

**TEMPORAL RESOLUTION OF UNCERTAINTY AND
RECURSIVE NON-EXPECTED UTILITY MODELS**

BY

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TEMPORAL RESOLUTION OF UNCERTAINTY AND RECURSIVE
NON-EXPECTED UTILITY MODELS¹

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1. INTRODUCTION

KREPS AND PORTEUS' (1978) recursive expected utility model allows an agent to care intrinsically about the timing of the resolution of uncertainty. For example, an anxious agent may prefer early resolution while a hopeful agent may prefer late. Recursive expected utility achieves this flexibility by relaxing the reduction of compound lottery axiom for temporal lotteries. The model remains tractable thanks to recursivity: preferences today are built up from preferences tomorrow that do not themselves depend on unrealized contingencies. In addition to recursivity, Kreps and Porteus assumed that preferences over the lotteries within each stage satisfy the standard independence axiom. Recursive non-expected utility models keep the tractability of Kreps and Porteus' analysis while allowing both for preferences about the timing of resolution of uncertainty, and for violations of independence.² That is, in evaluating lotteries at each stage, recursive non-expected utility models replace independence by some weaker axiom.

In this paper, we show that if an agent always (weakly) prefers early resolution of uncertainty, then the recursive forms of the two most commonly used non-expected utility models, betweenness and rank dependence, each almost collapses back to recursive expected utility. More precisely, in the case of betweenness, violations of expected utility are only possible in the first stage. For the case of rank dependence, such violations are only possible in the last stage. Our result can be interpreted as providing support for Kreps and Porteus' model. Loosely speaking, to escape the recursive expected utility model, either (a) the agent's within-stage preferences must fail to conform to either the betweenness or rank-dependence model; or (b) she must be quite inconsistent in her preferences for early or late resolution; or (c) she must violate recursivity.³

Chew and Epstein (1989) define a premium to measure the degree of preference for early resolution in recursive betweenness models. They show that a constant premium implies independence. A constant premium, however, is a strong assumption: think of the analogy to a risk premium. Our result shows that it suffices for the premium not to change sign. Sarin and Wakker (1998) examine recursive preferences that are "sequentially consistent": the agent uses the same family of preferences not only to evaluate each

¹ We thank Eddie Dekel and Boaz Moselle, three referees and an editor for helpful comments and suggestions.

² For surveys of the evidence against independence see, for example, Machina (1987) and Sugden (forthcoming). For evidence consistent with recursive non-expected utility, see Conlisk (1989), Starmer and Sugden (1991), and Bernasconi (1994). For discussions for and against the recursive approach, see Segal (1990) and Epstein (1992), and Machina (1989). Recursivity also arises in Kreps and Porteus (1979) two-period induced preference model, where the preference for early resolution is not intrinsic but for planning purposes. With more stages, however, recursivity fails unless action choices are restricted to just before the last stage.

³ Grant, Kajii, and Polak (1998) give necessary and sufficient conditions for general (not necessarily recursive) smooth preferences to satisfy preference for early resolution. That paper also provides an example of recursive preferences that are neither betweenness nor rank-dependence, that can accommodate preference for early resolution.

stage of a temporal lottery, but also to evaluate the lotteries induced by each (normal-form) strategy in a decision tree. They find that recursivity and sequential consistency together with betweenness (respectively, rank dependence) imply that deviations from expected utility are also only possible in the first (respectively, last) stage. Sequential consistency, however, does not restrict an agent to prefer early resolution of uncertainty, and preference for early resolution does not imply sequential consistency. It is an open question why these seemingly quite different restrictions have such similar consequences.

