IMPARTIALITY, PRIORITY, AND SOLIDARITY IN THE THEORY OF JUSTICE

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The ethic of priority is a compromise between the extremely compensatory ethic of outcome equality and the needs-blind ethic of resource equality. We propose an axiom of priority and characterize resource-allocation rules that are impartial, prioritarian, and solidaristic. They comprise a class of rules that equalize across individuals some index of outcome and resources. Consequently, we provide an ethical rationalization for the many applications in which such indices have been used (e.g., the human development index, the index of primary goods, etc.).

KEYWORDS: Impartiality, priority, solidarity, allocation rules, characterization result.

1. INTRODUCTION

IN THIS PAPER, we study the ethics of resource allocation in a basic and common problem. There is a resource, available in given quantity, to be allocated among individuals, each of whom possesses a capability to transform the resource into some given valued outcome, and the achievements of individuals with regard to that outcome are interpersonally comparable.[2]

In many resource-allocation problems of this sort, there are two focal points of distribution: to distribute the available resource equally among all who need it and to distribute the resource among the population so as to equalize the outcomes among them. There is a view that lies in between these two (extreme) focal points: to give *priority* to those who are less capable of transforming resource into outcome. We formalize this view as an axiom of allocation rules that indicate that no individual can dominate another in both resources and outcomes. We also formalize two other principles that we believe characterize fairness in many problems: *impartiality* and *solidarity*. The former says that in deciding how to allocate the resource, we ignore all personal attributes that are irrelevant, according to our moral standard, to the problem at hand.[3] The latter says that, if an allocation rule is fair, then when new individuals join a society (e.g., through birth or immigration), then the resources allocated to all

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[2]Think, for instance, of future earning power as a function of investment in education, longevity as a function of investment in health, etc.

[3]For instance, if the problem is one of allocating scarce rescuer time to saving earthquake victims, we ignore the victim's religion and race (though perhaps not age).

the *original* members should change in the same direction. Examples where we believe these principles either apply in common practice or, arguably, morality suggests that they should apply include the allocation of parental time to children, the distribution of a parent's estate among children, the allocation of the budget of educational finance, and the Americans with Disabilities Act.

In this paper, we characterize the set of allocation rules that jointly satisfy impartiality, priority, and solidarity. Our characterization result shows that the combination of these three notions is equivalent to a kind of egalitarianism, where the equality in question is equality of a conception of well-being that is some general *index* of resources and outcomes, where the index is not determined without further assumptions. Thus, there is a large class of rules that satisfy the three notions, and the equal-resource and equal-outcome rules are polar cases in that class, being on the conservative and radical ends.

2. THE MODEL

Let $\mathbb{I}$ represent a population of individuals who produce an objectively measurable output from a resource called *wealth*. For each $i \in \mathbb{I}$, let $u_i : \mathbb{R}_+ \to \mathbb{R}_+$ be the individual function that transforms wealth into the outcome. We assume that, for each i, u_i is continuous, strictly increasing, unbounded, and satisfies $u_i(0) = 0$.[4] We shall also assume that $\{u_i : i \in \mathbb{I}\}$ constitutes a *covering domain*, i.e., the graphs of these functions cover the positive quadrant. We say that an individual i is *disabled* with respect to another individual j if the former always needs at least the same wealth as the latter to reach the same level of the outcome, i.e., if $u_i \leq u_j$.

Let $\mathcal{I}$ be the family of all finite subsets of $\mathbb{I}$. We define an *economy* e as a triple (N, u, W), where $N = \{i_1, i_2, \ldots, i_n\} \in \mathcal{I}$ is the set of individuals, $u = (u_i)_{i \in N}$ is the profile of their outcome functions (defined in the preceding text), and $W \in \mathbb{R}_+$ represents the available wealth. The family of all economies is $\mathcal{E}$.

An *allocation rule* is a function F that associates to each economy $e = (N, u, W) \in \mathcal{E}$ a unique point $F(e) = (F_i(e))_{i \in N} \in \mathbb{R}^n_+$ such that $\sum_{i \in N} F_i(e) = W$. That is, an allocation rule indicates how to distribute the wealth available in an economy among its members.

Examples of rules are the following. First, the rule that awards each agent the same amount:

EQUAL-RESOURCE RULE: Equal resource is represented as $ER_i(N, u, W) = W/n$.

