forward by  $\tau$  time units. Property (5) is a restriction on functionals built from an underlying Markov process, and we will call functionals that satisfy property (5), and have initial value 1, *multiplicative functionals*. Later we will give convenient representations of such functionals.

Our approach is motivated by this multiplicative property of the stochastic discount factor and uses the connection between this multiplicative property and the semigroup property of the family of pricing operators. We will also use this multiplicative property to study the valuation of payoffs with stochastic growth components. To accommodate these other processes we set up a more general framework in the next couple of sections.

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<sup>4</sup>Garman (1984) allows for non-Markov environments. In this case the family of operators forms an “evolution semigroup”. We adopt a Markov formulation of the Law of One Price for tractability.

### 3 Markov and related processes

We first describe the underlying Markov process, and then build other convenient processes from this underlying process. These additional processes are used to represent stochastic discounting, stochastic growth and returns over long-horizons.

#### 3.1 Baseline process

Let  $\{X_t : t \geq 0\}$  be a continuous time, strong Markov process defined on a probability space  $\{\Omega, \mathcal{F}, Pr\}$  with values on a state space  $\mathcal{D}_0 \subset \mathbb{R}^n$ . The sample paths of  $X$  are continuous from the right and with left limits, and we will sometimes also assume that this process is stationary and ergodic. Let  $\mathcal{F}_t$  be completion of the sigma algebra generated by  $\{X_u : 0 \leq u \leq t\}$ .

One simple example is:

**Example 3.1. Finite-state Markov chain** *Consider a finite state Markov chain with states  $x_j$  for  $j = 1, 2, \dots, N$ . The local evolution of this chain is governed by an  $N \times N$  intensity matrix  $\mathbb{U}$ . An intensity matrix encodes all of the transition probabilities. The matrix  $\exp(t\mathbb{U})$  is the matrix of transition probabilities over an interval  $t$ . Since each row of a transition matrix sums to unity, each row of  $\mathbb{U}$  must sum to zero. The diagonal entries are non-positive and represent minus the intensity of jumping from the current state to a new one. The remaining row entries, appropriately scaled, represent the conditional probabilities of jumping to the respective states.*

When treating infinite state spaces we restrict the Markov process  $X$  to be a semimartingale. As a consequence, we can extract a continuous component  $X^c$  and what remains is a pure jump process  $X^j$ . The evolution of the jump component is given by:

$$dX_t^j = \int_{\mathbb{R}^n} y \zeta(dy, dt)$$

where  $\zeta = \zeta(\cdot, \cdot; \omega)$  is a random counting measure. That is, for each  $\omega$ ,  $\zeta(b, [0, t]; \omega)$  gives the total number of jumps in  $[0, t]$  with a size in the Borel set  $b$  in the realization  $\omega$ . In general, the associated Markov stochastic process  $X$  may have an infinite number of small jumps in any time interval. In what follows we will assume that this process has a finite number of jumps over a finite time interval. This rules out most Lévy processes, but greatly simplifies the notation. In this case, there is

a finite measure  $\eta(dy|x)dt$  that is the *compensator* of the random measure  $\zeta$ . It is the (unique) predictable random measure, such that for each predictable stochastic function  $f(x, t; \omega)$ , the process

$$\int_0^t \int_{\mathbb{R}^n} f(y, s; \omega) \zeta(dy, ds; \omega) - \int_0^t \int_{\mathbb{R}^n} f(y, s; \omega) \eta[dy|X_{s-}(\omega)] ds$$

is a martingale. The measure  $\eta$  encodes both a jump intensity and a distribution of the jump size given that a jump occurs. The jump intensity is the implied conditional measure of the entire state space  $\mathcal{D}_0$ , and the jump distribution is the conditional measure divided by the jump intensity.

We presume that the continuous sample path component satisfies the stochastic evolution:

$$dX_t^c = \xi(X_{t-})dt + \Gamma(X_{t-})dB_t$$

where  $B$  is a multivariate  $\mathcal{F}_t$ -Brownian motion and  $\Gamma(x)' \Gamma(x)$  is nonsingular. Given the rank condition, the Brownian increment can be deduced from the sample path of the state vector via:

$$dB_t = [\Gamma(X_{t-})' \Gamma(X_{t-})]^{-1} \Gamma(X_{t-})' [dX_t^c - \xi(X_{t-})dt].$$

### Example 3.2. Markov Diffusion

*In what follows we will often use the following example. Suppose the Markov process  $X$  has two components,  $X^f$  and  $X^o$ , where  $X^f$  is a Feller square root process and is positive and  $X^o$  is an Ornstein-Uhlenbeck process and ranges over the real line:*

$$\begin{aligned} dX_t^f &= \xi_f(\bar{x}_f - X_t^f)dt + \sqrt{X_t^f} \sigma_f dB_t^f, \\ dX_t^o &= \xi_o(\bar{x}_o - X_t^o)dt + \sigma_o dB_t^o \end{aligned}$$

with  $\xi_i > 0$ ,  $\bar{x}_i > 0$  for  $i = f, o$  and  $2\xi_f \bar{x}_f \geq (\sigma_f)^2$  where  $B = \begin{bmatrix} B^f \\ B^o \end{bmatrix}$  is a bivariate standard Brownian motion. The parameter restrictions guarantee that there is a stationary distribution associated with  $X^f$  with support contained in  $\mathbb{R}_+$ .<sup>5</sup> For conve-

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<sup>5</sup>We could accommodate the case where  $B^f$  or  $B^o$  are each multi-dimensional, by considering a filtration  $\{\mathcal{F}_t\}$  larger than the one generated by  $X$ . In effect, we would enlarge the state space in ways that were inconsequential to the computations that interest us. However, for simplicity we have assumed throughout this paper that  $\{\mathcal{F}_t\}$  is the filtration generated by  $X$ .

nience, we make the two processes independent. We use  $X^o$  to model predictability in growth rates and  $X^f$  to model predictability in volatility. We normalize  $\sigma_o$  to be positive and  $\sigma_f$  to be negative. We specify  $\sigma_f$  to be negative in order that a positive increment to  $B_t^f$  reduces volatility.

## 3.2 Multiplicative functionals

A functional is a stochastic process constructed from the original Markov process:

**Definition 3.1.** A **functional** is a real-valued process  $\{M_t : t \geq 0\}$  that is adapted ( $M_t$  is  $\mathcal{F}_t$  measurable for all  $t$ .) We will assume that  $M_t$  has a version with sample paths that are continuous from the right with left limits.

Recall that  $\theta$  denotes the shift operator. Using this notation, write  $M_u(\theta_t)$  for the corresponding function of the  $X$  process shifted forward  $t$  time periods. Since  $M_u$  is constructed from the Markov process  $X$  between dates zero and  $u$ ,  $M_u(\theta_t)$  depends only on the process between dates  $t$  and date  $t + u$ .

**Definition 3.2.** The functional  $M$  is **multiplicative** if  $M_0 = 1$ , and  $M_{t+u} = M_u(\theta_t)M_t$ .

Products of multiplicative functionals are multiplicative functionals. We are particularly interested in strictly positive multiplicative functionals. In this case, one may define a new functional  $A = \log(M)$ , that will satisfy an additive property. It turns out that it is more convenient to parameterize  $M$  using its logarithm  $A$ . The functional  $A$  will satisfy the following definition:

**Definition 3.3.** A functional  $A$  is **additive** if  $A_0 = 0$  and  $A_{t+u} = A_u(\theta_t) + A_t$ , for each nonnegative  $t$  and  $u$ .<sup>6</sup>

Exponentials of additive functionals are strictly positive multiplicative functionals.

While the joint process  $\{(X_t, A_t) : t \geq 0\}$  is Markov, by construction the additive functional does not *Granger cause* the original Markov process. Instead it is constructed from that process. No additional information about the future values of  $X$  are revealed by current and past values of  $A$ . When  $X$  is restricted to be stationary, an additive functional can be nonstationary but it has stationary increments. The following are examples of additive functionals:

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<sup>6</sup>Notice that we do not restrict additive functionals to have bounded variation as, e.g. Revuz and Yor (1994).

**Example 3.3.** Given any continuous function  $\psi$ ,  $A_t = \psi(X_t) - \psi(X_0)$ .

**Example 3.4.** Let  $\beta$  be a Borel measurable function on  $\mathcal{D}_0$  and construct:

$$A_t = \int_0^t \beta(X_u) du$$

where  $\int_0^t \beta(X_u) du < \infty$  with probability one for each  $t$ .

**Example 3.5.** Form:

$$A_t = \int_0^t \gamma(X_{u-})' dB_u$$

where  $\int_0^t |\gamma(X_u)|^2 du$  is finite with probability one for each  $t$ .

**Example 3.6.** Form:

$$A_t = \sum_{0 \leq u \leq t} \kappa(X_u, X_{u-})$$

where  $\kappa : \mathcal{D}_0 \times \mathcal{D}_0 \rightarrow \mathbb{R}$ ,  $\kappa(x, x) = 0$ .

Sums of additive functionals are additive functionals. We may thus use examples 3.4, 3.5 and 3.6 as building blocks in a parameterization of additive functionals. This parameterization uses a triple  $(\beta, \gamma, \kappa)$  that satisfies:

- a)  $\beta : \mathcal{D}_0 \rightarrow \mathbb{R}$  and  $\int_0^t \beta(X_u) du < \infty$  for every positive  $t$ ;
- b)  $\gamma : \mathcal{D}_0 \rightarrow \mathbb{R}^m$  and  $\int_0^t |\gamma(X_u)|^2 du < \infty$  for every positive  $t$ ;
- c)  $\kappa : \mathcal{D}_0 \times \mathcal{D}_0 \rightarrow \mathbb{R}$ ,  $\kappa(x, x) = 0$  for all  $x \in \mathcal{D}_0$ ,  $\int \exp(\kappa(y, x)) \eta(dy|x) < \infty$  for all  $x \in \mathcal{D}_0$ .

Form:

$$\begin{aligned} A_t &= \int_0^t \beta(X_u) du + \int_0^t \gamma(X_{u-})' dB_u + \sum_{0 \leq u \leq t} \kappa(X_u, X_{u-}), \\ &= \int_0^t \beta(X_u) du + \int_0^t \gamma(X_{u-})' [\Gamma(X_{u-})' \Gamma(X_{u-})]^{-1} \Gamma(X_{u-})' [dX_u^c - \xi(X_{u-}) du] \\ &\quad + \sum_{0 \leq u \leq t} \kappa(X_u, X_{u-}). \end{aligned}$$

This additive functional is a semimartingale.

While we will use extensively these parameterizations of an additive functional, they are not exhaustive as the following example illustrates.

**Example 3.7.** Suppose that  $\{X_t : t \geq 0\}$  is a standard scalar Brownian motion,  $b$  a Borel set in  $\mathbb{R}$ , and define the occupation time of  $b$  up to time  $t$  as

$$A_t \doteq \int_0^t \mathbf{1}_{\{X_u \in b\}} du.$$

$A_t$  is an additive functional. As a consequence, the local time at a point  $r$  defined as

$$L_t \doteq \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{\{X_u \in (r-\epsilon, r+\epsilon)\}} du,$$

is also an additive functional.

Since the logarithm of a strictly positive multiplicative process is an additive process we will consider parameterized versions of strictly positive multiplicative processes by parameterizing the corresponding additive process. For instance, if  $M = \exp(A)$  when  $A$  is parameterized by  $(\beta, \gamma, \kappa)$ , we will say that the multiplicative process  $M$  is parameterized by  $(\beta, \gamma, \kappa)$ . Notice that Ito's Lemma guarantees that:

$$\frac{dM_t}{M_{t-}} = \left[ \beta(X_{t-}) + \frac{|\gamma(X_{t-})|^2}{2} \right] dt + \gamma(X_{t-})' dB_t + \exp[\kappa(X_t, X_{t-})] - 1.$$

The multiplicative process  $\{M_t : t \geq 0\}$  of this form is a local martingale if, and only if,

$$\beta + \frac{|\gamma|^2}{2} + \int (\exp[\kappa(y, \cdot)] - 1) \eta(dy|\cdot) = 0. \quad (6)$$

For the remainder of this section we describe three types of multiplicative functionals that we use in our subsequent analysis.

### 3.3 Stochastic discount factors

In this section we write down two parameterized examples of multiplicative stochastic discount factors that we will use to illustrate our results.

#### Example 3.8. Breeden model

Using the Markov process given in example 3.2, we consider a special case of Breeden (1979)'s consumption-based asset pricing model. Suppose that equilibrium

consumption evolves according to:

$$dc_t = X_t^o dt + \sqrt{X_t^f} \vartheta_f dB_t^f + \vartheta_o dB_t^o. \quad (7)$$

where  $c_t$  is the logarithm of consumption  $C_t$ . Given our previous sign convention that  $\sigma_o > 0$ , when  $\vartheta_o > 0$  a positive shock  $dB_t^o$  is unambiguously good news because it gives a favorable movement for consumption and its growth rate. Similarly, since  $\sigma_f < 0$ , when  $\vartheta_f > 0$  a positive shock  $dB_t^f$  is unambiguously good news because it reduces volatility while increasing consumption. Suppose also that investor's preferences are given by:

$$E \int_0^\infty \exp(-bt) \frac{C_t^{1-a} - 1}{1-a} dt$$

for  $\mathbf{a}$  and  $\mathbf{b}$  strictly positive. The implied stochastic discount factor is  $S_t = \exp(A_t^s)$  where

$$A_t^s = -\mathbf{a} \int_0^t X_s^o ds - bt - \mathbf{a} \int_0^t \sqrt{X_s^f} \vartheta_f dB_s^f - \mathbf{a} \int_0^t \vartheta_o dB_s^o.$$

### Example 3.9. Kreps-Porteus model

When investors have time separable logarithmic utility and perfect foresight, the continuation value process  $W^*$  for the consumption process satisfies the differential equation:

$$\frac{dW_t^*}{dt} = \mathbf{b}(W_t^* - c_t) \quad (8)$$

where  $\mathbf{b}$  is the subjective rate of time discount. This equation is solved forward with an appropriate "terminal" condition. In constructing this differential equation we have scaled the logarithm of consumption by  $\mathbf{b}$  for convenience. Let

$$W_t = \frac{1}{1-\mathbf{a}} \exp[(1-\mathbf{a})W_t^*]$$

for  $\mathbf{a} > 1$  and notice that  $W_t$  is an increasing transformation of  $W_t^*$ . Thus for the purposes of representing preferences,  $W$  can be used as an ordinally equivalent continuation value process. The process  $W$  satisfies the differential equation:

$$\begin{aligned} \frac{dW_t}{dt} &= \mathbf{b}(1-\mathbf{a})W_t \left( \frac{1}{1-\mathbf{a}} \log[(1-\mathbf{a})W_t] - c_t \right) \\ &= \mathbf{b}W_t \{(\mathbf{a}-1)c_t + \log[(1-\mathbf{a})W_t]\} \end{aligned} \quad (9)$$

Next suppose that investors do not have perfect foresight. We may now think of the

right-hand sides of (8) and (9) as defining the drift or local means of the continuation values. As we know from Kreps and Porteus (1978) and Duffie and Epstein (1992), the resulting preferences cease to be ordinally equivalent. The first gives the recursive equation for continuation values that are expectations of the discounted logarithmic utility. Instead we use the counterpart to the second differential equation:

$$\lim_{\epsilon \downarrow 0} \frac{E(W_{t+\epsilon} - W_t | \mathcal{F}_t)}{\epsilon} = \mathbf{b}W_t \{(\mathbf{a} - 1)c_t + \log[(1 - \mathbf{a})W_t]\}$$

where  $\mathbf{a} > 1$ . The resulting preferences can be viewed as a special case of the continuous-time version of the preferences suggested by Kreps and Porteus (1978) and of the stochastic differential utility model of Duffie and Epstein (1992) and Schroder and Skiadas (1999). If we were to take the continuation value process  $W$  as a starting point in a stochastic environment and transform back to the utility index  $W^*$  using

$$W_t^* = \frac{1}{1 - \mathbf{a}} \log[(1 - \mathbf{a})W_t]$$

the resulting drift would include a contribution of the local variance as an application of Ito's Lemma. For these preferences the intertemporal composition of risk matters. Bansal and Yaron (2004) have used this feature of preferences in conjunction with predictable components in consumption and consumption volatility as a device to amplify risk premia. This particular utility recursion we use imposes a unitary elasticity of intertemporal substitution as in the original preference specification with logarithmic utility. The parameter  $\mathbf{a}$  alters risk prices as we will illustrate.<sup>7</sup>

Suppose again that consumption evolves according to equation (7). Conjecture a continuation value process of the form:

$$W_t = \frac{1}{1 - \mathbf{a}} \exp \left[ (1 - \mathbf{a})(\mathbf{w}_f X_t^f + \mathbf{w}_o X_t^o + c_t + \bar{\mathbf{w}}) \right]$$

The coefficients satisfy:

$$\begin{aligned} -\xi_f \mathbf{w}_f + \frac{(1 - \mathbf{a})\sigma_f^2}{2} (\mathbf{w}_f)^2 + (1 - \mathbf{a})\vartheta_f \sigma_f \mathbf{w}_f + \frac{(1 - \mathbf{a})\vartheta_f^2}{2} &= \mathbf{b}w_f \\ -\xi_o \mathbf{w}_o + 1 &= \mathbf{b}w_o \end{aligned}$$

---

<sup>7</sup>Epstein and Zin (1989) use a more general discrete-time version of these preferences as a way to distinguish risk aversion from intertemporal substitution.

$$\xi_f \bar{x}_f \mathbf{w}_f + \xi_o \bar{x}_o \mathbf{w}_o + \frac{(1-a)\sigma_o^2}{2} (\mathbf{w}_o)^2 + (1-a)\vartheta_o \sigma_o \mathbf{w}_o + \frac{(1-a)\vartheta_o^2}{2} = \mathbf{b}\bar{w}.$$

The equation for  $\mathbf{w}_f$  is quadratic and there are potentially two solutions. The solution that interests us is

$$\mathbf{w}_f = \frac{(a-1)\sigma_f \vartheta_f + \mathbf{b} + \xi_f - \sqrt{[(a-1)\sigma_f \vartheta_f + \mathbf{b} + \xi_f]^2 - (a-1)^2 \sigma_f^2 \vartheta_f^2}}{(1-a)\sigma_f^2}.$$

See appendix A. Further,  $\mathbf{w}_0 > 0$  and, as we show in appendix A,  $\mathbf{w}_f < 0$ .