2. FRAMEWORK AND RESULT

Suppose that, two periods from now, an agent will consume a risky outcome. We are interested in the agent's preferences about when the uncertainty concerning this outcome is resolved. One extreme case is that all uncertainty is resolved in the first stage, by the end of the first period. Alternatively, the agent might learn nothing until she gets the outcome at the end of the second period, or part of the uncertainty about the outcome could be resolved in each period. In more general problems, there could also be uncertainty about consumption at the end of the first period. It is enough for our purposes, however, to consider the case where there is no such uncertainty and, indeed, it is sufficient to look at the special case where there is no consumption at all before the end. Thus, the objects of choice for our agent are isomorphic to the set of two-stage lotteries, where each stage denotes a time period when uncertainty could be resolved.

Formally, let \mathcal{X} (with typical element x) be a set of outcomes. Unless otherwise stated, we take \mathcal{X} to be the interval $[0, 1]$.⁴ Let $\mathcal{L}(\mathcal{X})$ (with typical element X) denote the set of Borel probability measures on \mathcal{X} . We will sometimes refer to elements of $\mathcal{L}(\mathcal{X})$ as distributions or one-stage lotteries. Let $\mathcal{L}_0(\mathcal{X})$ be the subset of $\mathcal{L}(\mathcal{X})$ with finite support; the set of simple one-stage lotteries. With the weak topology of probability measures, $\mathcal{L}_0(\mathcal{X})$ is a dense subset of the compact metrizable space $\mathcal{L}(\mathcal{X})$. Let $\delta_x \in \mathcal{L}(\mathcal{X})$ be the degenerate one-stage lottery that assigns probability one to the outcome x . With slight abuse of notation, we will use $[(x_i, p_i)_{i=1}^n]$ to denote the simple one-stage lottery $\sum_{i=1}^n p_i \delta_{x_i}$, even though the outcomes (the x_i 's) need not be distinct. Let $\mathcal{L}(\mathcal{L}(\mathcal{X}))$ (with generic element X^2) denote the set of probability measures on $\mathcal{L}(\mathcal{X})$, the set of two-stage lotteries. Define $\mathcal{L}_0(\mathcal{L}(\mathcal{X}))$ similarly. With the weak topology, $\mathcal{L}_0(\mathcal{L}(\mathcal{X}))$ is a dense subset of the compact metrizable space $\mathcal{L}(\mathcal{L}(\mathcal{X}))$. Let $\delta_X^2 \in \mathcal{L}(\mathcal{L}(\mathcal{X}))$ be the degenerate two-stage lottery that assigns first-stage probability one to the second-stage lottery X . And let $X^2 := [(X_j, q_j)_{j=1}^M]$ denote the simple two-stage lottery $\sum_{j=1}^M q_j \delta_{X_j}^2$, where the second-stage lotteries (the X_j 's) need not be distinct. Loosely speaking, X^2 is a dynamic two-stage process where, in the first stage, a lottery X_i is chosen with probability q_i , and, in the second stage, an outcome is obtained according to X_i .

We can identify two special subclasses of two-stage lotteries: early-resolution lotteries where all uncertainty is resolved in the first stage, and late-resolution lotteries where no uncertainty is resolved in the first stage.⁵ Since both early- and late-resolution lotteries

⁴ The analysis can readily be extended to outcome sets that are general compact metric spaces if we assume that all welfare-relevant risk can be characterized as risk over the ranks of outcomes; see Grant, Kajii, and Polak (1992).

⁵ The latter set is a two-stage analogy of Kreps and Porteus' (1978) set $P_t(y_t)$, except we again restrict consumption to the end.

are degenerate in all but one stage, they are isomorphic to the set of one-stage lotteries. Each one-stage lottery X in $\mathcal{L}(\mathcal{X})$ is associated with the late-resolution lottery δ_X^2 . Also, for each such X , define the early-resolution lottery $\Phi_{er}(X)$ by the rule, for each Borel subset B of \mathcal{X} , $\Phi_{er}(X)(\{\delta_x: x \in B\}) := X(B)$. In particular, if $X = [(x_i, p_i)_{i=1}^n]$, then $\delta_X^2 = [[(x_i, p_i)_{i=1}^n], 1]$, and $\Phi_{er}(X) = [(\delta_{x_i}, p_i)_{i=1}^n]$. For all outcomes x , the trivial two-stage lottery $[\delta_x, 1]$, is both an early- and a late-resolution lottery.