[4] This is a way to model that the level of outcome achieved is strictly increasing in wealth for every individual and that a wealth level of zero can be thought of as inducing death, which is an equally bad outcome for all individuals.

An alternative to the equal-resource rule is obtained by focusing on the levels of outcome individuals achieve, as opposed to what resources they receive, and choosing the vector at which these outcome levels are equal.

EQUAL-OUTCOME RULE: Equal outcome is represented as $EO_i(N, u, W) = u_i^{-1}(\lambda)$, where $\lambda > 0$ is chosen so that $\sum_{i \in N} u_i^{-1}(\lambda) = W$.

Note that, for all $i \in \mathbb{I}$, u_i^{-1} is a continuous, strictly increasing, and unbounded function that satisfies $u_i^{-1}(0) = 0$. From here, it follows that the equal-outcome rule is well defined.

We now present the axioms we want the rules to satisfy.

Impartiality is modeled by the domain axiom, because by defining rules on the class of economies $\mathcal{E}$, we are excluding much information about individuals that we consider ethically irrelevant.

We now turn to *priority*. Our axiom of *priority* says that no agent can dominate another agent in resources and outcome.

AXIOM 1—Priority (PR): *Let* $e = (N, u, W) \in \mathcal{E}$ *and* $i, j \in N$ *such that* $F_i(e) < F_j(e)$. *Then* $u_i(F_i(e)) \geq u_j(F_j(e))$.

Note that this axiom guarantees that disabled agents receive at least as much wealth as abler ones. That is, we discriminate positively toward the disabled. In other words, priority implies the so-called *weak equity* axiom introduced by Sen (1973). On the other hand, priority also says that the obligation toward the unfortunate is limited, because a disabled person is never resourced to the extent that her outcome exceeds that of an able agent. It is also straightforward to show that priority implies a weak version of anonymity that says that individuals who are equally able are rewarded equally. This is usually referred to as *symmetry*.

We conclude with *solidarity*. Here we rely upon a literature that has formulated various solidarity axioms in the past twenty years (e.g., Thomson (1983), Roemer (1986), Moulin (1987), Sprumont (1996)). Our notion of solidarity says that the arrival of immigrants, whether or not accompanied by changes in the available wealth, should affect all original agents in the same direction: all gain, all lose, or all receive the same as before.

AXIOM 2—Solidarity (SL): *Let* $e = (N, u, W) \in \mathcal{E}$ *and* $e' = (N', u', W') \in \mathcal{E}$ *such that* $N' \subseteq N$. *Let* $F_{N'}(e)$ *denote the projection of* $F(e)$ *onto the set of coordinates that correspond to* N'. *Then either* $F(e') = F_{N'}(e)$, $F(e') > F_{N'}(e)$, *or* $F(e') < F_{N'}(e)$.[5]

[5]Note that for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we write $x > y$ if $x_i > y_i$ for all $i = 1, \ldots, n$.

Solidarity can be decomposed into two axioms that appear frequently in the literature: *resource monotonicity* and *consistency*. Resource monotonicity (e.g., Roemer (1986)) says that when a bad or a good shock comes to an economy, all its members should share in the calamity or windfall. This concept is formally stated as follows:

AXIOM 3—Resource Monotonicity (RM): *Let* $e = (N, u, W) \in \mathcal{E}$ *and* $e' = (N', u', W') \in \mathcal{E}$ *such that* $N' = N$, $u' = u$, *and* $W' > W$. *Then* $F(e') > F(e)$.

Consistency (e.g., Young (1987)) says that if a subgroup of individuals secedes with the resource allocated to it under a rule, then in the smaller economy, the rule allocates the resource in the same way. This concept is formally stated as follows:

AXIOM 4—Consistency (CY): *Let* $e = (N, u, W) \in \mathcal{E}$. *Let* $N' \subset N$ *and* $e' = (N', u', W')$, *where* $u' = (u_i)_{i \in N'}$ *and* $W' = \sum_{i \in N'} F_i(e)$. *Then* $F_i(e) = F_i(e')$ *for all* $i \in N'$.