The stochastic discount factor is the product of two multiplicative functionals. One has the same form as the Breeden model with a logarithmic instantaneous utility function. It is the exponential of:

$$A_t^B = - \int_0^t X_s^o ds - \mathbf{b}t - \int_0^t \sqrt{X_s^f} \vartheta_f dB_s^f - \int_0^t \vartheta_o dB_s^o.$$

The other is a martingale constructed from the forward-looking continuation value process. It is the exponential of:

$$A_t^w = (1-a) \int_0^t \sqrt{X_s^f} (\vartheta_f + \mathbf{w}_f \sigma_f) dB_s^f + (1-a) \int_0^t (\vartheta_o + \mathbf{w}_o \sigma_o) dB_s^o - \frac{(1-a)^2}{2} \int_0^t X_s^f (\vartheta_f + \mathbf{w}_f \sigma_f)^2 ds - \frac{(1-a)^2 (\vartheta_o + \mathbf{w}_o \sigma_o)^2}{2} t$$

We next consider a variety of ways in which multiplicative functionals can be used to build models of asset prices and to characterize the resulting implications.

### 3.4 Valuation functionals and returns

We use a special class of multiplicative functionals called valuation functionals to characterize local pricing. The result of this analysis will be the Markov version of a local risk-return tradeoff. A valuation functional is constructed to have the following property. If the future value of the process is the payout, the current value is the price of that payout. For instance a valuation process could be the result of continually reinvesting dividends in a primitive asset. Equivalently, it can be constructed by continually compounding realized returns to an investment. To characterize local pricing, we use valuation processes that are multiplicative functionals. Recall that the

product of two multiplicative functionals is a multiplicative functional. The following definition is motivated by the connection between the absence of arbitrage and the martingale properties of properly normalized prices.

**Definition 3.4.** *A valuation functional  $\{V_t : t \geq 0\}$  is a multiplicative functional such that the product functional  $\{V_t S_t : t \geq 0\}$  is a martingale.*

Provided that  $V$  is strictly positive, the associated gross returns over any horizon  $u$  can be calculated by forming ratios:

$$R_{t,t+u} = \frac{V_{t+u}}{V_{t-}}$$

Thus increment in the value functional scaled by the current (pre-jump) value gives an instantaneous net return. The martingale property of the product  $VS$  gives a local pricing restriction for returns.

To deduce a convenient and familiar risk-return relation, consider the multiplicative functional  $M = VS$  where  $V$  is parameterized by  $(\beta_v, \gamma_v, \kappa_v)$  and  $\{S_t : t \geq 0\}$  is parameterized by  $(\beta_s, \gamma_s, \kappa_s)$ . In particular, the implied net return evolution is:

$$\frac{dV_t}{V_{t-}} = \left[ \beta_v(X_{t-}) + \frac{|\gamma_v(X_{t-})|^2}{2} \right] dt + \gamma_v(X_{t-})' dB_t + \exp[\kappa_v(X_t, X_{t-})] - 1.$$

Thus the expected net rate of return is:

$$\varepsilon_v \doteq \beta_v + \frac{|\gamma_v|^2}{2} + \int (\exp[\kappa_v(y, \cdot)] - 1) \eta(dy|\cdot).$$

Since both  $V$  and  $S$  are exponentials of additive processes, their product is the exponential of an additive process and is parameterized by:

$$\begin{aligned} \beta &= \beta_v + \beta_s \\ \gamma &= \gamma_v + \gamma_s \\ \kappa &= \kappa_v + \kappa_s \end{aligned}$$

**Proposition 3.1.** *A valuation functional parameterized by  $(\beta_v, \gamma_v, \kappa_v)$  satisfies the pricing restriction:*

$$\beta_v + \beta_s = -\frac{|\gamma_v + \gamma_s|^2}{2} - \int (\exp[\kappa_v(y, \cdot) + \kappa_s(y, \cdot)] - 1) \eta(dy|\cdot). \quad (10)$$

*Proof.* This follows from the definition of a valuation functional and the martingale restriction (6).  $\square$

This restriction is local and determines the instantaneous risk-return relation. The parameters  $(\gamma_v, \kappa_v)$  determine the Brownian and jump risk exposure. The following corollary gives the required local mean rate of return:

**Corollary 3.1.** *The required mean rate of return for the risk exposure  $(\gamma_v, \kappa_v)$  is:*

$$\varepsilon_v = -\beta_s - \gamma_v \cdot \gamma_s - \frac{|\gamma_s|^2}{2} - \int (\exp[\kappa_v(y, \cdot) + \kappa_s(y, \cdot)] - \exp[\kappa_v(y, \cdot)]) \eta(dy, \cdot)$$

The vector  $-\gamma_s$  contains the factor risk prices for the Brownian motion components. The function  $\kappa_s$  is used to price exposure to jump risk. Then  $\varepsilon_v$  is the required expected rate of return expressed as a function of the risk exposure. This local relation is familiar from the extensive literature on continuous-time asset pricing.<sup>8</sup> In the case of Brownian motion risk, the local risk price vector of the exposure to risk is given by  $-\gamma_s$ .

A valuation functional is typically constructed from the values of a self-financing strategy. Not every self-financing strategy results in a valuation which can be written as a multiplicative functional, but the class of (multiplicative) valuation functionals is sufficiently rich to extract the implied local risk prices. For this reason we restrict ourselves in this paper to (multiplicative) valuation functionals.

**Example 3.10. Breeden example continued**

*Consider again the Markov diffusion example 3.2 with the stochastic discount factor given in example 3.8. This is a Markov version of Breeden's model. The local risk price for exposure to the vector of Brownian motion increments is:*

$$-\gamma_s = \begin{bmatrix} \mathbf{a}\sqrt{x_f}\vartheta_f \\ \mathbf{a}\vartheta_o \end{bmatrix}$$

*and the instantaneous risk-free rate is:*

$$\mathbf{b} + \mathbf{a}x_o - \frac{\mathbf{a}^2(x_f(\vartheta_f)^2 + (\vartheta_o)^2)}{2}.$$

---

<sup>8</sup>Shaliastovich and Tauchen (2005) present a structural model of asset prices in discrete time with a Levy component to the risk exposure. The continuous-time counterpart would include Markov processes with an infinite number jumps expected in any finite time interval.

Consider a family of valuation processes parameterized by  $(\beta, \gamma)$  where:  $\gamma(x) = (\sqrt{x_f}\gamma_f, \gamma_o)$ . To satisfy the martingale restriction, we must have:

$$\beta(x) = \mathbf{b} + \mathbf{a}x_o - \frac{1}{2} [x_f(\gamma_f - \mathbf{a}\vartheta_f)^2 + (\gamma_o - \mathbf{a}\vartheta_o)^2]$$

### Example 3.11. Kreps-Porteus model continued

Consider again the Markov diffusion example 3.2 with the stochastic discount factor given in example 3.9. The local risk price for exposure to the vector of Brownian motion increments is:

$$-\gamma_s = \begin{bmatrix} \mathbf{a}\sqrt{x_f}\vartheta_f + (\mathbf{a} - 1)\sqrt{x_f}\mathbf{w}_f\sigma_f \\ \mathbf{a}\vartheta_o + (\mathbf{a} - 1)\mathbf{w}_o\sigma_o \end{bmatrix},$$

and the instantaneous risk-free rate is:

$$\mathbf{b} + x_o - \frac{1}{2} [x_f(\vartheta_f)^2 + (\vartheta_o)^2] - (\mathbf{a} - 1)x_f\vartheta_f(\vartheta_f + \mathbf{w}_f\sigma_f) - (\mathbf{a} - 1)\vartheta_o(\vartheta_o + \mathbf{w}_o\sigma_o).$$

In particular, the local risk prices are larger than for their counterpart in the Breeden (1979) model when  $\vartheta_o$  and  $\vartheta_f$  are both positive.<sup>9</sup>

As we have seen, alternative valuation functionals reflect alternative risk exposures. The examples we just discussed show how the required expected rate of return ( $\beta_v$ ) for a given local risk-exposures  $(\gamma_v, \kappa_v)$  depends on the underlying economic model and the associated parameter values. The methods we will describe allow us to characterize the behavior of expectations of valuation functionals over long horizons.

## 3.5 Growth functionals

In our analysis of valuation, we will investigate the pricing of cash flows as we extend the payoff horizon. To investigate the value implications for cash flows that grow stochastically, we introduce a reference growth process:  $\{G_t : t \geq 0\}$  that is a positive multiplicative functional. Consider a cash flow that can be represented as

$$D_t = G_t\psi(X_t)D_0 \tag{11}$$

---

<sup>9</sup>When  $\vartheta_f$  is positive Kleshchelski and Vincent (2007) show that the real term structure will be often downward sloping.

for some initial condition  $D_0$  where  $G$  is a multiplicative functional. The initial condition is introduced to offset the restriction that multiplicative functionals are normalized to be unity at date zero. Heuristically, we may think of  $\psi(X)$  as the stationary component of the cash flow and  $G$  as the growth component.<sup>10</sup>As we will illustrate, however, the covariance between components sometimes makes this interpretation problematic.

The fact that the product of multiplicative functionals is a multiplicative functional implies that the product of a stochastic discount factor functional and a growth functional is itself multiplicative. This property facilitates the construction of valuation operators designed for cash flow processes that grow stochastically over time.

In Section 2 we emphasized the connection between the multiplicative property of stochastic discount factors and the semigroup property of pricing operators. In the next section we discuss how multiplicative functionals give rise to semigroups. This development lays the groundwork for considering a variety of ways in which multiplicative functionals and their implied semigroups can be used to characterize the implications of asset pricing models over long horizons.

## 4 Multiplicative functionals and semigroups

Given a multiplicative functional  $M$ , our aim is to establish properties of the family of operators:

$$\mathbb{M}_t\psi(x) = E[M_t\psi(X_t)|X_0 = x]. \quad (12)$$

### 4.1 Semigroups

Let  $L$  be a Banach space with norm  $\|\cdot\|$ , and let  $\{\mathbb{T}_t : t \geq 0\}$  be a family of operators on  $L$ . The operators in these family are linked according to the following property:

**Definition 4.1.** *A family of linear operators  $\{\mathbb{T}_t : t \geq 0\}$  is a one-parameter **semigroup** if  $\mathbb{T}_0 = \mathbb{I}$  and  $\mathbb{T}_{t+s} = \mathbb{T}_t\mathbb{T}_s$  for all  $s, t \geq 0$ .*

One possibility is that these operators are conditional expectations operators, in which case this link typically follows from the Law of Iterated Expectations restricted to

---

<sup>10</sup>One can easily write down securities with a payout that can not be represented as in equation (11), but we are interested in deriving properties of the pricing of securities with a payout as in (11) to evaluate alternative models and parameter configurations.

Markov processes. We will also use such families of operators to study valuation and pricing. As we argued in section 2, from a pricing perspective, the semigroup property follows from the Markov version of the Law of Iterated Values, which holds when there is frictionless trading at intermediate dates.

We will often impose further restrictions on semigroups such as:

**Definition 4.2.** *The semigroup  $\{\mathbb{T}_t : t \geq 0\}$  is **positive** if for any  $t \geq 0$ ,  $\mathbb{T}_t\psi \geq 0$  whenever  $\psi \geq 0$ .*

## 4.2 Multiplicative semigroup

The semigroups that interest us are constructed from multiplicative functionals.

**Proposition 4.1.** *Let  $M$  be a multiplicative functional such that for each  $\psi \in L$ ,  $E[M_t\psi(X_t)|X_0 = x] \in L$ . Then*

$$\mathbb{M}_t\psi(x) = E[M_t\psi(X_t)|X_0 = x]$$

*is a semigroup in  $L$ .*

*Proof.* For  $\psi \in L$ ,  $\mathbb{M}_0\psi = \psi$  and:

$$\begin{aligned} \mathbb{M}_{t+u}\psi(x) &= E[E[M_{t+u}\psi(X_{t+u})|\mathcal{F}_t]|X_0 = x] \\ &= E[E[M_t M_u(\theta_t)\psi(X_u(\theta_t))|\mathcal{F}_t]|X_0 = x] \\ &= E[M_t E[M_u(\theta_t)\psi(X_u(\theta_t))|X_0(\theta_t)]|X_0 = x] \\ &= E[M_t \mathbb{M}_u\psi(X_t)|X_0 = x] \\ &= \mathbb{M}_t \mathbb{M}_u\psi(x), \end{aligned}$$

which establishes the semigroup property. □

In what follows we will refer to semigroups constructed from multiplicative functionals as in this proposition as *multiplicative semigroups*. If the multiplicative process is a stochastic discount factor we will refer to the corresponding multiplicative semigroup as the *pricing semigroup*. Other semigroups also interest us.

## 4.3 Valuation semigroups

Associated with a valuation functional  $V$  is a semigroup  $\{\mathbb{V}_t : t \geq 0\}$ . For any such a valualational functional, we will derive the asymptotic growth rates of the implied

cumulative return over long time horizons. The limiting growth rate expressed as a function of the risk exposures  $(\gamma_v, \kappa_v)$  gives one version of long-term risk-return tradeoff. While measurement of long-horizon returns in log-linear environments has commanded much attention, operator methods can accommodate volatility movements as well. (See Bansal et al. (2007) for a recent addition to this literature.)

Our characterization of the long-run expected rate of return is motivated by our aim of quantifying a risk-return relation. In contrast Stutzer (2003) uses the conditional expectation of a valuation functional raised to a negative power in developing a large deviation criterion for portfolio evaluation over large horizons. He also relates this formulation to the familiar power utility model applied to terminal wealth appropriately scaled. Since a multiplicative functional raised to a negative power remains multiplicative, the limits we characterize are also germane to his analysis.

In what follows we will suggest another way to represent a long-term risk return tradeoff.