Let \succeq_2 denote a preference relation over the set of two-stage lotteries. These preferences pertain to an agent at time 0, looking forward at a two-stage process during which uncertainty about the final two-period-away outcome will be resolved; that is, these preferences refer to time before any uncertainty has yet had a chance to be resolved.⁶ We will assume throughout that the preference relation \succeq_2 is complete and transitive, and is continuous on $\mathcal{L}(\mathcal{L}(\mathcal{X}))$. Let \succeq_{er} and \succeq_{lr} be, respectively, the restrictions of \succeq_2 to early- and late-resolution lotteries. Since we want to allow the timing of resolution of uncertainty to matter for the agent, we do *not* require that $\Phi_{er}(X) \sim_2 \delta_X^2$ for all one-stage lotteries X in $\mathcal{L}(\mathcal{X})$; that is, we do not require the preference relations \succeq_{er} and \succeq_{lr} to be the same. Both \succeq_{er} and \succeq_{lr} inherit continuity from \succeq_2 . Given our isomorphisms, the preference relations \succeq_{er} and \succeq_{lr} can be endowed with properties of preferences over one-stage lotteries. We assume throughout that both \succeq_{er} and \succeq_{lr} respect first-order stochastic dominance. Thus, we can define their certainty-equivalent functions. For example, the late-resolution certainty-equivalent function, $CE_{lr}(\cdot): \mathcal{L}(\mathcal{X}) \rightarrow \mathcal{X}$, maps each one-stage lottery X in $\mathcal{L}(\mathcal{X})$ to the outcome in \mathcal{X} such that $\delta_X^2 \sim_{lr} [\delta_{CE_{lr}(X)}, 1]$.

The following properties of preferences over (sets isomorphic to) one-stage lotteries can be axiomatized but since they are well-known, we define them in terms of their representations. Exploiting the isomorphisms, we define a representation of a preference relation \succeq_{lr} or \succeq_{er} , as a function with domain $\mathcal{L}(\mathcal{X})$.

DEFINITION: We say that the preference relation \succeq_{er} (respectively, \succeq_{lr}) is *represented by a functional* $V: \mathcal{L}(\mathcal{X}) \rightarrow \mathbb{R}$ if for all pairs of lotteries X and Y in $\mathcal{L}(\mathcal{X})$, $V(X) \geq V(Y)$ if and only if $\Phi_{er}(X) \succeq_{er} \Phi_{er}(Y)$ (respectively, $\delta_X^2 \succeq_{lr} \delta_Y^2$). We say that the preference relation \succeq_{er} (respectively, \succeq_{lr}) satisfies:

- (a) *expected utility* if there exists a function $u: \mathcal{X} \rightarrow \mathbb{R}$, and a representation V given by $V(X) = \int u(x)X(dx)$ for each X in $\mathcal{L}(\mathcal{X})$;
- (b) *betweenness*⁷ if there exists a function $v: \mathcal{X} \times \mathbb{R} \rightarrow \mathbb{R}$, and a representation V implicitly defined by $V(X) = \int v(x, V(X))X(dx)$ for each X in $\mathcal{L}(\mathcal{X})$;
- (c) *rank dependence*⁸ if there exists a function $u: \mathcal{X} \rightarrow \mathbb{R}$, a strictly increasing function $g: [0, 1] \rightarrow [0, 1]$, with $g(0) = 0$ and $g(1) = 1$, and a representation V given by $V(X) = -\int u(x)d[g \circ G_X(x)]$, where G_X is the decumulative function of X , for each X in $\mathcal{L}(\mathcal{X})$.

The following substitution property (recursivity) involves replacing one one-stage lottery X_j by another Y_j within a two-stage lottery.