It is straightforward to show that the axiom of solidarity (Axiom 2) is equivalent to the combination of resource monotonicity (Axiom 3) and consistency (Axiom 4). It can also be shown (e.g., Moreno-Ternero and Roemer (2005)) that resource monotonicity implies the following technical axiom that will be useful in the ensuing discussion:

AXIOM 5—Resource Continuity (RC): *If* $W^n \to W$, *then* $F(N, u, W^n) \to F(N, u, W)$.

3. THE RESULTS

3.1. *The Characterization Result*

Both the equal-resource and the equal-outcome rules presented satisfy impartiality, priority, and solidarity. In this section, we identify all the remaining existing rules that satisfy these axioms. To do so, it is worth noting first that the equal-resource and the equal-outcome rules are somehow extreme rules among those that satisfy impartiality and priority. More precisely, the equal-resource rule is the best (worst) impartial and prioritarian rule for the ablest (disablest) agent in an economy, whereas the equal-outcome rule is the best (worst) impartial and prioritarian rule for the disablest (ablest) agent in an economy. This suggests that the set of all rules that satisfy impartiality, priority, and solidarity should consist of all rules that result from a compromise between the equal-resource and the equal-outcome rule. Our proposal for this compromise is the following.

Let Φ be the class of all functions $\varphi : \mathbb{R}^2_{++} \cup \{(0, 0)\} \to \mathbb{R}_+$, continuous on its domain and nondecreasing, such that $\inf\{\varphi(x, y)\} = \varphi(0, 0) = 0$ and, for all

$(x, y) > (z, t)$, $\varphi(x, y) > \varphi(z, t)$. Let φ be a function in the class Φ. For all $i \in \mathbb{I}$, define the function $\psi_i : \mathbb{R}_+ \to \mathbb{R}_+$ that determines the φ-value that agent i achieves, depending on the wealth she receives, i.e., $\psi_i(w) = \varphi(w, u_i(w))$ for all $w \in \mathbb{R}_+$. Then we define the corresponding *index-egalitarian* rule as the rule that equalizes the φ-value across individuals in an economy.

INDEX-EGALITARIAN RULE: Index egalitarianism is represented as $E_i^\varphi(e) = \psi_i^{-1}(\lambda)$, where $\lambda > 0$ is chosen so that $\sum_{i \in N} \psi_i^{-1}(\lambda) = W$.

Note that, for all $i \in I$, ψ_i^{-1} is a continuous, strictly increasing, and unbounded function that satisfies $\psi_i^{-1}(0) = 0$. From here, it follows that E^φ is well defined. Note also that applied in this manner to an agent's wealth and outcome, φ can be considered to be a generalized index of resources and outcomes. So the rules just defined equalize a generalized index of resources and outcomes.[6] The equal-resource rule is the E^{φ_1} rule, where $\varphi_1(x, y) = x$. The equal-outcome rule is the E^{φ_2} rule, where $\varphi_2(x, y) = y$.

All the rules within the family $\{E^\varphi\}_{\varphi \in \Phi}$ satisfy impartiality, priority, and solidarity. More remarkably, there is no other rule that satisfies these properties simultaneously, as the next result shows.

THEOREM 1: *A rule F satisfies impartiality, priority, and solidarity if and only if $F \in \{E^\varphi\}_{\varphi \in \Phi}$.*

Theorem 1 shows, in particular, that prioritarianism, at least in conjunction with solidarity, does not preclude equality, but it modifies the equalisandum from outcome to an index of resources and outcomes. Using indices of resources and outcomes to measure the success of an allocation procedure is a fairly common practice. The United Nations Development Program's human development indicator is an index of a country's gross domestic product, literacy rate, and infant mortality rate. Rawls (1971) worked, famously, with an index of primary goods: some of those goods were resources and some were outcomes. Sen (1980) has written of using an index of functionings as a possible measure of a person's welfare. In these examples, the social welfare supremum is thought to be the allocation of resources that equalizes the index in question at the highest possible level. This is our characterization theorem.