#### 4.4 Semigroups induced by cash flow growth

We study cash flows with a common growth component using the semigroup:

$$\mathbb{Q}_t\psi(x) = E[G_t S_t \psi(X_t) | X_0 = x]$$

instead of the pricing semigroup  $\{\mathbb{S}_t\}$  constructed previously. The date zero price assigned to  $D_t$  is  $D_0\mathbb{Q}_t\psi(X_0)$ . More generally, the date  $\tau$  price assigned to  $D_{t+\tau}$  is  $D_0G_\tau\mathbb{Q}_t\psi(X_\tau)$ . Thus the date  $\tau$  price to (current period) payout ratio is

$$\frac{D_0G_\tau\mathbb{Q}_t\psi(X_\tau)}{D_\tau} = \frac{\mathbb{Q}_t\psi(X_\tau)}{\psi(X_\tau)}$$

provided that  $\psi(X_\tau)$  is different from zero. For a security such as an equity with a perpetual process of cash payouts or dividends, the price-dividend ratio is the integral of all such terms for  $t \geq 0$ . Our subsequent analysis will characterize the limiting contribution to this value. The rate of decay of  $\mathbb{Q}_t\psi(X_\tau)$  as  $t$  increases will give a measure of the *duration* of the cash flow as it contributes to the value of the asset.

This semigroup assigns values to cash flows with common growth component  $G$  but alternative transient contributions  $\psi$ . To study how valuation is altered when we change stochastic growth, we will be led to alter the semigroup.

When the growth process is degenerate and equal to unity, the semigroup is identical to the one constructed previously in section 2. This semigroup is useful in studying the valuation of stationary cash flows including discount bonds and the term structure of interest rates. It supports local pricing and generalizations of the analyses in Backus and Zin (1994) and Alvarez and Jermann (2005) that use fixed income securities to make inferences about economic fundamentals. This semigroup offers a convenient benchmark for the study of long-term risk just as a risk free rate offers a convenient benchmark in local pricing.

The decomposition (11) used in this semigroup construction is not unique. For instance, let  $\varphi$  be a strictly positive function of the Markov state. Then

$$D_t = G_t \psi(X_t) D_0 = \left[ G_t \frac{\varphi(X_t)}{\varphi(X_0)} \right] \left[ \frac{\psi(X_t)}{\varphi(X_t)} \right] [D_0 \varphi(X_0)].$$

Since  $\frac{\psi(X_t)}{\varphi(X_t)}$  is a transient component, we can produce (infinitely) many such decompositions. For decomposition (11) to be unique, we must thus restrict the growth component.

A convenient restriction is to require that  $G_t = \exp(\delta t) \hat{G}_t$  where  $\hat{G}$  is a martingale. With this choice, by construction  $G$  has a constant conditional growth rate  $\delta$ . Later we show how to extract martingale components,  $\hat{G}$ 's, from a large class of multiplicative functionals  $G$ . In this way we will establish the existence of such a decomposition. Even with this restriction, the decomposition will not necessarily be unique, but we will justify a particular choice.

We investigate long-term risk by changing the reference growth functionals. These functionals capture the long-term risk exposure of the cash flow. Our approach extends the analysis of Hansen et al. (2008) beyond log-linear environments. As we will demonstrate, the valuation of cash flows with common reference growth functionals will be approximated by a single dominant component when the valuation horizon becomes long. Thus the contributions to value that come many periods into the future will be approximated by a single pricing factor that incorporates an adjustment for risk. Changing the reference growth functional alters the long-term risk exposure with a corresponding adjustment in valuation. Each reference growth functional will be associated with a distinct semigroup. We will characterize long-term risk formally by studying the limiting behavior of the corresponding semigroup.

As we have just seen, semigroups used for valuing growth claims are constructed

object	multiplicative functional	semigroup
stochastic discount factor	$S$	$\{S_t\}$
cumulated return	$V$	$\{V_t\}$
stochastic growth	$G$	$\{G_t\}$
valuation with stochastic growth	$Q = GS$	$\{Q_t\}$

Table 1: Alternative semigroups and multiplicative functionals

by forming products of two multiplicative functionals: a stochastic discount factor functional and a growth functional. Pricing stationary claims and constructing cumulative returns lead to the construction of alternative multiplicative functionals. Table 1 gives a summary of the alternative multiplicative functionals and semigroups. For this reason, we will study the behavior of a general multiplicative semigroup:

$$\mathbb{M}_t\psi(x) = E [M_t\psi(X_t)|X_0 = x]$$

for some strictly positive multiplicative functional  $M$ .

The next three sections establish some basic representation and approximation results for multiplicative semigroups that are needed for our subsequent economic analysis of long-term risk. An important vehicle in this study is the *extended generator* associated with a multiplicative process. This generator is a local (in time) construct. We develop its properties in section 5. In section 6 we show how to use a principal eigenfunction of this extended generator to construct our basic multiplicative decomposition (1). As we demonstrate in section 7, an appropriately chosen eigenfunction and its associated eigenvalue dictate the long-term behavior of a multiplicative semigroup and the corresponding multiplicative functional. After establishing these basic results, we turn to the featured application in our paper: how do we characterize the long-term risk-return tradeoff.

## 5 Generators

In this section we define a notion of an *extended generator* associated with a multiplicative functional. The definition parallels the definition of an extended (infinitesimal) generator associated with Markov processes as in *e.g.* Revuz and Yor (1994). Our extended generator associates to each function  $\psi$  a function  $\chi$  such that  $M_t\chi(X_t)$  is the “expected time derivative” of  $M_t\psi(X_t)$ .

**Definition 5.1.** A Borel function  $\psi$  belongs to the domain of the **extended generator**  $\mathbb{A}$  of the multiplicative functional  $M$  if there exists a Borel function  $\chi$  such that  $N_t = M_t\psi(X_t) - \psi(X_0) - \int_0^t M_s\chi(X_s)ds$  is a local martingale with respect to the filtration  $\{\mathcal{F}_t : t \geq 0\}$ . In this case the extended generator assigns the function  $\chi$  to  $\psi$ , and we write  $\chi = \mathbb{A}\psi$ .

For strictly positive multiplicative processes  $M$  the extended generator is (up to sets of measure zero) single valued and linear. In the remainder of the paper, if the context is clear, we often refer to the extended generator simply as the generator.

Our first example deals with Markov chains:

**Example 5.1. Markov chain generator**

Recall the finite state Markov chain example 3.1 with intensity matrix  $\mathbb{U}$ . Let  $u_{ij}$  denote entry  $(i, j)$  of this matrix. Consider a multiplicative functional that is the product of two components. The first component decays at rate  $\beta_i$  when the Markov state is  $x_i$ . The second component only changes when the Markov process jumps from state  $i$  to state  $j$ , in which case the multiplicative functional is scaled by  $\exp[\kappa(x_j, x_i)]$ . From this construction we can deduce the generator  $\mathbb{A}$  for the multiplicative semigroup depicted as a matrix with entry  $(i, j)$ :

$$a_{ij} = \begin{cases} u_{ii} - \beta_i & \text{if } i = j \\ u_{ij} \exp[\kappa(x_j, x_i)] & \text{if } i \neq j \end{cases}$$

This formula uses the fact that in computing the generator we are scaling probabilities by the potential proportional changes in the multiplicative functional. The matrix  $\mathbb{A}$  is not necessarily an intensity matrix. The row sums are not necessarily zero. The reason for this is that the multiplicative functional can include pure discount effects or pure growth effects. These effects can be present even when the  $\beta_i$ 's are zero since it is typically the case that

$$\sum_{j \neq i} u_{ij} \exp[\kappa(x_j, x_i)] \neq -u_{ii}.$$

The unit function is a trivial example of a multiplicative functional. In this case the extended generator is exactly what is called in the literature the extended generator of the Markov process  $X$ . When  $X$  is parameterized by  $(\eta, \xi, \Gamma)$  Ito's formula shows that the generator has the representation:

$$\mathbb{A}\phi(x) = \xi(x) \cdot \frac{\partial\phi(x)}{\partial x} + \frac{1}{2}\text{trace} \left( \Sigma(x) \frac{\partial^2\phi(x)}{\partial x \partial x'} \right) + \int [\phi(y) - \phi(x)] \eta(dy|x). \quad (13)$$

where  $\Sigma = \Gamma\Gamma'$  provided  $\phi$  is  $C^2$  and the integral in (13) is finite.

Recall our earlier parameterization of an additive functional  $A$  in terms of the triple  $(\beta, \gamma, \kappa)$ . The process  $M = \exp(A)$  is a multiplicative functional. We now display how to go from the extended generator of the Markov process  $X$ , that is the generator associated with  $M \equiv 1$ , to the extended generator of the multiplicative functional  $M$ . The formulas below use the parameterization for the multiplicative process to transform the generator of the Markov process into the generator of the multiplicative semigroup and are consequences of Ito's lemma:

- a) jump measure:  $\exp[\kappa(y, x)] \eta(dy|x)$ .
- b) first derivative term:  $\xi(x) + \Gamma(x)\gamma(x)$ ;
- c) second derivative term:  $\Sigma(x)$ ;
- d) level term:  $\beta(x) + \frac{|\gamma(x)|^2}{2} + \int (\exp[\kappa(y, x)] - 1) \eta(dy, x)$ ;

The Markov chain example that we discussed above can be seen as a special case where  $\gamma$ ,  $\xi$ , and  $\Gamma$  are all null.

There are a variety of direct applications of this analysis. In the case of the stochastic discount factor introduced in section 3.3, the generator encodes the local prices reflected in the local risk-return tradeoff of Proposition 3.1. The level term that arises gives the instantaneous version of a risk free rate. In the absence of jump risk, the increment to the drift gives the factor risk prices. The function  $\kappa$  shows us how to value jump risk in small increments in time.

In a further application, Anderson et al. (2003) use this decomposition to characterize the relation among four alternative semigroups, each of which is associated with an alternative multiplicative process. Anderson et al. (2003) feature models of robust decision making. In addition to the generator for the original Markov process, a second generator depicts the worst case Markov process used to support the robust equilibrium. There is a third generator of an equilibrium pricing semigroup, and a fourth generator of a semigroup used to measure the statistical discrepancy between the original model and the worst-case Markov model.

## 6 Principal eigenfunctions and martingales

As stated in the introduction, we use a decomposition of the multiplicative functional to study long-run behavior. We construct this decomposition using an appropriate eigenfunction of the generator associated to the multiplicative functional.

**Definition 6.1.** *A Borel function  $\phi$  is an **eigenfunction** of the extended generator  $\mathbb{A}$  with **eigenvalue**  $\rho$  if  $\mathbb{A}\phi = \rho\phi$ .*

Intuitively if  $\phi$  is an eigenfunction, the “expected time derivative” of  $M_t\phi(X_t)$  is  $\rho M_t\phi(X_t)$ . Hence the expected time derivative of  $\exp(-\rho t)M_t\phi(X_t)$  is zero. The next proposition formalizes this intuition.

**Proposition 6.1.** *Suppose that  $\phi$  is an eigenfunction of the extended generator associated with the eigenvalue  $\rho$ . Then*

$$\exp(-\rho t)M_t\phi(X_t)$$

*is a local martingale.*

*Proof.*  $N_t = M_t\psi(X_t) - \psi(X_0) - \rho \int_0^t M_s\psi(X_s)ds$  is a local martingale that is continuous from the right with left limits and thus a semimartingale (Protter (2005), Chapter 3, Corollary to Theorem 26) and hence  $Y_t = M_t\phi(X_t)$  is also a semimartingale. Since  $dN_t = dY_t - \rho Y_{t-}dt$ , integration by parts yields:

$$\exp(-\rho t)Y_t - Y_0 = - \int_0^t \rho \exp(-\rho s)Y_s ds + \int_0^t \exp(-\rho s)dY_s = \int_0^t \exp(-\rho s)dN_s.$$

□

It is the strictly positive eigenfunctions that interest us.

**Definition 6.2.** *A **principal eigenfunction** of the extended generator is an eigenfunction that is strictly positive.*

**Corollary 6.1.** *Suppose that  $\phi$  is a principal eigenfunction with eigenvalue  $\rho$  for the extended generator of the multiplicative functional  $M$ . Then this multiplicative functional can be decomposed as:*

$$M_t = \exp(\rho t)\hat{M}_t \left[ \frac{\phi(X_0)}{\phi(X_t)} \right].$$

where  $\hat{M}_t = \exp(-\rho t)M_t \frac{\phi(X_t)}{\phi(X_0)}$  is a local martingale and a multiplicative functional.

Let  $\hat{M}$  be the local martingale from Corollary 6.1. Since  $\hat{M}$  is bounded from below, the local martingale is necessarily a supermartingale and thus for  $t \geq u$ ,

$$E\left(\hat{M}_t | \mathcal{F}_u\right) \leq \hat{M}_u.$$

We are primarily interested in the case in which this local martingale is actually a martingale:

**Assumption 6.1.** *The local martingale  $\hat{M}$  is a martingale.*

For Assumption 6.1 to hold it suffices that the local martingale  $N$  introduced in the proof of Proposition 6.1 is a martingale. In appendix C we give primitive conditions that imply Assumption 6.1.

When Assumption 6.1 holds we may use  $\hat{M}$  to a new probability on the sigma algebra generated by  $\mathcal{F}_t$  for each  $t$ . Later, we will use this new probability to establish approximation results that hold for long horizons.

We did not restrict  $\phi$  to belong to the Banach space  $L$  where the semigroup  $\{\mathbb{M}_t : t \geq 0\}$  was defined. Since  $\hat{M}$  is a supermartingale,  $\mathbb{M}_t \phi := \exp(\rho t) \phi(x) E[\hat{M}_t | X_0 = x]$  is always well defined. In addition, the semigroup  $\{\hat{\mathbb{M}}_t : t \geq 0\}$  is well defined at least on the Banach space of bounded Borel measurable functions. Moreover:

**Proposition 6.2.** *If  $\phi$  is a principal eigenfunction with eigenvalue  $\rho$  for the extended generator of the multiplicative functional  $M$ , then for each  $t \geq 0$ ,  $\exp(\rho t) \phi \geq \mathbb{M}_t \phi$ . If, in addition, Assumption 6.1 holds then, for each  $t \geq 0$*

$$\mathbb{M}_t \phi = \exp(\rho t) \phi. \tag{14}$$

*Conversely, if  $\phi$  is strictly positive,  $\mathbb{M}_t \phi$  is well defined for  $t \geq 0$ , and (14) holds, then  $\hat{M}$  is a martingale.*

*Proof.*

$$1 \geq E[\hat{M}_t | X_0 = x] = \frac{\exp(-\rho t)}{\phi(x)} E[M_t \phi(X_t) | X_0 = x],$$

with equality when  $\hat{M}$  is a martingale. Conversely, using (14) and the multiplicative property of  $M$  one obtains,

$$E[\exp(-\rho t) M_t \phi(X_t) | \mathcal{F}_s] = \exp(-\rho t) M_s E[M_{t-s}(\theta_s) \phi(X_t) | X_s] = \exp(-\rho s) M_s \phi(X_s)$$

□

This proposition guarantees that under Assumption 6.1 a principal eigenfunction of the extended generator also solves the principal eigenvalue problem given by (14). Conversely, a strictly positive solution to the principal eigenvalue problem (14) yields a decomposition as in Corollary 6.1, where the process  $\hat{M}$  is actually a martingale.

In light of the decomposition given by Corollary 6.1, when the local martingale  $\hat{M}$  is a martingale, we will sometimes refer to  $\rho$  as the growth rate of the multiplicative functional  $M$ ,  $\hat{M}$  as its martingale component and  $\frac{\phi(X_0)}{\phi(X_t)}$  as its transient or stationary component. This decomposition is typically not unique, however. As we have defined them, there may be multiple principal eigenfunctions even after a normalization. Each of these principal eigenfunctions implies a distinct decomposition, provided that we establish that the associated local martingales are martingales. Since the martingale and the stationary components are correlated, it can happen that  $E \left[ \hat{M}_t \frac{\phi(X_0)}{\phi(X_t)} | X_0 = x \right]$  diverges exponentially challenging the interpretation that  $\rho$  is the asymptotic growth rate of the semigroup. We take up this issue in the next section.

**Remark 6.1.** *There are well known martingale decompositions of additive functionals with stationary increments used in deducing central limit approximation and in characterizing the role of permanent shocks in time series. The nonlinear, continuous time Markov version of such a decomposition is:*

$$A_t = \omega t + m_t - v(X_t) + v(X_0)$$

where  $\{m_t : t \geq 0\}$  is a martingale with stationary increments (see Bhattacharya (1982) and Hansen and Scheinkman (1995)). Exponentiating this decomposition is of a similar type to that given in Corollary 6.1 except that the exponential of a martingale is not a martingale. When the martingale increments are constant functions of Brownian increments, then exponential adjustment has simple consequences.<sup>11</sup> In particular, the exponential adjustment is offset by changing  $\omega$ . With state dependent volatility in the martingale approximation, however, there is no longer a direct link between the additive and multiplicative decompositions. In this case the multiplicative decomposition of Corollary 6.1 is the one that is valuable for our purposes.