⁶ This preference relation is analogous to that defined in Kreps and Porteus (1978, Section 4, p. 195), specialized to two periods and to the case where no consumption takes place before the last period.

⁷ See Chew (1983, 1989) and Dekel (1986).

⁸ See Quiggin (1982), Yaari (1987).

DEFINITION: We say that an agent’s preference relation over two-stage lotteries, \succeq_2 , satisfies *recursivity* if for all pairs of two-stage lotteries in $\mathcal{L}_0(\mathcal{L}(\mathcal{X}))$ of the form $X^2 = [(X_i, q_i)_{i=1}^N]$ and $Y^2 = [(X_1, q_1; \dots; X_{j-1}, q_{j-1}; Y_j, q_j; X_{j+1}, q_{j+1}; \dots; X_N, q_N)]$ with $q_i > 0$: $X^2 \succeq_2 Y^2$ if and only if $\delta_{X_i}^2 \succeq_{lr} \delta_{Y_j}^2$.

Although the condition is written for the restriction of \succeq_2 over $\mathcal{L}_0(\mathcal{L}(\mathcal{X}))$, recursivity is also a property of \succeq_2 over $\mathcal{L}(\mathcal{L}(\mathcal{X}))$ since the continuous extension to $\mathcal{L}(\mathcal{L}(\mathcal{X}))$ of a continuous preference relation over $\mathcal{L}_0(\mathcal{L}(\mathcal{X}))$ is unique.

Recursivity in preferences over two-stage lotteries is a monotonicity property similar to first-order stochastic dominance in preferences over one-stage lotteries. First-order stochastic dominance says that preferences over one-stage lotteries respect the preference order over degenerate one-stage lotteries (that is, over outcomes). Recursivity says that the preferences over two-stage lotteries respect the preference order over degenerate two-stage lotteries (that is, over second-stage lotteries).⁹ The power of this property is that we can apply it effectively to reduce preferences over two-stage lotteries to preferences over one-stage lotteries. First, we replace each second-stage lottery by its late-resolution certainty equivalent. By definition, for all X in $\mathcal{L}(\mathcal{X})$, $[\delta_{CE_{lr}(X)}, 1] \sim_{lr} [X, 1]$. So, given recursivity, any two-stage lottery X^2 is indifferent to the early-resolution lottery $\Phi_{er}(X)$, where X is the element of $\mathcal{L}(\mathcal{X})$ such that, for any Borel subset B of \mathcal{X} , $X(B) = X^2(\{Y \in \mathcal{L}(\mathcal{X}): CE_{lr}(Y) \in B\})$.¹⁰ We then evaluate $\Phi_{er}(X)$ using \succeq_{er} . This trick is well-known and is sometimes called the fold-back or recursive method.¹¹ This method can be applied for any forms of preference we choose for \succeq_{er} and \succeq_{lr} (given continuity and respect for first order stochastic dominance), confirming that recursivity does not imply expected utility. That is, recursivity neither implies: for all X, Y , and Z in $\mathcal{L}(\mathcal{X})$ and all α in $(0, 1]$, $\Phi_{er}(X) \succeq_{er} \Phi_{er}(Y)$ if and only if $\Phi_{er}(\alpha X + (1 - \alpha)Z) \succeq_{er} \Phi_{er}(\alpha Y + (1 - \alpha)Z)$; nor does it imply the analogous axiom for \succeq_{lr} . Kreps and Porteus’s recursive expected utility model combines recursivity (“temporal consistency”) with these “temporal substitution” or expected utility axioms. By retaining recursivity but dropping “temporal substitution,” we move from recursive expected utility to more general recursive forms of preference.

The following substitution property captures the idea that the agent prefers uncertainty to be resolved earlier rather than later.