3.2. *Related Literature*

We conclude by relating our result to previous work on the literature on distributive justice. The simplest model in this literature is the widely studied *rationing model*. This model describes a situation in which a given amount of wealth has to be allocated among a group of agents, when the available

[6]We are indebted in a major way to Klaus Nehring, who suggested the E^φ rules.

amount is not enough to satisfy all their *claims*. Formally, a rationing problem is described by a triple (N, c, W), where N is the set of agents, $c \in \mathbb{R}^n_+$ is a vector of claims, and $W \in \mathbb{R}_+$ represents the amount of wealth to be divided. The very notion of the rationing problem requires $\sum_{i \in N} c_i \geq W > 0$. An allocation rule is a function F that associates with every (N, c, W) a unique point $F(N, c, W) \in \mathbb{R}^n$ such that $0 \leq F(N, c, W) \leq c$ and $\sum_{i \in N} F_i(N, c, W) = W$. One of the most influential results in this literature is owing to Young (1987), who characterizes the set of all rules that satisfy symmetry, resource continuity, and consistency. This set comprises the so-called *parametric rules*. Formally, a rule F is parametric if there exists a function $f : [a, b] \times \mathbb{R}_+ \to \mathbb{R}_+$, where $[a, b] \subseteq \mathbb{R}$, continuous and weakly monotonic in its first argument, such that (i) for all $x \in \mathbb{R}_+$, $f(a, x) = 0$, (ii) for all $x \in \mathbb{R}_+$, $f(b, x) = x$, and (iii) given $e = (N, c, W)$, there exists $\lambda_e \in [a, b]$ such that $F_i(e) = f(\lambda_e, c_i)$ for all $i \in N$.

The economic environments we use in this paper are not rationing problems, but they are quite related and therefore an approach similar to Young (1987) can be followed to construct parametric rules in our model. Let $\mathcal{V}$ denote the set of outcome functions we introduce in our model, i.e., $\mathcal{V} = \{v : \mathbb{R}_+ \to \mathbb{R}_+ :$ continuous, strictly increasing and such that $v(0) = 0$ and $\lim_{x \to \infty} v(x) = \infty\}$. We say that a rule F is *parametric* (in the model of our paper) if there exists a function $f : \mathbb{R}_+ \times \mathcal{V} \to \mathbb{R}_+$, continuous and strictly monotonic in its first argument, such that (i) for all $v \in \mathcal{V}$, $f(0, v) = 0$, (ii) for all $v \in \mathcal{V}$, $\lim_{\lambda \to \infty} f(\lambda, v) = \infty$, and (iii) given $e = (N, u, W) \in \mathcal{E}$, there exists $\lambda_e \in \mathbb{R}_+$ such that $F_i(e) = f(\lambda_e, u_i)$ for all $i \in N$.

Adapting an argument introduced by Young (1987) to our context we can show the following result[7]:

PROPOSITION 1: *If a rule satisfies impartiality, priority, and solidarity, then it is parametric.*

We show now, by means of an example, that the converse of Proposition 1 is not true. For each agent $i \in \mathbb{I}$, we define her *unity claim* as $u_i^{-1}(1)$, i.e., the wealth that i must receive to enjoy an outcome level of 1. We consider now the rule that allocates the wealth of an economy proportionally to agents' unity claims in the economy. Formally, for all $e = (N, u, W) \in \mathcal{E}$, $P(e) = \lambda \cdot (u_i^{-1}(1))_{i \in N}$, where $\lambda = W / (\sum_{i \in N} u_i^{-1}(1))$. Then P is a parametric rule that fails to satisfy priority.[8] To show this, let $u_1(x) = x/2$ and $u_2(x) = x^2$ for all $x \in \mathbb{R}_+$. Consider $e = (\{1, 2\}, \{u_1, u_2\}, 2)$. Then $P(e) = (4/3, 2/3)$ and $u_1(4/3) = 2/3 > 4/9 = u_2(2/3)$.

Proposition 1 and Theorem 1 show that the index-egalitarian rules are parametric, but the converse is not true. This makes our characterization result genuinely different from previous results in the literature.

[7]The reader is referred to Moreno-Ternero and Roemer (2005) for the proof.

[8]Its parametric representation would be $f(\lambda, u_i) = \lambda \cdot u_i^{-1}(1)$.

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APPENDIX: PROOF OF THEOREM 1

It is not difficult to show that all the E^φ rules are impartial and satisfy PR and SL. Conversely, let F be an impartial rule that satisfies PR and SL. Then, as mentioned previously, F satisfies RM, CY, and RC. The following claim, which holds thanks to PR and RC, will be used throughout the proof.