### Example 6.1. Markov chain example

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<sup>11</sup>This is the case studied by Hansen et al. (2008).

Recall that for a finite state space, we can represent the Markov process in terms of a matrix  $\mathbb{U}$  that serves as its generator. Previously we constructed the corresponding generator  $\mathbb{A}$  of the multiplicative semigroup. For this example, the generator is a matrix. A principal eigenvector is found by finding an eigenvector of  $\mathbb{A}$  with strictly positive entries. Standard Perron-Frobenius theory implies that if the chain is irreducible, since the multiplicative functional is strictly positive, there is such an eigenvector which is unique up to scale.

While there is uniqueness in the case of an irreducible finite state chain, there can be multiple solutions in more general settings.

**Example 6.2. Markov diffusion example continued**

Consider a multiplicative process  $M = \exp(A)$  where:

$$A_t = \bar{\beta}t + \int_0^t \beta_f X_s^f ds + \int_0^t \beta_o X_s^o ds + \int_0^t \sqrt{X_s^f} \gamma_f dB_s^f + \int_0^t \gamma_o dB_s^o, \quad (15)$$

where  $X^f$  and  $X^o$  are given in Example 3.2.

Guess an eigenfunction of the form:  $\exp(\mathbf{c}_f x_f + \mathbf{c}_o x_o)$ . The corresponding eigenvalue equation is:

$$\begin{aligned} \rho &= \bar{\beta} + \beta_f x_f + \beta_o x_o + \frac{\gamma_f^2}{2} x_f + \frac{\gamma_o^2}{2} \\ &\quad + \mathbf{c}_f [\xi_f (\bar{x}_f - x_f) + x_f \gamma_f \sigma_f] + \mathbf{c}_o [\xi_o (\bar{x}_o - x_o) + \gamma_o \sigma_o] \\ &\quad + (\mathbf{c}_f)^2 x_f \frac{\sigma_f^2}{2} + (\mathbf{c}_o)^2 \frac{\sigma_o^2}{2} \end{aligned}$$

This generates two equations: one that equates the coefficients of  $x_f$  to zero and another that equates the coefficients of  $x_o$  to zero:

$$\begin{aligned} 0 &= \beta_f + \frac{\gamma_f^2}{2} + \mathbf{c}_f (\gamma_f \sigma_f - \xi_f) + (\mathbf{c}_f)^2 \frac{\sigma_f^2}{2} \\ 0 &= \beta_o - \mathbf{c}_o \xi_o. \end{aligned}$$

The solution to the first equation is:

$$\mathbf{c}_f = \frac{(\xi_f - \gamma_f \sigma_f) \pm \sqrt{(\xi_f - \gamma_f \sigma_f)^2 - \sigma_f^2 (2\beta_f + \gamma_f^2)}}{(\sigma_f)^2} \quad (16)$$

provided that

$$(\xi_f - \gamma_f \sigma_f)^2 - \sigma_f^2 (2\beta_f + \gamma_f^2) \geq 0.$$

The solution to the second equation is:

$$c_o = \frac{\beta_o}{\xi_o}. \quad (17)$$

The resulting eigenvalue is:

$$\rho = \bar{\beta} + \frac{\gamma_o^2}{2} + c_f \xi_f \bar{x}_f + c_o (\xi_o \bar{x}_o + \gamma_o \sigma_o) + (c_o)^2 \frac{\sigma_o^2}{2}.$$

Write

$$\hat{M}_t = \exp(-\rho t) M_t \frac{\exp(c_f X_t^f + c_o X_t^o)}{\exp(c_f X_0^f + c_o X_0^o)},$$

Since  $\hat{M}$  is multiplicative we can express it as  $\hat{M}_t = \exp(\hat{A}_t)$  where:

$$\hat{A}_t = \int_0^t \sqrt{X_s^f} (\gamma_f + c_f \sigma_f) dB_s^f + \int_0^t (\gamma_o + c_o \sigma_o) dB_s^o - \frac{(\gamma_f + c_f \sigma_f)^2}{2} \int_0^t X_s^f ds - \frac{(\gamma_o + c_o \sigma_o)^2}{2} t.$$

Also, it can be shown that  $\hat{M}$  is always a martingale. Hence we may use this martingale to define a new probability measure. Using this new measure entails appending an extra drift to law of motion of  $X$ . The resulting distorted or twisted drift for  $X^f$  is

$$\xi_f (\bar{x}_f - x_f) + x_f \sigma_f (\gamma_f + c_f \sigma_f),$$

and the drift for  $X^o$  is:

$$\xi_o (\bar{x}_o - x_o) + \sigma_o (\gamma_o + c_o \sigma_o).$$

Later we will argue that only one of these solutions interests us. We will select a solution for  $c_f$  so that the implied distorted process for  $X^f$  remains stationary. Notice that

$$\xi_f (\bar{x}_f - x_f) + x_f \sigma_f (\gamma_f + c_f \sigma_f) = \xi_f \bar{x}_f \pm x_f \sqrt{(\xi_f - \gamma_f \sigma_f)^2 - \sigma_f^2 (2\beta_f + \gamma_f^2)}.$$

For mean reversion to exist, we require that the coefficient on  $x_f$  be negative.

**Remark 6.2.** At a cost of an increase in notational complexity we could add an “affine” jump component as in Duffie et al. (2000). Suppose that the state variable

$X^o$  instead of being an Ornstein-Uhlenbeck process satisfies:

$$dX_t^o = \xi_o(\bar{x}_o - X_t^o)dt + \sigma_o dB_t^o + dZ_t$$

where  $Z$  is a pure jump process whose jumps have a fixed probability distribution  $\nu$  on  $\mathbb{R}$  and arrive with intensity  $\varpi_1 x_f + \varpi_2$  with  $\varpi_1 \geq 0$ ,  $\varpi_2 \geq 0$ . Suppose that the additive functional  $A$  has an additional jump term modeled using  $\kappa(y, x) \doteq \bar{\kappa}(y - x_o)$  for  $y \neq x$  and  $\int \exp[\bar{\kappa}(z)] d\nu(z) < \infty$ .

The generator  $\mathbb{A}$  has now an extra term given by:

$$(\varpi_1 x_f + \varpi_2) \int [\phi(x_f, x_o + z) - \phi(x_f, x_o)] \exp[\bar{\kappa}(z)] d\nu(z).$$

Hence when  $\phi(x) = \exp(\mathbf{c}_f x_f + \mathbf{c}_o x_o)$  the extra term reduces to:

$$(\varpi_1 x_f + \varpi_2) \exp(\mathbf{c}_f x_f + \mathbf{c}_o x_o) \int [\exp(\mathbf{c}_o z) - 1] \exp[\bar{\kappa}(z)] d\nu(z).$$

As before we must have

$$\mathbf{c}_o = \frac{\beta_o}{\xi_o}$$

and hence  $\mathbf{c}_f$  must solve:

$$0 = \beta_f + \frac{\gamma_f^2}{2} + \mathbf{c}_f(\gamma_f \sigma_f - \xi_f) + (\mathbf{c}_f)^2 \frac{\sigma_f^2}{2} + \varpi_1 \int \left[ \exp\left(\frac{\beta_o}{\xi_o} z\right) - 1 \right] \exp[\bar{\kappa}(z)] d\nu(z).$$

The resulting eigenvalue is

$$\rho = \bar{\beta} + \frac{\gamma_o^2}{2} + \mathbf{c}_f \xi_f \bar{x}_f + \mathbf{c}_o (\xi_o \bar{x}_o + \gamma_o \sigma_o) + (\mathbf{c}_o)^2 \frac{\sigma_o^2}{2} + \varpi_2 \int \left[ \exp\left(\frac{\beta_o}{\xi_o} z\right) - 1 \right] \exp[\bar{\kappa}(z)] d\nu(z).$$

## 7 Long-run dominance

In this section we establish approximation results for semigroups that apply over long time horizons. The limiting result we justify is

$$\lim_{t \rightarrow \infty} \exp(-\rho t) \mathbb{M}_t \psi = \phi \int \frac{\psi}{\phi} d\hat{\zeta}, \quad (18)$$

where the limit is expressed in terms of principal eigenvalue  $\rho$  and eigenfunction  $\phi$  for a collection of functions  $\psi$  and a measure  $\hat{\varsigma}$  that we will characterize. We have illustrated that there may be multiple principle eigenfunctions. We show that at most one of these principle eigenfunctions is the one germane for establishing this limiting behavior. In light of (18), the eigenvalue  $\rho$  governs the growth (or decay) of the semigroup. When we rescale the semigroup to eliminate this growth (decay), the limiting state dependence is proportional to the dominant eigenfunction  $\phi$  (which is itself only determined up to a scale factor) for alternative functions  $\psi$ . The precise characterization of this limiting behavior of the semigroup are the fundamental inputs into our characterization of valuation over long time horizons. It provides us with a measure of long-term growth rates is asset payoffs and of long-term decay rates in the values assigned to these payoffs. It gives us ways to formalize long-term risk-return tradeoffs for nonlinear Markov models as we will show in the next section.

Prior to our more general investigation, we first illustrate the results in the case of a Markov chain.

## 7.1 Markov chain

Consider the finite state Markov chain example with intensity matrix  $\mathbb{U}$ . In this section we will study the long-run behavior of the semigroup by solving the eigenvalue problem:

$$\mathbb{A}\phi = \rho\phi$$

for an eigenvector  $\phi$  with strictly positive entries and a real eigenvalue  $\rho$ . This solution exists whenever the chain is irreducible and the multiplicative functional is strictly positive. Given this solution, then

$$\mathbb{M}_t\phi = \exp(t\mathbb{A})\phi = \exp(\rho t)\phi.$$

The beauty of Perron-Frobenius theory is that  $\rho$  is the eigenvalue that dominates in the long run. Its real part is strictly larger than the real parts of all of the other eigenvalues. This property dictates its dominant role. To see this, suppose for simplicity that the matrix  $\mathbb{A}$  has distinct eigenvalues:

$$\mathbb{A} = \mathbb{T}\mathbb{D}\mathbb{T}^{-1}.$$

where  $\mathbb{T}$  is a matrix with eigenvectors in each column and  $\mathbb{D}$  is a diagonal matrix of eigenvalues. Then

$$\exp(t\mathbb{A}) = \mathbb{T} \exp(t\mathbb{D}) \mathbb{T}^{-1}.$$

Let the first entry of  $\mathbb{D}$  be  $\rho$  and the first column of  $\mathbb{T}$  be  $\phi$ . Scaling by  $\exp(-\rho t)$  and taking limits:

$$\lim_{t \rightarrow \infty} \exp(-\rho t) \exp(t\mathbb{A}) \psi = (\phi^* \cdot \psi) \phi,$$

where  $\phi^*$  is the first row of  $\mathbb{T}^{-1}$ . Thus  $\rho$  determines the long-run growth rate of the semigroup. After adjusting for this growth, the semigroup has an approximate one factor structure in the long run. Provided that  $\phi^* \cdot \psi$  is not zero,  $\exp(-\rho t) \mathbb{M}_t \psi$  is asymptotically proportional to the dominant eigenvector  $\phi$ .

## 7.2 General analysis

To establish this dominance more generally, we use the martingale construction as in the decomposition of Corollary 6.1 to build an alternative family of distorted Markov transition operators and apply known results about Markov operators to this alternative family.

In what follows we will maintain Assumption 6.1 and let  $\hat{\mathbb{A}}$  denote the extended generator of the martingale  $\hat{M}$ . We will also call the semigroup  $\hat{\mathbb{M}}$  associated with  $\hat{M}$  the *principal eigenfunction semigroup*. This semigroup is well defined at least on the space  $L^\infty$ , and it maps constant functions into constant functions.

Consistent with the applications that interest us, we consider only multiplicative functionals that are strictly positive.

**Assumption 7.1.** *The multiplicative functional  $M$  is strictly positive with probability one.*

As we mentioned earlier, the martingale  $\hat{M}$  can be used to define a new measure on sets  $f \in \mathcal{F}_t$  for any  $t$ . We are interested in the case where we may initialize the process  $X$  such that, under the new probability measure, the process  $X$  is stationary. The next assumption guarantees that this is possible:

**Assumption 7.2.** *There exists a probability measure  $\hat{\zeta}$  such that*

$$\int \hat{\mathbb{A}} \psi d\hat{\zeta} = 0$$

for all  $\psi$  in the  $L^\infty$  domain of the generator  $\hat{\mathbb{A}}$ .

We write  $\hat{E}$  and  $\hat{Pr}$  for the expectation operator and the probability measure obtained when we use  $\hat{\zeta}$  as the distribution of the initial state  $X_0$  and the martingale  $\hat{M}$  to distort the transition probabilities, that is, for each event  $f \in \mathcal{F}_t$

$$\hat{Pr}(f) = \int E[\hat{M}_t \mathbf{1}_f | X_0 = x] d\hat{\zeta}(x).$$

For each  $x$ , the probability  $\hat{Pr}(\cdot | X_0 = x)$  is absolutely continuous with respect to probability measure implied by  $Pr$  conditioned on  $X_0 = x$  when restricted to  $\mathcal{F}_t$  for each  $t \geq 0$ . Assumption 7.2 guarantees that  $\hat{\zeta}$  is a stationary distribution for the distorted Markov process. (For example, see Proposition 9.2 of Ethier and Kurtz (1986).) Furthermore,

$$E[M_t \psi(X_t) | X_0 = x] = \exp(\rho t) \phi(x) \hat{E} \left[ \frac{\psi(X_t)}{\phi(X_t)} | X_0 = x \right]. \quad (19)$$

If we treat  $\exp(-\rho t) \phi(X_t)$  as a *numeraire*, equation (19) is reminiscent of the familiar risk-neutral pricing in finance. Note, however, that the numeraire depends on the eigenvalue-eigenfunction pair, and equation (19) applies even when the multiplicative process does not define a price.<sup>12</sup>

Let  $\hat{\Delta} > 0$  and consider the discrete time Markov process obtained by sampling the process at  $\hat{\Delta}j$  for  $j = 0, 1, \dots$ . This discrete process is often referred to as a *skeleton*. In what follows we assume that the resulting discrete time process is *irreducible*.

**Assumption 7.3.** *There exists a  $\hat{\Delta} > 0$  such that the discretely sampled process  $\{X_{\hat{\Delta}j} : j = 0, 1, \dots\}$  is **irreducible**. That is, for any Borel set  $\Lambda$  of the state space  $\mathcal{D}_0$  with  $\hat{\zeta}(\Lambda) > 0$ ,*

$$\hat{E} \left[ \sum_{j=0}^{\infty} \mathbf{1}_{\{X_{\hat{\Delta}j} \in \Lambda\}} | X_0 = x \right] > 0$$

for all  $x \in \mathcal{D}_0$ .

Under Assumption 7.1 it is equivalent to assume that this irreducibility restriction holds under the original probability measure.<sup>13</sup>

<sup>12</sup>The idea of using an appropriately chosen eigenfunction of an operator to construct and analyze a twisted probability measure is also featured in the work of Kontoyiannis and Meyn (2003).

<sup>13</sup>Irreducibility and Harris recurrence are defined relative to a measure. This claim uses the

We establish approximation results by imposing a form of stochastic stability under the distorted probability measure. We assume that the distorted Markov process satisfies:

**Assumption 7.4.** *The process  $X$  is **Harris recurrent** under the measure  $\hat{P}r$ . That is, for any Borel set  $\Lambda$  of the state space  $\mathcal{D}_0$  with positive  $\hat{\zeta}$  measure,*

$$\hat{P}r \left\{ \int_0^\infty \mathbf{1}_{\{X_t \in \Lambda\}} dt = \infty \mid X_0 = x \right\} = 1$$

for all  $x \in \mathcal{D}_0$ .

Among other things, this assumption guarantees that the stationary distribution  $\hat{\zeta}$  is unique.

Under these assumptions, we characterize the role of the principal eigenvalue and function on the long-run behavior of the semigroup. The proof is in appendix B.

**Proposition 7.1.** *Suppose that  $\hat{M}$  satisfies Assumptions 6.1, 7.1 - 7.4, and let  $\Delta > 0$ .*

a. *For any  $\psi$  for which  $\int (|\psi|/\phi) d\hat{\zeta} < \infty$*

$$\lim_{j \rightarrow \infty} \exp(-\rho \Delta j) \mathbb{M}_{\Delta j} \psi = \phi \int \frac{\psi}{\phi} d\hat{\zeta}$$

for almost all ( $\hat{\zeta}$ )  $x$ .

b. *For any  $\psi$  for which  $\psi/\phi$  is bounded,*

$$\lim_{t \rightarrow \infty} \exp(-\rho t) \mathbb{M}_t \psi = \phi \int \frac{\psi}{\phi} d\hat{\zeta}$$

for  $x \in \mathcal{D}_0$ .

The approximation implied by Proposition 7.1, among other things, gives a formal sense in which  $\rho$  is a long-run growth rate. It also provides more precise information, namely that after eliminating the deterministic growth, application of the semigroup to  $\psi$  is approximately proportional to  $\phi$  where the scale coefficient is  $\int \frac{\psi}{\phi} d\hat{\zeta}$ . Subsequently, we will consider other versions of this approximation. We will also impose

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$\hat{\zeta}$  measure when verifying irreducibility for the original probability measure. Since irreducibility depends only on the probability distribution conditioned on  $X_0$ , it does not require that the  $X$  process be stationary under the original measure.

additional regularity conditions that will guarantee convergence without having to sample the Markov process.

### 7.2.1 Uniqueness

As we mentioned earlier, there may exist more than one principal eigenfunction of the extended generator even after a scale normalization is imposed. To be of interest to us, a principal eigenfunction must generate a twisted probability measure, that is  $\hat{M}$  must be a martingale. As we showed in example 6.2 this requirement is not enough to guarantee uniqueness - there may exist more than one principal eigenfunction for which the implied  $\hat{M}_t$  is a martingale. However, in that example, only one of the two solutions we exhibited implies a Markov evolution for  $X$  that is stochastically stable. The other solution will also result in a Markov process, but it fail to be stationary. Recall that the two candidate drift distortions are:

$$\xi_f \bar{x}_f \pm x_f \sqrt{(\xi_f - \gamma_f \sigma_f)^2 - \sigma_f^2 (2\beta_f + \gamma_f^2)}.$$

Only when we select the solution associated with the negative root do we obtain a process that has a stationary density.

This approach to uniqueness works much more generally. The next proposition establishes that stochastic stability requirements will typically eliminate the multiplicity of principal eigenvectors that generate appropriate twisted probabilities. More generally, it states that the eigenvalue of interest to us is always the smallest one.

**Proposition 7.2.** *Assume that Assumption 7.1 is satisfied and that there exists a sampling interval  $\Delta$  such that  $\{X_{\Delta j} : j = 0, 1, \dots\}$  is irreducible. Suppose  $\phi$  is a principal eigenfunction of the extended generator  $\mathbb{A}$  of a multiplicative process  $M$  for which the associated process  $\{\hat{M}_t : t \geq 0\}$  satisfies Assumptions 6.1, 7.2 with a stationary distribution  $\hat{\zeta}$ , 7.3, and 7.4. Then the associated eigenvalue  $\rho$  is the smallest eigenvalue associated with a principal eigenfunction. Furthermore, if  $\tilde{\phi}$  is another positive eigenfunction associated with  $\rho$  then  $\tilde{\phi}$  is proportional to  $\phi$  ( $\hat{\zeta}$  almost surely).*

The proof can be found in appendix B.

This Proposition guarantees that once we find a positive eigenfunction that generates a martingale that satisfies the required stochastic stability restriction, then we

have found the only eigenfunction of interest (up to a constant scale factor). For instance in Example 6.2 we only examined candidate eigenfunctions of a particular functional form but found one that satisfies the assumptions of Proposition 7.2. Hence there exists no other eigenfunctions that satisfy these assumptions.

### 7.2.2 $L^p$ approximation

When there exists a stationary distribution, it follows from Nelson (1958) that the semigroup  $\{\hat{\mathbb{M}}_t : t \geq 0\}$  can be extended to  $\hat{L}^p$  for any  $p \geq 1$  constructed using the measure  $d\hat{\zeta}$ . The semigroup is a weak contraction. That is, for any  $t \geq 0$ ,

$$\|\hat{\mathbb{M}}_t \psi\|_p \leq \|\psi\|_p$$

where  $\|\cdot\|_p$  is the  $\hat{L}^p$  norm.

**Proposition 7.3.** *Under Assumption 7.2, for  $p \geq 1$*

$$\left[ \int |\hat{\mathbb{M}}_t \psi|^p \left( \frac{1}{\phi^p} \right) d\hat{\zeta} \right]^{\frac{1}{p}} \leq \exp(\rho t) \left[ \int |\psi|^p \left( \frac{1}{\phi^p} \right) d\hat{\zeta} \right]^{\frac{1}{p}}$$

*provided that  $\int |\psi|^p \left( \frac{1}{\phi^p} \right) d\hat{\zeta} < \infty$ .*

*Proof.* This follows from the weak contraction property established by Nelson (1958) together with the observation that

$$\exp(-\rho t) \left( \frac{1}{\phi} \right) \hat{\mathbb{M}}_t \psi = \hat{\mathbb{M}}_t \left( \frac{\psi}{\phi} \right).$$

□

**Remark 7.1.** *This proposition establishes an approximation in an  $L^p$  space constructed using the transformed measure  $\frac{1}{\phi^p} d\hat{\zeta}$ . Notice that  $\phi$  itself is always in this space. In particular, we may view the semigroup  $\{\hat{\mathbb{M}}_t : t \geq 0\}$  as operating on this space.*

Proposition 7.3 shows that when the distorted Markov process constructed using the eigenfunction is stationary,  $\rho$  can be interpreted as an asymptotic growth rate of the multiplicative semigroup. The eigenfunction is used to characterize the

space of functions over which the bound applies. We now produce a more refined approximation.

Let  $\hat{Z}^p$  denote the set of Borel measurable functions  $\psi$  such that  $\int \psi d\hat{\zeta} = 0$  and  $\int |\psi|^p d\hat{\zeta} < \infty$ . Suppose that

**Assumption 7.5.** *For any  $t > 0$ ,*

$$\sup_{\psi \in \hat{Z}^p: \|\psi\| \leq 1} \|\hat{\mathbb{M}}_t \psi\|_p < 1.$$

In the case of  $p = 2$ , Hansen and Scheinkman (1995) give sufficient conditions for Assumption 7.5 to be satisfied.<sup>14</sup>

**Proposition 7.4.** *Under Assumptions 7.2 and 7.5, for any  $\psi$  such that  $\int \left| \frac{\psi}{\phi} \right|^p d\hat{\zeta} < \infty$ ,*

$$\left[ \int \left| \exp(-\rho t) \mathbb{M}_t \psi - \phi \int \frac{\psi}{\phi} d\hat{\zeta} \right|^p \frac{1}{\phi^p} d\hat{\zeta} \right]^{\frac{1}{p}} \leq c \exp(-\eta t).$$

for some rate  $\eta > 0$  and positive constant  $c$ .

### 7.2.3 Lyapunov functions

Meyn and Tweedie (1993a) establish, under an additional mild continuity condition, sufficient conditions for the assumptions in this section using a “Lyapunov function” method. In this subsection we will assume:

**Assumption 7.6.** *The process  $X$  is a Feller process under the probability measure associated with  $\hat{M}$ .<sup>15</sup>*

We use Lyapunov functions that are restricted to be norm-like.

**Definition 7.1.** *A continuous function  $V$  is called **norm-like** if the set  $\{x : V(x) \leq r\}$  is precompact for each  $r > 0$ .*

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<sup>14</sup>Assumption 7.5 for  $p = 2$  is equivalent to requiring that the distorted Markov process be *rho-mixing*.

<sup>15</sup>By a Feller process we presume that the implied conditional expectation operators map continuous functions on the one-point compactification of  $\mathcal{D}$  into continuous functions. In fact, Meyn and Tweedie (1993b) permit more general processes. The restriction that the process be Feller implies that all compact subsets are what Meyn and Tweedie (1993b) refer to as “petite sets.”

A norm-like function converges to  $+\infty$  along any sequence  $\{x_j\}$  that converges to  $\infty$ . We will consider here only norm-like functions  $V$  for which  $\hat{\mathbb{A}}V$  is continuous.

A sufficient condition for the existence of stationary distribution (Assumption 7.2) and for Harris recurrence (Assumption 7.4) is that there exists a norm-like function  $V$  for which

$$\frac{\mathbb{A}(\phi V)}{\phi} - \rho V = \hat{\mathbb{A}}V \leq -1$$

outside a compact subset of the state space. (See Theorem 4.2 of Meyn and Tweedie (1993b).)

In subsection 7.2.2 we established  $L^p$  approximations results. The space  $\hat{L}^p$  is largest for  $p = 1$ . It is of interest to ensure that the constant functions are in the corresponding domain for the semigroup  $\{\mathbb{M}_t : t \geq 0\}$ . This requires that  $1/\phi$  have a finite first moment under the stationary distribution  $\hat{\zeta}$ . A sufficient condition for this is the existence of a norm-like function  $V$  such that

$$\mathbb{A}(\phi V) - \rho\phi V = \phi\hat{\mathbb{A}}(V) \leq -\max\{1, \phi\}$$

for  $x$  outside a compact set. (Again see Theorem 4.2 of Meyn and Tweedie (1993b).)

Finally, Proposition 7.4 only applies when the process is *weakly dependent* under the stationary distribution.<sup>16</sup> By weakening the sense of approximation we can expand the range of applicability. Consider some function  $\hat{\psi} \geq 1$ . For any  $t$ , we use

$$\sup_{|\psi| \leq \hat{\psi}} \left| \hat{\mathbb{M}}_t \psi - \int \psi d\hat{\zeta} \right|$$

for each  $x$  as a measure of approximation. When  $\hat{\psi} = 1$  this is equivalent to what is called the total variation norm by viewing  $\hat{\mathbb{M}}_t \psi$  and  $\int \psi d\hat{\zeta}$  applied to indicator functions as measures for each  $x$ . It follows from Meyn and Tweedie (1993b) Theorem 5.3 that if there exists a norm-like function  $V$  and a real number  $\mathbf{a}$  such that

$$\begin{aligned} \frac{\mathbb{A}(\phi V)}{\phi} - \rho V &= \hat{\mathbb{A}}V \leq -\hat{\psi} \\ \frac{\mathbb{A}(\phi \hat{\psi})}{\phi} - \rho \hat{\psi} &= \hat{\mathbb{A}}\hat{\psi} \leq \mathbf{a}\hat{\psi} \end{aligned} \tag{20}$$

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<sup>16</sup>In contrast, Proposition 7.1 applies more generally.

outside a compact set, then

$$\phi \lim_{t \rightarrow \infty} \sup_{|\psi| \leq \phi \hat{\psi}} \left| \hat{\mathbb{M}}_t \frac{\psi}{\phi} - \int \frac{\psi}{\phi} d\hat{\zeta} \right| = \lim_{t \rightarrow \infty} \sup_{|\psi| \leq \phi \hat{\psi}} \left| \exp(-\rho t) \mathbb{M}_t \psi - \phi \int \frac{\psi}{\phi} d\hat{\zeta} \right| = 0.$$

Note that in inequality (20) the constant  $\mathbf{a}$  can be positive. Hence this inequality only requires the existence of an upper bound on rate of growth of the conditional expectation of the function  $\hat{\psi}$  under the distorted probability. While the approximation is uniform in functions dominated by  $\phi \hat{\psi}$  it is pointwise in the Markov state  $x$ .

The approximation results obtained in this section have a variety of applications depending upon our choice of the multiplicative functional  $M$ . In these applications  $M$  is constructed using stochastic discount factor functionals, growth functionals or valuation functionals. These applications are described in the next section.

## 8 Long-term risk

A familiar result from asset pricing is the characterization of the short-term risk-return tradeoff. The tradeoff reflects the compensation, expressed in terms of expected returns, from being exposed to risk over short time horizons. Continuous-time models of financial markets are revealing because they give a sharp characterization of this tradeoff by looking at the instantaneous limits. Our construction of valuation functionals in section 3.4 reflects this tradeoff in a continuous-time Markov environment. Formally, the tradeoff is given in Corollary 3.1. In this section we explore another extreme, the tradeoff pertinent for the long run.

In the study of dynamical systems, a long-run analysis gives an alternative characterization that reveals different features from the short-run dynamics. For linear systems it is easy to move from the short run to the long run. Nonlinearity makes this transformation much less transparent. This is precisely why operator methods are of value. Specifically, we study growth or decay rates in semigroups constructed from alternative multiplicative functionals. Asset values are commonly characterized in terms of growth rates in the cash flows and risk-adjusted interest rates. By using results from the previous section, we have a way to provide long-term counterparts to growth rates and risk-adjusted interest rates. By changing the long-term cash flow exposure to risk, we also have a way to study how long-term counterparts to risk-adjusted interest rates change with cash-flow risk exposure. In this section we

show how to apply the methods of section 7 to support characterizations of long-term growth and valuation.

It has long been recognized that steady state analysis provides a useful characterization of a dynamical system. For Markov processes the counterpart to steady state analysis is the analysis of a stationary distribution. We are led to a related but distinct analysis for two reasons. First, we consider economic environments with stochastic growth. Second, our interest is in the behavior of valuation, including valuation of cash flows with long-run risk exposure. These differences lead us to study stochastic steady distributions under alternative probability measures.

As we have seen, these considerations lead naturally to the study of multiplicative semigroups that display either growth in expectation or decay in value. The counterpart to steady state analysis is the analysis of the principal eigenvalues and eigenfunctions, the objects that characterize the long-run behavior of multiplicative semigroups. We use appropriately chosen eigenvalues and eigenfunctions to change probability measures. Changing probability measures associated with positive martingales are used extensively in asset pricing. Our use of this tool is distinct from the previous literature because of its role in long-run approximation.

We now explore three alternative applications of the methods developed in this paper.

## 8.1 Decomposition of stochastic discount factors ( $M = S$ )

Alvarez and Jermann (2005) characterize the long-run behavior of stochastic discount factors. Their characterization is based on a multiplicative decomposition on a permanent and a transitory component (see their Proposition 1). Corollary 6.1 delivers this decomposition, which we write as:

$$S_t = \exp(\rho t) \hat{M}_t \frac{\phi(X_0)}{\phi(X_t)} \tag{21}$$

for some martingale  $\hat{M}$ . The eigenvalue  $\rho$  is typically negative. We illustrated that such a decomposition is not unique. For such a decomposition to be useful in long-run approximation, the probability measure implied by martingale  $\hat{M}$  must imply that the process  $X$  remains stationary. Proposition 7.2 shows that only one such representation implies that the process  $X$  remains recurrent and stationary under the

change of measure.

Decomposition (21) of a stochastic discount factor functional shows how to extract a deterministic growth component and a martingale component from the stochastic discount factor functional. Long-run behavior is dominated by these two components *vis a vis* a transient component. Building in part on representations in Bansal and Lehmann (1997), Hansen (2008) shows that the transient component can often include contributions from habit persistence or social externalities as modeled in the asset pricing literature. This stochastic discount factor decomposition can be used to approximate prices of long-term discount bonds:

$$\exp(-\rho t)E(S_t|X_0 = x) = \hat{E}\left(\frac{1}{\phi(X_t)}|X_0 = x\right)\phi(x) \approx \hat{E}\left[\frac{1}{\phi(X_t)}\right]\phi(x)$$

where the approximation on the right-hand side becomes arbitrarily accurate as the horizon  $t$  becomes large. Prices of very long-term bonds depend on the current state only through  $\phi(x)$ . Thus  $\phi$  is the dominant pricing factor in the long run. This approximation result extends more generally to stationary cash flows as characterized by Proposition 7.1.<sup>17</sup>

## 8.2 Changing valuation functionals ( $M = V$ )

Alternative valuation functionals imply alternative risk exposures and growth trajectories. For one version of a long-term risk-return frontier, we change the risk exposure of the valuation functional subject to pricing restriction (10). This gives a family of valuation functionals that are compatible with a single stochastic discount factor. We may then apply the decomposition in Corollary 6.1 restricted so that the distorted Markov process is stationary to find a corresponding growth rate associated with each of these valuation functionals. Thus alternative valuation functionals as parameterized by the triple  $(\beta_v, \gamma_v, \kappa_v)$  and restricted by the pricing restriction of Proposition 3.1 imply return processes with different long-run growth rates. The principal eigenvalues of the corresponding semigroups give these rates. In effect the valuation functionals can be freely parameterized by their risk exposure pair  $(\gamma_v, \kappa_v)$  with  $\beta_v$  determined by the local pricing restriction. The vector  $\gamma_v$  gives the exposure to Brownian risk and  $\kappa_v$  the exposure to jump risk.