CLAIM: *For any $(i, \alpha) \in \mathbb{I} \times \mathbb{R}_+$, there exists $e \in \mathcal{E}$ such that $F_i(e) = \alpha$.*

Let $i \in \mathbb{I}$ and $\alpha \in \mathbb{R}_+$ be given. Define $E(F, i, \alpha) = \{e \in \mathcal{E} : F_i(e) = \alpha\}$ and $C(F, i, \alpha) = \{(a, b) \in \mathbb{R}^2_+ : a = F_j(e); b = u_j(a) \text{ for some } e \in E(F, i, \alpha)\}$. By the Claim, it follows that $C(F, i, \alpha)$ is not empty. Our aim is to show that the family of curves $\{C(F, i, \alpha) : \alpha \in \mathbb{R}_+\}$ is the isoquant map of an appropriate function $\varphi \in \Phi$ and to show from here that $F = E^\varphi$.

We show first that any $C(F, i, \alpha)$ is *downward sloping*, i.e., if $(a, b), (a', b') \in C(F, i, \alpha)$ and $a' > a$, then $b' \leq b$. Suppose, to the contrary, that $b' > b$. By definition, there exist $e = (N, u, W)$, $e' = (N', u', W') \in E(F, i, \alpha)$, and $j \in N$, $k \in N'$ such that $(a, b) = (F_j(e), u_j(F_j(e)))$ and $(a', b') = (F_k(e'), u_k(F_k(e')))$. Let $e^j = (\{i, j\}, (u_i, u_j), \alpha + a)$ and $e^k = (\{i, k\}, (u_i, u_k), \alpha + a')$. Then, by CY, $F(e^j) = (\alpha, a)$ and $F(e^k) = (\alpha', a')$. By the Claim, there exists $e^* = (\{i, j, k\}, (u_i, u_j, u_k), W^*) \in E(F, i, \alpha)$. Then $F_i(e^*) = F_i(e^j) = F_i(e^k) = \alpha$. Thus, by SL applied to e^* and e^j, $F_j(e^*) = F_j(e^j) = a$ and, by SL applied to e^* and e^k, $F_k(e^*) = F_k(e^k) = a'$. Therefore, $F_j(e^*) = a$, $F_k(e^*) = a'$, and so $F_j(e^*) < F_k(e^*)$. Thus, by PR, $u_j(F_j(e^*)) \geq u_k(F_k(e^*))$. However, we also know, by hypothesis, that $b = u_j(F_j(e^*)) < u_k(F_k(e^*)) = b'$, a contradiction.

We show now that $\{C(F, i, \alpha) : \alpha \in \mathbb{R}_+\}$ is a collection of *disjoint sets*. Let $\alpha_1 > \alpha_2$. Suppose $(a, b) \in C(F, i, \alpha_1) \cap C(F, i, \alpha_2)$. Let $e_1 = (N_1, u_1, \alpha_1) \in E(F, i, \alpha_1)$, $e_2 = (N_2, u_2, \alpha_2) \in E(F, i, \alpha_2)$, and $j \in N_1$, $k \in N_2$ such that $(a, b) = (F_j(e_1), u_j(F_j(e_1))) = (F_k(e_2), u_k(F_k(e_2)))$. Let $\widehat{e}_1 = (\{i, j\}, (u_i, u_j), a + \alpha_1)$ and $\widehat{e}_2 = (\{i, k\}, (u_i, u_k), a + \alpha_2)$. The CY axiom implies that $F_i(\widehat{e}_1) = \alpha_1$ and $F_i(\widehat{e}_2) = \alpha_2$. By the Claim, there exists $\widetilde{e}_2 = (\{i, k\}, (u_i, u_k), W) \in E(F, i, \alpha_1)$ with $W > a + \alpha_2$. By RM, applied to $\widehat{e}_2$ and $\widetilde{e}_2$, we know that $F_k(\widetilde{e}_2) > F_k(\widehat{e}_2) = a$. Therefore, $(a, b) < (F_k(\widetilde{e}_2), u_k(F_k(\widetilde{e}_2))) \in C(F, i, \alpha_1)$. This contradicts the fact that $C(F, i, \alpha_1)$ is downward sloping.