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<sup>17</sup>Alvarez and Jermann (2005) refer to an earlier version of our paper for the link to eigenfunctions.

Thus a long-run risk-return frontier is given by the mapping from the risk exposure pair  $(\gamma_v, \kappa_v)$  to the long-run growth rate of the valuation process. The growth rate may be computed by solving an eigenvalue problem that exploits the underlying Markovian dynamics. This characterizations allows us to move beyond the log-linear, log-normal specification implicit in many studies of long-horizon returns. The dominant eigenvalue calculation allows for conditional heteroskedasticity with long-run consequences and it allows jumps that might occur infrequently. The principal eigenfunction (along with the eigenvalue) can be used to construct the martingale component as in Corollary 6.1.

**Example 8.1. Application to the Markov diffusion example**

Recall that in the Breeden model and the Kreps-Porteus model, the implied stochastic discount factor is  $S_t = \exp(A_t^s)$  where

$$A_t^s = \bar{\beta}^s t + \int_0^t \beta_f^s X_s^f ds + \int_0^t \beta_o^s X_s^o ds + \int_0^t \sqrt{X_s^f} \gamma_f^s dB_s^f + \int_0^t \gamma_o^s dB_s^o \quad (22)$$

where the alternative models give rise to alternative interpretations of the parameters. To parameterize a valuation functional  $V = \exp(A^v)$ , we construct

$$A_t^v = \bar{\beta}^v t + \int_0^t \beta_f^v X_s^f ds + \int_0^t \beta_o^v X_s^o ds + \int_0^t \sqrt{X_s^f} \gamma_f^v dB_s^f + \int_0^t \gamma_o^v dB_s^o$$

where

$$\bar{\beta}^v + \beta_f^v x_f + \beta_o^v x_o = -\bar{\beta}^s - \beta_f^s x_f - \beta_o^s x_o - \frac{x_f}{2} (\gamma_f^s + \gamma_f^v)^2 - \frac{1}{2} (\gamma_o^s + \gamma_o^v)^2.$$

This equation imposes the local risk-return relation and determines  $\bar{\beta}^v$ ,  $\beta_f^v$  and  $\beta_o^v$  as a function of the stochastic discount factor parameters and the risk exposure parameters  $\gamma_f^v$  and  $\gamma_o^v$ .

To infer the growth rates of valuation processes parameterized by  $(\gamma_f^v, \gamma_o^v)$ , we find the principal eigenvalue for the multiplicative semigroup formed by setting  $M = V$ . Applying the calculation in Example 6.2, this eigenvalue is given by

$$\begin{aligned} \rho &= \bar{\beta}^v + \frac{(\gamma_o^v)^2}{2} + \mathbf{c}_f^v \xi_f \bar{x}_f + \mathbf{c}_o^v (\xi_o \bar{x}_o + \gamma_o^v \sigma_o) + (\mathbf{c}_o^v)^2 \frac{\sigma_o^2}{2} \\ &= -\bar{\beta}^s - \frac{(\gamma_o^s)^2}{2} - \gamma_o^s \gamma_o^v + \mathbf{c}_f^v \xi_f \bar{x}_f + \mathbf{c}_o^v (\xi_o \bar{x}_o + \gamma_o^v \sigma_o) + (\mathbf{c}_o^v)^2 \frac{\sigma_o^2}{2} \end{aligned}$$

where  $c_f^v$  and  $c_o^v$  are given by formulas (16) and (17) respectively. The terms on the right-hand side exclusive of  $c_f^v \xi_f \bar{x}_f$  give the continuous time log-normal adjustments, while  $c_f^v \xi_f \bar{x}_f$  adjusts for the stochastic volatility in the cumulative return. A long-run risk-return tradeoff is given by mapping of  $(\gamma_f^v, \gamma_o^v)$  into the eigenvalue  $\rho$ .

Note that  $\rho^v$  is a linear function of  $\gamma_o^v$ . One notion of a long-run risk price is obtained by imputing the marginal change in the rate of return given a marginal change in the risk exposure as measured by  $\rho$ :

$$\frac{\partial \rho}{\partial \gamma_o^v} = -\gamma_o^s + c_o^v \sigma_o = -\gamma_o^s - \beta_o^s \frac{\sigma_o}{\xi_o}. \quad (23)$$

In contrast  $\rho$  depends nonlinearly on  $\gamma_f^v$ , although risk prices can still be constructed by computing marginal changes in the implied rates of return at alternative values of  $\gamma_f^v$

### 8.3 Changing cash flows ( $M = G, M = S$ and $M = GS$ )

Consider next a risky cash flow of the form:

$$D_t = G_t \psi(X_t) D_0$$

where  $G$  is a multiplicative functional. This cash flow grows over time. We could parameterize the multiplicative functional as the triple  $(\beta_g, \gamma_g, \kappa_g)$ , but this over-parameterizes the long-term risk exposure. The transient components to cash flows will not alter the long-run risk calculation. One attractive possibility is to apply Corollary 6.1 and Propositions 7.1 and 7.2 with  $(M = G)$  and use the martingale from that decomposition for our choice of  $G$ . Thus we could impose the following restriction on the parametrization of  $G$ :

$$\beta_g + \frac{|\gamma_g|^2}{2} + \int (\exp[\kappa_g(y, \cdot)] - 1) \eta(dy, \cdot) = \delta$$

for some positive growth rate  $\delta$ . Given  $\delta$  this relation determines a unique  $\beta_g$ . In addition we restrict these parameters so that the distorted probability measure associated with an extended generator built from:

a) jump measure:  $\exp[\kappa_g(y, x)] \eta(dy|x)$ .

b) first derivative term:  $\xi(x) + \Gamma(x)\gamma_g(x)$ ;

c) second derivative term:  $\Sigma(x)$ ;

implies a semigroup conditional expectation operators that converge to the corresponding unconditional expectation operator.

Hansen et al. (2008) explore the valuation consequences by constructing a semigroup using  $M = GS$  where  $S$  is a stochastic discount factor functional. They only consider the log-linear/log-normal model, however. Provided that we can apply Proposition 7.1 for this choice of  $M$  and  $\psi$ , the negative of the eigenvalue  $-\rho$  is the overall rate of decay in value of the cash flow.

Consider an equity with cash flow  $D$ . For appropriate specifications of  $\psi$ , the values of the cash flows far into the future are approximately proportional to the eigenfunction  $\phi$ . Thus we may view  $\frac{1}{-\rho}$  as the limiting contribution to the price dividend ratio. The decay rate  $\rho$  reflects both a growth rate effect and discount rate effect. To net out the growth rate effect, we compute  $-\rho + \delta$  as an asymptotic rate of return that encodes a risk adjustment. Heuristically, this is linked to Gordon growth model because  $-\rho$  is the difference between the asymptotic rate of return  $-\rho + \delta$  and the growth rate  $\delta$ .

Following Hansen et al. (2008), we explore the consequences of altering the cash flow risk exposure. Such alterations induce changes in the asymptotic decay rate in value ( $-\rho$ ) and hence in the long-run dividend price ratio  $\frac{1}{-\rho}$  and the asymptotic rate of return  $-\rho + \delta$ . The long-run cash flow risk-return relation is captured by the mapping from the cash flow risk exposure pair  $(\gamma_g, \kappa_g)$  to the corresponding required rate of return  $-\rho + \delta$ .

Hansen et al. (2008) use this apparatus to produce such a tradeoff using empirical inputs in a discrete-time, log linear environment. The formulation developed here allows for extensions to include nonlinearity in conditional means, heteroskedasticity that contributes to long-run risk and large shocks modeled as jump risk.

### **Example 8.2. Application to the Markov diffusion example**

*Returning to the Breeden model or the Kreps-Porteus model, suppose the growth process  $G$  is the exponential of the additive functional:*

$$A_t^g = \delta t + \int_0^t \sqrt{X_s^f} \gamma_f^g dB_s^f + \int_0^t \gamma_o^g dB_s^o - \int_0^t \frac{X_s^f (\gamma_f^g)^2 + (\gamma_o^g)^2}{2} ds.$$

The parameters  $\gamma_f^g$  and  $\gamma_f^o$  parameterize the cash flow risk exposure. We limit the cash flow risk exposure by the inequality:

$$2(\xi_f + \sigma_f \gamma_f^g) \bar{x}_f \geq \sigma_f^2.$$

This limits the martingale component so that it induces stationarity under the probability measure induced by the  $\hat{M}$  associated with  $M = G$ .

We use again the parameterization  $S_t = \exp(A_t^s)$  where  $A_t^s$  is given by (22). Hence  $A = A^s + A^g$  is given by

$$A_t = \bar{\beta}t + \int_0^t \beta_f X_s^f ds + \int_0^t \beta_o X_s^o ds + \int_0^t \sqrt{X_s^f} \gamma_f dB_s^f + \int_0^t \gamma_o dB_s^o$$

here  $\bar{\beta} = \delta - \frac{(\gamma_o^g)^2}{2} + \bar{\beta}^s$ ,  $\beta_f = -\frac{(\gamma_f^g)^2}{2} + \beta_f^s$ ,  $\beta_o = \beta_o^s$ ,  $\gamma_f = \gamma_f^g + \gamma_f^s$  and  $\gamma_o = \gamma_o^g + \gamma_o^s$ . The formulas given in Example 6.2 discussed previously give us an asymptotic, risk adjusted rate of return,

$$-\rho + \delta = -\frac{(\gamma_o^s)^2}{2} - \gamma_o^s \gamma_o^g - \bar{\beta}^s - \mathbf{c}_f \xi_f \bar{x}_f - \mathbf{c}_o [\xi_o \bar{x}_o + (\gamma_o^g + \gamma_o^s) \sigma_o] - (\mathbf{c}_o)^2 \frac{\sigma_o^2}{2}.$$

Recall that  $\mathbf{c}_o = \beta_o^s / \xi_o$  and  $\mathbf{c}_f$  is a solution to a quadratic equation (16). This allows us to map exposures to the risks  $B^o$  and  $B^f$  into asymptotic rates of return. For instance, the long-run risk price for the exposure to the  $B^o$  risk is:

$$\frac{d\rho}{d\gamma_o^g} = -\gamma_o^s - \frac{\beta_o^s}{\xi_o} \sigma_o$$

This risk price vector coincides with the one imputed from valuation functionals (see (23)). The long run contribution is reflected in the parameter  $\xi_o$  that governs the persistence of the Ornstein-Uhlenbeck process. This limit gives a continuous-time counterpart to the discrete-time log-normal model studied by Hansen et al. (2008).

The cash flow risk exposure to  $B^f$  is encoded in the coefficient  $\mathbf{c}_f$  of the eigenfunction. Since this coefficient depends on  $\gamma_f^g$  in a nonlinear manner, there is nonlinearity in the long-run risk price, the marginal prices depend on the magnitude of the exposure. The prices of the cash flow exposure to  $B^f$  risk differ from their counterparts from valuation functions. The cash flow prices feature the exposure at a specific horizon while the valuation prices value the cumulative exposure over the horizon when

*payoffs are reinvested. In general these prices will be differ even though they happen to agree for log-normal specifications.*

Hansen (2008) gives some other continuous-time examples drawing on alternative contributions from the asset pricing literature.

Hansen et al. (2008) also decompose the one-period return risk to equity into a portfolio of one-period holding period returns to cash flows for log-linear models. To extend their analysis, consider a cash flow of the form:

$$D_t = D_0 G_t \psi(X_t)$$

where  $G$  is a multiplicative growth functional. The limiting gross return is given by:

$$\lim_{t \rightarrow \infty} \frac{E(S_t D_t / S_1 | \mathcal{F}_1)}{E(S_t D_t | \mathcal{F}_0)} = \exp(-\rho) G_1 \frac{\phi(X_1)}{\phi(X_0)}$$

where  $\rho$  and  $\phi$  are the principal eigenvalue and eigenfunction of the semigroup constructed using  $M = GS$ . This limit presumes that  $\hat{E}\psi(X_t)$  is positive. The limiting holding period return has a cash flow growth component  $G_1$ , an eigenvalue component  $\exp(-\rho)$  and an eigenfunction component  $\frac{\phi(X_1)}{\phi(X_0)}$ . The limit is independent of the transient contribution to the cash flow provided that assumptions of Proposition 7.1 are satisfied.

## 9 Existence

In our analysis thus far we supposed that we can find solutions to the principal eigenvalue problem and then proceeded to check the alternative solutions. We also exhibited solutions to this problem for specific examples. We now discuss some sufficient conditions for the existence of a solution to the eigenvalue problem. We return to our study of a generic semigroup represented with a multiplicative functional  $M$ . As we know from Table 1, there are a variety of constructions of a multiplicative functional depending upon which of the applications described in section 8 is the focal point of an investigation.

Our analysis in this section builds on work of Nummelin (1984) and and Kontoyiannis and Meyn (2005). We impose the following drift condition:

**Assumption 9.1.** *There exists a function  $V \geq 1$  and constant  $\underline{a}$  such that for  $x \in \mathcal{D}_0$ ,*

$$\frac{\mathbb{A}V(x)}{V(x)} \leq \underline{a}$$

Since  $M$  is a multiplicative functional, so is  $\left\{ \frac{M_t V(X_t)}{V(X_0)} \right\}$ . We show in appendix D that when Assumption 9.1 holds the operator:

$$\mathbb{F}\psi(x) = \int_0^\infty \exp(-at) E \left[ M_t \frac{V(X_t)}{V(X_0)} \psi(X_t) \mid X_0 = x \right]$$

is bounded on  $L^\infty$  whenever  $\underline{a} > \underline{a}$ . As an alternative, we could have constructed an analogous operator applied to  $V\psi$ , in which case we should also scale the outcome:  $V\mathbb{F}\psi$ . In what follows it will be convenient to work directly with the operator  $\mathbb{F}$ .<sup>18</sup>

We impose the following positivity condition on  $\mathbb{F}$ :

**Assumption 9.2.** *There exists a measure  $\nu$  on the state space  $\mathcal{D}_0$  such that for every Borel set  $\Lambda$  for which  $\nu(\Lambda) > 0$ ,*

$$\mathbb{F}\mathbf{1}_\Lambda(x) > 0.$$

This irreducibility assumption on  $\mathbb{F}$  can be obtained from more primitive hypotheses as verified in appendix D.

If Assumption 9.2 holds, it follows from Theorem 2.1 of Nummelin (1984) that there exists a function  $s \geq 0$  with  $\int s d\nu > 0$  such that for any  $\psi \geq 0$ ,

$$\mathbb{F}\psi \geq s \int \psi d\nu. \tag{24}$$

The function  $s$  is necessarily bounded and hence we may scale it to have an  $L^\infty$  norm equal to unity by adjusting the measure  $\nu$  accordingly.

Next form

$$\int \left( \sum_{j=0}^{\infty} r^{-j} \mathbb{F}^j s \right) d\nu \tag{25}$$

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<sup>18</sup>The operator  $\mathbb{F}$  is a resolvent operator associated with the semigroup built with the multiplicative functional  $\left\{ \frac{M_t V(X_t)}{V(X_0)} \right\}$ .

for alternative values of the real number  $r$ . Theorem 3.2 of Nummelin (1984) states that  $\lambda$  is a critical value  $\lambda \geq \|\mathbb{F}\|$  for which the quantity in (25) is finite for  $r > \lambda$  and infinite for  $r < \lambda$  and that such an  $\lambda$  is independent of the particular function  $s$  chosen. When  $(r\mathbb{I} - \mathbb{F})^{-1}$  does not exist as a bounded operator,  $r$  is in the *spectrum* of  $\mathbb{F}$ . The spectrum is closed and thus  $\lambda$  is necessarily an element of the spectrum.

Our goal is to state sufficient conditions under which  $\lambda$  is an eigenvalue of  $\mathbb{F}$  associated with a non-negative eigenfunction  $\phi$ . This result is of interest because, as we show in appendix D, when  $\phi$  is a non-negative eigenfunction of  $\mathbb{F}$  then  $V\phi$  is a positive eigenfunction of the semigroup  $\{\mathbb{M}_t : t \geq 0\}$ . Proposition 6.2 thus guarantees that we can produce the desired decomposition of the multiplicative functional  $M$ .

Construct the non-negative operator

$$\mathbb{G}\psi = \sum_{j=0}^{\infty} \lambda^{-j} (\mathbb{F} - s \otimes \nu)^j \psi.$$

where  $s \otimes \nu$  is the operator

$$(s \otimes \nu)\psi = s \int \psi d\nu.$$

Following Nummelin (1984), our candidate eigenfunction is the non-negative function  $\mathbb{G}s$ . Provided that  $\mathbb{G}$  is a bounded operator,  $\mathbb{G}s$  is an eigenfunction of  $\mathbb{F}$ . (See D.)

This leaves open how verify that  $\mathbb{G}$  is a bounded operator.<sup>19</sup> Instead of assuming that  $\mathbb{G}$  is bounded we may suppose that

**Assumption 9.3.**  $\phi = \mathbb{G}s$  is bounded.<sup>20</sup>

In addition we strengthen Assumption 9.1.

**Assumption 9.4.** *There exists a function  $V \geq 1$  such that for any  $r > 0$ , there exists a positive number  $c$  such that*

$$\frac{\mathbb{A}V}{V} \leq -r + cs$$

for all  $x \in \mathcal{D}_0$ .

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<sup>19</sup>It follows from Nummelin (1984) Proposition 4.7 that  $\mathbb{G}s$  is finite except on set of  $\nu$  measure zero and from Propositions 4.7 and 2.1 of Nummelin (1984) (applied to the kernel  $\lambda^{-1}(\mathbb{F} - s \otimes \nu)$ ) that:  $\mathbb{F}\phi \leq \lambda\phi$ .

<sup>20</sup>Alternatively, we could assume that exists a function  $0 \leq s^* \leq 1$  such that  $Fs^* \leq \lambda s^*$  and  $\int s^* d\nu > 0$ . It may then be shown that  $\phi \leq \frac{\lambda}{\int s^* d\nu}$  as in say Proposition 4.11 of Kontoyiannis and Meyn (2003).

In appendix D we show that these two assumptions guarantee that the operator  $\mathbb{G}$  is bounded using an argument that follows in part the proof of Proposition 4.11 in Kontoyiannis and Meyn (2003).

**Remark 9.1.** *We may apply the Multiplicative Mean Ergodic Theorem (Theorem 4.16) of Kontoyiannis and Meyn (2003) if we decompose the generator:*

$$\frac{\mathbb{A}\phi V}{V} = \mathbb{B}\phi + \phi \frac{\mathbb{A}V}{V}. \quad (26)$$

where  $\mathbb{B}\phi \doteq \left(\frac{\mathbb{A}\phi V}{V} - \phi \frac{\mathbb{A}V}{V}\right)$ . Notice that  $\mathbb{B}1 = 0$ . Typically,  $\mathbb{B}$  will be the generator of a semigroup for a Markov process. Kontoyiannis and Meyn (2003) impose restrictions on this process and bounds on their counterpart to  $\frac{\mathbb{A}V}{V}$  to establish existence of a positive eigenfunction.<sup>21</sup>

**Remark 9.2.** *Alternatively, we may establish existence of an eigenfunction by showing that  $\mathbb{F}$  is a compact operator on an appropriately weighted  $L^2$  space by using the approach of Chen et al. (2007). Chen et al. (2007) focus on the case of a multivariate diffusion implying that  $\mathbb{B}$  is a second-order differential operator.*

## 10 Conclusions

In this paper we characterized the long-run risk-return relationship for nonlinear continuous time Markov environments. This long-term relationship shows how alternative cash flow risk exposures are encoded in asymptotic risk-adjusted discount rates. To achieve this characterization we decompose a multiplicative functional built from the Markov process into the product of three components: (i) a deterministic exponential trend, (ii) a martingale and (iii) a transitory component. The martingale and transitory components are constructed from a principal eigenfunction associated with the multiplicative functional, and the rate of growth of the exponential trend is given by the corresponding eigenvalue. The multiplicative functional represents a semigroup of valuation operators that accommodate stochastic growth in consumption or cash flows. Thus the decomposition of the multiplicative functional allows us to characterize transitory and permanent components to valuation. Specifically, the

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<sup>21</sup>The Kontoyiannis and Meyn (2003) establish more refined results motivated by their interest in large deviation theory.

martingale component gives an alternative distorted or twisted probability that we used to characterize approximation over long time horizons.

This long-horizon apparatus is a complement to the short-term risk-return trade-offs familiar from asset pricing and is tailored to accommodate stochastic growth. It supports an analysis of the term structure of risk prices. We explore this term structure for two reasons. First, a variety of recent theories of asset prices feature investor preferences in which the intertemporal decomposition of risk is an essential ingredient, as in models in which separability over states of nature or time is relaxed. A second motivation is that the arguably simplified models that we use to construct evidence are likely to be misspecified when pricing over short intervals of time. Pricing models can be repaired by appending *ad hoc* dynamics, but then it becomes valuable to understand which repairs have long-run consequences.

There are several natural extensions of this work. First, while we presented results concerning existence and uniqueness of principal eigenvalues and eigenfunctions, it remains important to develop methods for computing these objects. There are only a limited array of examples for which quasi-analytical solutions are currently available. Second, while we have focused on dominant eigenvalues, more refined characterizations are needed to understand how well long-run approximation works and how it can be improved. Results in Chen et al. (2007) and Kontoyiannis and Meyn (2003) could be extended and applied to achieve more refined characterizations.<sup>22</sup> Third we considered only processes with a finite number of jumps in any finite interval of time. Extending the results presented in this paper to more general Lévy processes may add new insights into characterizing long-term risk. Shaliastovich and Tauchen (2005) motivate for such extensions when building structural models of asset pricing.

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<sup>22</sup>Lewis (1998), Linetsky (2004) and Boyarchenko and Levendorskii (2007) use spectral methods of this type to study of the term structure of interest rates and option pricing by applying and extending quasi-analytical characterizations of eigenfunctions.

## A Value function in the example economy

Recall the following equation :

$$-\xi_f w_f + \frac{(1-a)\sigma_f^2}{2}(w_f)^2 + (1-a)\vartheta_f \sigma_f w_f + \frac{(1-a)\vartheta_f^2}{2} = b w_f$$

that we solved when constructing the value function for the Kreps and Porteus (1978) example with  $a > 1$ . This quadratic equation has two solutions:

$$w_f = \frac{(a-1)\sigma_f \vartheta_f + b + \xi_f \pm \sqrt{[(a-1)\sigma_f \vartheta_f + b + \xi_f]^2 - (a-1)^2 \sigma_f^2 \vartheta_f^2}}{(1-a)\sigma_f^2}$$

Solutions will exist provided

$$|\xi_f + b + (a-1)\sigma_f \vartheta_f| > (a-1)|\sigma_f \vartheta_f|,$$

which will always be satisfied when  $\vartheta_f \sigma_f > 0$ . Solutions will also exist when  $\vartheta_f \sigma_f < 0$  and

$$\xi_f + b \geq 2(a-1)|\vartheta_f \sigma_f|$$

In both cases

$$\xi_f + b + (a-1)\sigma_f \vartheta_f \geq 0,$$

and thus both zeros are negative.

As claimed in the text, the solution that interests us is

$$w_f = \frac{(a-1)\sigma_f \vartheta_f + b + \xi_f - \sqrt{[(a-1)\sigma_f \vartheta_f + b + \xi_f]^2 - (a-1)^2 \sigma_f^2 \vartheta_f^2}}{(1-a)\sigma_f^2}.$$

To see why, we note that a finite time horizon solution is given by a value function with the same functional form but coefficients that depend on the gap of time between the terminal period and the current period. This lead us to study the slope of the quadratic function:

$$-\xi_f w_f + \frac{(1-a)\sigma_f^2}{2}(w_f)^2 + (1-a)\vartheta_f \sigma_f w_f + \frac{(1-a)\vartheta_f^2}{2} - b w_f$$

at the two zeros of this function. This function is concave. We pick the zero associated

with a negative slope, which will always be the right most zero since this is the only “stable” solution.

## B Approximation

In this appendix we present additional proofs for some of the Propositions in section 7. We start with the proof of Proposition 7.1

*Proof.* Note that

$$\exp(-\rho t)\mathbb{M}_t\psi(x) = \hat{\mathbb{M}}_t\left(\frac{\psi}{\phi}\right)\phi(x).$$

It follows from Theorem 6.1 of Meyn and Tweedie (1993a) that,

$$\lim_{t \rightarrow \infty} \sup_{0 \leq \psi \leq \phi} \left| \hat{\mathbb{M}}_t\left(\frac{\psi}{\phi}\right) - \int \frac{\psi}{\phi} d\hat{\zeta} \right| = 0,$$

which proves (b). Consider any sample interval  $\Delta > 0$ . Then

$$\lim_{j \rightarrow \infty} \sup_{0 \leq \psi \leq \phi} \left| \hat{\mathbb{M}}_{\Delta j}\left(\frac{\psi}{\phi}\right) - \int \frac{\psi}{\phi} d\hat{\zeta} \right| = 0.$$

From Proposition 6.3 of Nummelin (1984), the sampled process  $\{X_{\Delta j} : j = 0, 1, \dots\}$  is aperiodic and Harris recurrent with stationary density  $\hat{\zeta}$ . Hence if  $\int \left| \frac{\psi}{\phi}(x) \right| d\hat{\zeta}(x) < \infty$ ,

$$\lim_{j \rightarrow \infty} \hat{\mathbb{M}}_{\Delta j}\left(\frac{\psi}{\phi}\right) = \int \frac{\psi}{\phi} d\hat{\zeta}$$

for almost all ( $\hat{\zeta}$ )  $x$ , which proves (a). (See for example, Theorem 5.2 of Meyn and Tweedie (1992).) □

Next, we prove Proposition 7.2.

*Proof.* Consider another principal eigenfunction  $\phi^*$  with associated eigenvalue  $\rho^*$ . By Proposition 6.2 the eigenfunction-eigenvalue pairs must solve

$$\begin{aligned} \mathbb{M}_t\phi(x) &= \exp(\rho t)\phi(x) \\ \mathbb{M}_t\phi^*(x) &\leq \exp(\rho^* t)\phi^*(x). \end{aligned}$$

If  $\hat{M}$  is the martingale associated with the eigenvector  $\phi$ , then

$$E \left[ \hat{M}_t \frac{\phi^*(X_t)}{\phi(X_t)} \mid X_0 = x \right] \leq \exp[(\rho^* - \rho)t] \frac{\phi^*(x)}{\phi(x)}.$$

Since the discrete-time sampled Markov process associated with  $\hat{M}$  is Harris recurrent, aperiodic and has a unique stationary distribution, the left-hand side converges to:

$$\hat{E} \left[ \frac{\phi^*(X_0)}{\phi(X_0)} \right] > 0,$$

for  $t = \Delta j$  as the integer  $j$  tends to  $\infty$ , whenever this expected value is finite. If this expected value is not finite, then since  $\frac{\phi^*}{\phi} > 0$ , the left hand side must diverge to  $+\infty$ . In any case, this requires that  $\rho \leq \rho^*$ . If  $\rho^* = \rho$  then

$$\hat{E} \left[ \frac{\phi^*(X_0)}{\phi(X_0)} \right] \leq \frac{\phi^*(x)}{\phi(x)}.$$

Hence the ratio of the two eigenfunctions is constant, ( $\hat{\zeta}$  almost surely).  $\square$

The following argument establishes Proposition 7.4:

*Proof.* Notice that

$$\left| \exp(-\rho t) \mathbb{M}_t \psi - \phi \int \frac{\psi}{\phi} d\hat{\zeta} \right| = \phi \left| \hat{\mathbb{M}}_t \left( \frac{\psi}{\phi} \right) - \int \frac{\psi}{\phi} d\hat{\zeta} \right|.$$

Moreover,

$$\int \phi^p \left| \hat{\mathbb{M}}_t \left( \frac{\psi}{\phi} \right) - \int \frac{\psi}{\phi} d\hat{\zeta} \right|^p \left( \frac{1}{\phi^p} \right) d\hat{\zeta} = \int \left| \hat{\mathbb{M}}_t \left( \frac{\psi}{\phi} \right) - \int \frac{\psi}{\phi} d\hat{\zeta} \right|^p d\hat{\zeta}.$$

Assumption 7.5 implies that the right-hand side converges to zero as  $t$  gets large. By the semigroup property, this convergence is necessarily exponentially fast.  $\square$

## C Martingales and absolute continuity

In this appendix we state some conditions that insure that Assumption 6.1 holds. Our result is inspired by the approach developed in Chapter 7 of Liptser and Shiryaev (2000). Let  $\hat{M}$  denote a multiplicative functional parameterized by  $(\hat{\beta}, \hat{\gamma}, \hat{\kappa})$  that is

restricted to be a local martingale. Thus  $\hat{M} = \exp(\hat{A})$  where

$$\hat{A}_t = \int_0^t \hat{\beta}(X_u) du + \int_0^t \hat{\gamma}(X_{u-})' [\Gamma(X_{u-})' \Gamma(X_{u-})]^{-1} \Gamma(X_{u-})' [dX_u^c - \xi(X_{u-}) du] + \sum_{0 \leq u \leq t} \hat{\kappa}(X_u, X_{u-}),$$

and

**Assumption C.1.**

$$-\hat{\beta} - \frac{|\hat{\gamma}|^2}{2} - \int (\exp[\hat{\kappa}(y, x)] - 1) \eta(dy|x) = 0.$$

The extended generator for  $\hat{M}$  is given by:

$$\begin{aligned} \hat{\mathbb{A}}\phi(x) &= [\xi(x) + \Gamma(x)\hat{\gamma}(x)] \cdot \frac{\partial\phi(x)}{\partial x} + \frac{1}{2} \text{trace} \left( \Sigma(x) \frac{\partial^2\phi(x)}{\partial x \partial x'} \right) \\ &\quad + \int [\phi(y) - \phi(x)] \exp[\hat{\kappa}(y, x)] \eta(dy|x). \end{aligned}$$

**Assumption C.2.** *There exists a probability space  $(\check{\Omega}, \check{\mathcal{F}}, \check{P}_r)$ , a filtration  $\check{\mathcal{F}}_t$ , an  $n$ -dimensional  $\check{\mathcal{F}}_t$  Brownian motion  $\check{B}$  and a semi-martingale  $\check{X} = \check{X}^c + \check{X}^j$ , where*

$$d\check{X}_t^c = [\xi(\check{X}_{t-}) + \Gamma(\check{X}_{t-})\hat{\gamma}(\check{X}_{t-})] dt + \Gamma(\check{X}_{t-}) d\check{B}_t \quad (27)$$

and  $\check{X}^j$  is a pure jump process with a finite number of jumps in any finite interval that has a compensator  $\exp[\hat{\kappa}(y, \check{X}_{t-})] \eta(dy|\check{X}_{t-}) dt$

In this case,

$$d\check{B}_t = [\Gamma(\check{X}_{u-})' \Gamma(\check{X}_{u-})]^{-1} \Gamma(\check{X}_{u-})' [d\check{X}_u^c - \xi(\check{X}_{u-}) du - \Gamma(\check{X}_{u-}) \hat{\gamma}(\check{X}_{u-}) du].$$

Use the process  $\check{X}$  to construct a multiplicative functional  $\check{M} = \exp(\check{A})$  where

$$\begin{aligned} \check{A}_t &= - \int_0^t \hat{\beta}(\check{X}_u) du - \int_0^t \hat{\gamma}(\check{X}_{u-})' [\Gamma(\check{X}_{u-})' \Gamma(\check{X}_{u-})]^{-1} \Gamma(\check{X}_{u-})' [d\check{X}_u^c - \xi(\check{X}_{u-}) du] \\ &\quad - \sum_{0 \leq u \leq t} \hat{\kappa}(\check{X}_u, \check{X}_{u-}) \\ &= - \int_0^t \left[ \hat{\beta}(\check{X}_u) + |\hat{\gamma}(\check{X}_{u-})|^2 \right] du \end{aligned}$$

$$\begin{aligned}
& - \int_0^t \hat{\gamma}(\check{X}_{u-})' [\Gamma(\check{X}_{u-})' \Gamma(\check{X}_{u-})]^{-1} \Gamma(\check{X}_{u-})' [d\check{X}_u^c - \xi(\check{X}_{u-}) du - \Gamma(\check{X}_{u-}) \hat{\gamma}(\check{X}_{u-}) du] \\
& - \sum_{0 \leq u \leq t} \hat{\kappa}(\check{X}_u, \check{X}_{u-}).
\end{aligned}$$

The multiplicative functional  $\check{M}$  is parameterized by:

$$\begin{aligned}
\check{\beta} &= -\hat{\beta} - |\hat{\gamma}|^2 \\
\check{\gamma} &= -\hat{\gamma} \\
\check{\kappa} &= -\hat{\kappa}.
\end{aligned}$$

**Assumption C.3.** *The parameterization  $(\check{\beta}, \check{\gamma}, \check{\kappa})$  of the multiplicative functional  $\check{M}$  satisfies:*

- a)  $\int_0^t \check{\beta}(\check{X}_u) du < \infty$  for every positive  $t$ ;
- b)  $\int_0^t |\check{\gamma}(\check{X}_u)|^2 du < \infty$  for every positive  $t$ .