Next, we show that if $\alpha_1 > \alpha_2$, then $C(F, i, \alpha_1)$ *lies above* $C(F, i, \alpha_2)$, i.e., (i) for all $(a, b) \in C(F, i, \alpha_2)$ there exists $(a', b') \in C(F, i, \alpha_1)$ such that $(a, b) < (a', b')$; (ii) there is no $(a'', b'') \in C(F, i, \alpha_2)$ and $(a, b) \in C(F, i, \alpha_1)$ such that $(a'', b'') < (a, b)$.

(i) Let $(a, b) \in C(F, i, \alpha_2)$. Then there exists $e = (\{i, j\}, (u_i, u_j), W) \in E(F, i, \alpha_2)$ such that $F_j(e) = a$ and $u_j(F_j(e)) = b$. By the Claim and because $F_i(e) = \alpha_2$, there exists $e^* = (\{i, j\}, (u_i, u_j), W^*) \in E(F, i, \alpha_1)$ with $W^* > W$. Let $(a', b') = (F_j(e^*), u_j(F_j(e^*)))$. Then $(a', b') \in C(F, i, \alpha_1)$. Furthermore, because F satisfies RM and u_j is strictly increasing, $(a', b') > (a, b)$.

(ii) Conversely, let $(a, b) \in C(F, i, \alpha_1)$. Suppose there were a point $(a'', b'') \in C(F, i, \alpha_2)$ such that $(a'', b'') > (a, b)$. We know, by (i), that there is a point $(a''', b''') \in C(F, i, \alpha_1)$ such that $(a''', b''') > (a'', b'')$. Thus, $(a''', b''') > (a, b)$, which contradicts that $C(F, i, \alpha_1)$ is downward sloping.

Let $(a, b) \in \mathbb{R}^2_+$ and $i \in \mathbb{I}$ be given. By the assumption of covering domain and the preceding discussion, there exists a unique $\alpha \in \mathbb{R}_+$ such that $(a, b) \in C(F, i, \alpha)$. Define $\varphi : \mathbb{R}^2_{++} \cup \{(0, 0)\} \to \mathbb{R}_+$ by $\varphi(a, b) = \alpha$, where $\alpha \in \mathbb{R}_+$ is the unique number for which $(a, b) \in C(F, i, \alpha)$. We show now that $\varphi \in \Phi$.

By PR, it follows that $\varphi(0, 0) = 0 \leq \varphi(x, y)$ for all $(x, y) \in \mathbb{R}^2_+$.

Let $x, x', y \in \mathbb{R}_{++}$ such that $x < x'$. If $\varphi(x, y) > \varphi(x', y)$, then $C(F, i, \varphi(x, y))$ lies above $C(F, i, \varphi(x', y))$. In such a case, because $(x', y) \in C(F, i, \varphi(x', y))$, there exists $(z, t) \in C(F, i, \varphi(x, y))$ such that $(x', y) < (z, t)$. Then $(z, t) > (x, y)$. This contradicts that $C(F, i, \varphi(x, y))$ is downward sloping. Similarly, we show that $\varphi(x, y) \leq \varphi(x', y)$ for all $x, x', y \in \mathbb{R}_{++}$ such that $y < y'$.

Let $(x, y), (z, t) \in \mathbb{R}^2_+$ such that $(x, y) > (z, t)$. By downward sloppiness, $\varphi(x, y) \neq \varphi(z, t)$. If $\varphi(x, y) < \varphi(z, t)$, then $C(F, i, \varphi(z, t))$ lies above $C(F, i, \varphi(x, y))$. We have, however, that $(x, y) \in C(F, i, \varphi(x, y))$, $(z, t) \in C(F, i, \varphi(z, t))$, and $(x, y) > (z, t)$, which represents a contradiction.