Notice that

$$\int \exp[\check{\kappa}(y, x)] \hat{\eta}(dy|x) = \int \eta(dy|x) < \infty$$

for all  $x \in \mathcal{D}_0$ . Moreover,

$$\check{\beta} + \frac{|\check{\gamma}|^2}{2} + \int (\exp[\check{\kappa}(y, x)] - 1) \hat{\eta}(dy|x) = -\hat{\beta} - \frac{|\hat{\gamma}|^2}{2} - \int (\exp[\hat{\kappa}(y, x)] - 1) \eta(dy|x) = 0.$$

Thus the multiplicative functional  $\check{M}$  is a local martingale.

**Proposition C.1.** *Suppose that assumptions C.1, C.2 and C.3 are satisfied. Then the local martingale  $\hat{M}$  is a martingale.*

*Proof.* We show that  $\hat{M}$  is a martingale in three steps:

- i) Since  $\check{M}$  is a local martingale, there is an increasing sequence of stopping times  $\{\check{\tau}_N : N = 1, \dots\}$  that converge to  $\infty$  such that

$$\check{M}_t^N = \begin{cases} \check{M}_t & t \leq \check{\tau}_N \\ \check{M}_{\check{\tau}_N} & t > \check{\tau}_N \end{cases}$$

is a martingale and

$$\check{E}(\check{M}_t^N | \check{X}_0 = x) = 1$$

for all  $t \geq 0$ .

- ii) Next we obtain an alternative formula for  $\check{E}(\mathbf{1}_{\{t \leq \check{\tau}_N\}} | X_0 = x)$  represented in terms of the original  $X$  process. The stopping time  $\check{\tau}_N$  can be represented as a function of  $\check{X}$ . Let  $\tau_N$  be the corresponding function of  $X$ , and construct:

$$\hat{M}_t^N = \begin{cases} \hat{M}_t & t \leq \tau_N \\ \hat{M}_{\tau_N} & t > \tau_N \end{cases}.$$

Recall that

$$\hat{M}_t^N = \Phi_t(X)$$

for some Borel measurable function  $\Phi_t$ . By construction,

$$\Phi_t(\check{X}) = \frac{1}{\check{M}_t^N}$$

Then

$$\begin{aligned} E(\hat{M}_t \mathbf{1}_{\{t \leq \tau_N\}} | X_0 = x) &= E(\hat{M}_t^N \mathbf{1}_{\{t \leq \tau_N\}} | X_0 = x) \\ &= \check{E}\left[\check{M}_t^N \left(\frac{1}{\check{M}_t^N}\right) \mathbf{1}_{\{t \leq \check{\tau}_N\}} | \check{X}_0 = x\right] \\ &= \check{E}(\mathbf{1}_{\{t \leq \check{\tau}_N\}} | \check{X}_0 = x) \end{aligned}$$

where the second equality follows from the Girsanov Theorem.

- iii) Note that

$$\lim_{N \rightarrow \infty} \check{E}(\mathbf{1}_{\{\check{\tau}_N \leq t\}} | \check{X}_0 = x) = 1$$

by the Dominated Convergence Theorem. Thus

$$E(\hat{M}_t | X_0 = x) \geq \lim_{N \rightarrow \infty} E(\hat{M}_t \mathbf{1}_{\{\tau_N \leq t\}} | X_0) = \lim_{N \rightarrow \infty} \check{E}(\mathbf{1}_{\{\check{\tau}_N \leq t\}} | \check{X}_0 = x) = 1.$$

Since  $\hat{M}$  is a nonnegative local martingale, we know that

$$E(\hat{M}_t | X_0 = x) \leq 1.$$

Therefore  $E(\hat{M}_t | X_0 = x) = 1$  for all  $t \geq 0$  and  $\hat{M}$  is a martingale.

□

## D Existence

We next establish the existence results discussed in Section 9. We divide our analysis into four lemmas. The first lemma states that under Assumption 9.1,  $\mathbb{F}$  is bounded. The second lemma verifies that if Assumptions 9.1 and 9.2 are satisfied and  $\mathbb{F}$  has a non-negative eigenfunction, there exists a strictly positive solution for the principal eigenvalue problem for the semigroup  $\mathbb{M}$  and as a consequence of Proposition 6.2 we obtain the desired decomposition of the multiplicative functional  $M$ . The third lemma shows that if  $\mathbb{G}$  is bounded  $\mathbb{G}s$  is an eigenfunction of  $\mathbb{F}$ . The fourth lemma shows that the boundedness of  $\mathbb{G}$  follows from Assumptions 9.2, 9.3 and 9.4.

**Lemma D.1.** *Suppose Assumption 9.1 is satisfied. Then  $\mathbb{F}$  is bounded.*

*Proof.* Let  $\mathbf{a} = \underline{\mathbf{a}} + \epsilon$  with  $\epsilon > 0$ . Construct the multiplicative process:

$$M_t^* = \exp(-\mathbf{a}t) M_t \frac{V(X_t)}{V(X_0)}.$$

Then

$$N_t^* = M_t^* - 1 - \int_0^t M_u^* \left[ \frac{\mathbb{A}V(X_u)}{V(X_u)} - \mathbf{a} \right] du$$

is a local martingale, as we now verify. Note that

$$N_t = M_t V(X_t) - V(X_0) - \int_0^t M_u \mathbb{A}V(X_u) du$$

is a local martingale. Thus  $\frac{1}{V(X_0)} \int_0^t \exp(-\mathbf{a}u) dN_u$  is also a local martingale and,

$$\begin{aligned} & \frac{1}{V(X_0)} \int_0^t \exp(-\mathbf{a}u) dN_u \\ &= \exp(-\mathbf{a}t) M_t \frac{V(X_t)}{V(X_0)} - 1 \\ & \quad + \mathbf{a} \int_0^t \exp(-\mathbf{a}u) M_u \frac{V(X_u)}{V(X_0)} du \\ & \quad - \int_0^t \exp(-\mathbf{a}u) M_u \frac{V(X_u)}{V(X_0)} \frac{\mathbb{A}V(X_u)}{V(X_u)} du = N_t^* \end{aligned}$$

Since  $N^*$  is a local martingale, Fatou's Lemma implies that

$$E(M_t^* | X_0 = x) + \int_0^t \epsilon E[M_u^* | X_0 = x] du \leq 1.$$

Since this holds for any  $t$ ,

$$\int_0^\infty \exp(-at) E \left[ M_t \frac{V(X_t)}{V(X_0)} | X_0 = x \right] dt \leq \frac{1}{\epsilon}. \quad (28)$$

Inequality (28) guarantees that

$$\mathbb{F}\psi = \int_0^\infty \exp(-at) E \left[ M_t \frac{V(X_t)}{V(X_0)} \psi(X_t) | X_0 = x \right] dt$$

defines a bounded operator in  $L^\infty$ . □

**Remark D.1.** *The irreducibility Assumption 9.2 on  $\mathbb{F}$  that we use in our next lemma can be obtained from more primitive hypotheses. Write  $\mathbb{K}(x, \Lambda) = \int_0^\infty \exp(-at) E [\mathbf{1}_\Lambda(X_t) | X_0 = x]$ , and suppose that  $\mathbb{K}$  satisfies the counterpart to Assumption 9.2. Then since  $V \geq 1$ ,  $\mathbb{F}$  will satisfy Assumption 9.2 whenever  $M$  is bounded below by a positive number. Another set of sufficient conditions is obtained by first assuming that there exists a function  $p(t, x, y)$  such that  $p(t, x, \cdot)$  is the conditional density (with respect to  $\nu$ ) of  $X_t$  given  $X_0 = x$ , and that  $p$  satisfies the following restriction. Let  $\{\tilde{\Lambda}_k : k = 1, 2, \dots\}$  be an increasing sequence of compact subsets of the state space  $\mathcal{D}_0$  whose union is the entire space. Suppose that for each integer  $k$  and  $x$  in the Markov state space, there exists a  $T$  such that for  $t \geq T$ , and  $y \in \tilde{\Lambda}_k$   $p(t, x, y) > 0$ . In this case we may define for each  $t \geq T$  and each  $y \in \tilde{\Lambda}_k$ ,  $f(t, x, y) = E[M_t | X_0 = x, X_t = y]$ . If we further assume that there exists a version of this conditional expectation that is a continuous function of  $(t, y)$  and that  $M > 0$ , then Assumption 9.2 must hold. To see this, notice that if  $\nu(\Lambda) > 0$  and  $\Lambda_k = \tilde{\Lambda}_k \cap \Lambda$  then there must exist a positive integer  $k$  for which  $\nu(\Lambda_k) > 0$ . Choose a  $T' > T$ , and set*

$$c = \inf_{T \leq t \leq T', y \in \Lambda_k} f(t, x, y) > 0.$$

Then

$$\begin{aligned} \int_T^{T'} \exp(-at) E [M_t \mathbf{1}_\Lambda(X_t) | X_0 = x] &\geq \int_T^{T'} \exp(-at) E [M_t \mathbf{1}_{\Lambda_k}(X_t) | X_0 = x] = \\ &\int_T^{T'} \exp(-at) E \{ \mathbf{1}_{\Lambda_k}(X_t) E [M_t | X_0 = x, X_t] | X_0 = x \} > 0, \end{aligned}$$

since  $\nu(\Lambda_k) > 0$ . Given our restriction that  $V \geq 1$ , assumption 9.2 must hold.

**Lemma D.2.** *Suppose Assumptions 9.1 and 9.2 are satisfied and  $\mathbb{F}\phi = \lambda\phi$  for some nonnegative bounded  $\phi$ . Then  $V\phi$  is a strictly positive eigenfunction for the semigroup  $\{\mathbb{M}_t : t \geq 0\}$ .*

*Proof.* Assumption 9.2 guarantees that  $\phi$  is strictly positive. Moreover,

$$\lambda\tilde{\mathbb{M}}_t\phi(x) = \tilde{\mathbb{M}}_t\mathbb{F}\phi(x) = \int_0^\infty \exp(-as)\tilde{\mathbb{M}}_{t+s}\phi(x)ds,$$

where the right-side follows from Tonelli's Theorem. Hence

$$\begin{aligned} \lambda\tilde{\mathbb{M}}_t\phi(x) &= \exp(at)\mathbb{F}\phi(x) - \exp(at) \int_0^t \exp(-as)\tilde{\mathbb{M}}_s\phi(x)ds \\ &= \exp(at)\lambda\phi(x) - \exp(at) \int_0^t \exp(-as)\tilde{\mathbb{M}}_s\phi(x)ds \end{aligned}$$

For a fixed  $x$ , define the function of  $t$ :

$$g(t) = \exp(-at)\tilde{\mathbb{M}}_t\phi(x).$$

Then

$$\lambda g(t) = \lambda\phi - \int_0^t g(s)ds$$

and  $g(0) = \phi(x)$ . The unique solution to this integral equation is

$$g(t) = \exp\left(-\frac{t}{\lambda}\right)\phi(x).$$

Hence  $\phi$  solves the principal eigenvalue problem for  $\tilde{\mathbb{M}}_t$  and  $V\phi$  solves the principal eigenvalue problem for  $\mathbb{M}$ . □

**Lemma D.3.** *Suppose  $\mathbb{F}$  and  $\mathbb{G}$  are bounded and assumption 9.2 is satisfied. Then  $\mathbb{F}\mathbb{G}s = \lambda\mathbb{G}s$ .<sup>23</sup>*

*Proof.* Since  $\mathbb{G}$  is bounded, for any bounded  $\psi$ ,

$$\psi = (\lambda\mathbb{I} - \mathbb{F} + \nu \otimes s)\mathbb{G}\psi = (\lambda\mathbb{I} - \mathbb{F})\mathbb{G}\psi + s \int \mathbb{G}\psi d\nu.$$

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<sup>23</sup>This result is essentially a specialization of Proposition 4.6 of Kontoyiannis and Meyn (2003) and closely related to Proposition 5.2 of Nummelin (1984). We include a proof for sake of completeness.

Further since  $\lambda$  is in the spectrum of  $\mathbb{F}$ , choose a sequence  $\{\psi_j : j = 1, 2, \dots\}$  such that  $\mathbb{G}\psi_j$  has  $L^\infty$  norm one and that

$$\lim_{j \rightarrow \infty} (\lambda \mathbb{I} - \mathbb{F}) \mathbb{G}\psi_j = 0.$$

The sequence  $\{\int \mathbb{G}\psi_j d\nu\}$  is bounded and hence has a convergent subsequence with a limit  $r$ . Thus there is a subsequence of  $\{\psi_j : j = 1, 2, \dots\}$  that converges to  $rs$ , and in particular  $r \neq 0$ . As a consequence,  $\mathbb{G}s = \phi$  is an eigenfunction associated with  $\lambda$ .  $\square$

**Lemma D.4.** *Under Assumptions 9.2, 9.3 and 9.4, the operator  $\mathbb{G}$  is bounded on  $L^\infty$ .*

*Proof.* Assumption 9.4 implies that

$$\mathbb{F}1 \leq \frac{1}{r+a} + \tilde{c}\mathbb{F}s.$$

where  $\tilde{c} = c/(r+a)$ . Moreover,

$$\mathbb{G}\mathbb{F}s \leq \left( \int s d\nu + \lambda \right) \mathbb{G}s,$$

and, in particular, since Assumption 9.3 is satisfied  $\mathbb{G}\mathbb{F}s$  is a bounded function.

Given that Assumption 9.4 applies for any  $r$ , it applies for an  $r$  such that  $\frac{1}{r+a}$  is less than  $\lambda$ . Moreover,

$$\lambda^{-1}(\mathbb{F} - s \otimes d\nu)1 \leq \frac{1}{(r+a)\lambda} + \frac{\tilde{c}}{\lambda}\mathbb{F}s.$$

Thus

$$\lambda^{-n}(\mathbb{F} - s \otimes \nu)^n 1 \leq 1 - \epsilon \sum_{j=0}^{n-1} \lambda^{-j-1}(\mathbb{F} - s \otimes \nu)^j 1 + \tilde{c} \sum_{j=0}^{n-1} \lambda^{-j-1}(\mathbb{F} - s \otimes \nu)^j \mathbb{F}s$$

where  $\epsilon = 1 - \frac{1}{r+a}$ . Rearranging terms and using that fact that  $\lambda^{-n}(\mathbb{F} - s \otimes \nu)^n 1 \geq 0$ ,

$$\epsilon \sum_{j=0}^{n-1} \lambda^{-j-1}(\mathbb{F} - s \otimes \nu)^j 1 \leq 1 + \tilde{c} \sum_{j=0}^{n-1} \lambda^{-j-1}(\mathbb{F} - s \otimes \nu)^j \mathbb{F}s.$$

Therefore,

$$\sum_{j=0}^{\infty} \lambda^{-j-1} (\mathbb{F} - s \otimes \nu)^j \mathbf{1} \leq \frac{1}{\epsilon} + \frac{\tilde{\mathbf{C}}}{\epsilon} \mathbb{G} \mathbb{F} s,$$

and hence  $\mathbb{G}$  is a bounded operator on  $L^\infty$ . □

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