Finally, we show that φ is continuous on $\mathbb{R}^2_+$. Let $\{(a_n, b_n)\} \to (a, b)$. We must show that $\{\alpha_n\} = \{\varphi(a_n, b_n)\} \to \alpha = \varphi(a, b)$. If such is not the case, then there exists a subsequence $\{\alpha_{k_n}\}$ that converges to $\overline{\alpha} \neq \alpha$. We assume that $\overline{\alpha} < \alpha$.[9] Then, for k_n sufficiently large, $\alpha_{k_n} < \frac{\alpha+\overline{\alpha}}{2} < \alpha$ and, therefore, $C(F, i, \alpha_{k_n})$ lies below $C(F, i, \frac{\alpha+\overline{\alpha}}{2})$ and this one lies below $C(F, i, \alpha)$. In particular, there exists a ball B about $(a, b) \in C(F, i, \alpha)$ that lies above $C(F, i, \frac{\alpha+\overline{\alpha}}{2})$. Whereas $(a_n, b_n) \to (a, b)$, it follows that for large k_n, $(a_{k_n}, b_{k_n}) \in B$. On the other hand, $(a_{k_n}, b_{k_n}) \in C(F, i, \alpha_{k_n})$, which, for large k_n, lies below $C(F, i, \frac{\alpha+\overline{\alpha}}{2})$. This represents a contradiction.

The proof of the theorem concludes by showing that $F = E^{\varphi}$, i.e., $F(N, u, W) = E^{\varphi}(N, u, W)$ for all $(N, u, W) \in \mathcal{E}$. Fix $e = (N, u, W) \in \mathcal{E}$. Two cases are distinguished.

[9]The proofs for the cases $\alpha < \overline{\alpha} < \infty$ and $\overline{\alpha} = \infty$ are similar.

CASE 1—$i \in N$: Let $\lambda = F_i(e)$. Then $(F_j(e), u_j(F_j(e))) \in C(F, i, \lambda)$ for all $j \in N$. Thus, $\psi_j(F_j(e)) = \lambda$ for all $j \in N$. Because $\sum_{j \in N} F_j(e) = W$, it follows that $F(e) = E^{\varphi}(e)$.

CASE 2—$i \notin N$: Let $j, k \in N$, and define $w_j = F_j(e)$ and $w_k = F_k(e)$. By the Claim, there exist $\widehat{e} = (\{i, j\}, (u_i, u_j), \widehat{W})$ and $\widetilde{e} = (\{i, k\}, (u_i, u_k), \widetilde{W})$ such that $w_j = F_j(\widehat{e})$ and $w_k = F_k(\widetilde{e})$. Let $\widehat{w}_i = F_i(\widehat{e})$ and $\widetilde{w}_i = F_i(\widetilde{e})$. We show that $C(F, j, w_j) = C(F, i, \widehat{w}_i)$. To do so, let $(a, b) \in C(F, j, w_j)$. Then there exists $l \in \mathbb{I}$ such that $b = u_l(a)$ and $(w_j, a) = (F_j(e^2), F_l(e^2))$, where $e^2 = (\{j, l\}, (u_j, u_l), w_j + a)$. By the Claim, there exists $e^3 = (\{i, j, l\}, (u_i, u_j, u_l), W^3)$ such that $F_i(e^3) = \widehat{w}_i$. Let $\widehat{e}^* = (\{i, j\}, (u_i, u_j), \widehat{w}_i + F_j(e^3))$. Then, by CY, $F(\widehat{e}^*) = (\widehat{w}_i, F_j(e^3))$. Thus, by RM applied to $\widehat{e}$ and $\widehat{e}^*$, $F_j(e^3) = w_j$. Let $e^{2*} = (\{j, l\}, (u_j, u_l), w_j + F_l(e^3))$. Then, by CY, $F(e^{2*}) = (w_j, F_l(e^3))$. Thus, by RM, $F_l(e^3) = a$. Thus, $F(e^3) = (\widehat{w}_i, w_j, a)$. Consequently, $(a, b) \in C(F, i, \widehat{w}_i)$, showing that $C(F, j, w_j) \subseteq C(F, i, \widehat{w}_i)$. The proof of the converse inclusion goes along the same lines. Analogously, it can be shown that $C(F, k, w_k) = C(F, i, \widetilde{w}_i)$. Thus, it follows that $(w_j, u_j(w_j)) \in C(F, i, \widehat{w}_i) \cap C(F, i, \widetilde{w}_i)$, which implies that $\widehat{w}_i = \widetilde{w}_i$. Let $\lambda = \widehat{w}_i = \widetilde{w}_i$. Then $C(F, j, w_j) = C(F, i, \lambda) = C(F, k, w_k)$. This shows, in particular, that $(F_l(e), u_l(F_l(e))) \in C(F, i, \lambda)$ for all $l \in N$. From here, the proof of Case 1 concludes. *Q.E.D.*

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