DEFINITION: For any two stage lotteries, we say that an agent’s preference relation over two-stage lotteries, \succeq_2 , exhibits *preference for early resolution of uncertainty* if for all pairs of two-stage lotteries of the form $X^2 = [(X_i, q_i)_{i=1}^N]$ and $Y^2 = [(X_1, q_1; \dots; X_{j-1}, q_{j-1}; Y_1, \beta q_j; Y_2, (1 - \beta)q_j; X_{j+1}, q_{j+1}; \dots; X_N, q_N)]$ in $\mathcal{L}_0(\mathcal{L}(\mathcal{X}))$ with β in $[0, 1]$: if $X_j = \beta Y_1 + (1 - \beta)Y_2$ then $Y^2 \succeq_2 X^2$.

As with recursivity, although the above condition is written for the restriction of \succeq_2 over $\mathcal{L}_0(\mathcal{L}(\mathcal{X}))$, preference for early-resolution of uncertainty is also a property of \succeq_2 over $\mathcal{L}(\mathcal{L}(\mathcal{X}))$. Loosely speaking, whereas recursivity involves the replacement of one

⁹ Kreps and Porteus’ original recursivity axiom, “temporal consistency,” requires that preferences over parent lotteries respect preference relations over sublotteries.

¹⁰ Since \succeq_{lr} is continuous, this construction is well-defined.

¹¹ The fact that this method can be used recursively has more bite if there are more than two stages.

“branch” of a lottery by another branch, preference for early resolution involves “splitting” one branch into two.

We can now state our result.

PROPOSITION 1: *Suppose that an agent’s preference relation, \succeq_2 , over two-stage lotteries satisfies recursivity and preference for early resolution, and that the restricted preference relations, \succeq_{er} and \succeq_{lr} , each respect first-order stochastic dominance. Then:*

- (i) *if the preference relations \succeq_{er} and \succeq_{lr} satisfy betweenness and \succeq_{er} either satisfies risk aversion or smoothness (Gateaux differentiability), then \succeq_{lr} satisfies expected utility;*
- (ii) *if the preference relations \succeq_{er} and \succeq_{lr} satisfy rank dependence and \succeq_{er} satisfies risk aversion, then \succeq_{er} satisfies expected utility.*

Since the proof in Section 3 is long and involved, we provide an outline. It is known that, given recursivity and preference for early resolution, if preferences over early-resolution lotteries are smooth, then preferences for late-resolution lotteries have a convex representation. A lemma shows that if preferences over early-resolution lotteries are risk averse, then we can dispense with smoothness; in this case, the late-resolution certainty-equivalent function is itself convex. For part (i) then, it is enough to show that betweenness preferences cannot be convexified unless they are expected utility. To do this, we formalize an intuitive argument of Aumann (1975, p. 629). For rank-dependence preferences, risk aversion (respectively, loving) implies that the distortion of the decumulative distribution function, g , is convex (concave). It is enough then to show that preference for early resolution implies that the distortion function for preferences over early-resolution lotteries is also concave. To do this, we exploit the idea that preference for early resolution is analogous to loving a “riskier” (more spread out) distribution of second-stage lotteries,¹² and hence involves concave distortion functions.

It is straightforward to extend the results to n -stage lotteries.¹³ For betweenness, violations of independence are only possible in the last stage; and, for rank dependence, only in the first. In each case, the risk aversion or smoothness assumptions are only required for preferences over the first-stage. Indeed, for the rank-dependence case, even without risk aversion in any stage, violations of independence are only possible in the first and last stage. Preference for early resolution and recursivity imply that preferences over later-stage lotteries are quasi-convex.¹⁴ And this, without requiring risk aversion, is enough to ensure that the associated later-stage rank-dependent distortion functions are convex.¹⁵ For all but the last stage, however, the analogy between preference for early resolution and risk loving ensures that the distortion functions must also be concave.

We state the above results for the case of preference for early resolution but similar results apply if the agent always prefers late resolution.¹⁶ More generally, whether an

¹² This is not to say that preferences over late-resolution lotteries are risk loving.

¹³ In our earlier working paper, Grant, Kajii, and Polak (1997), the analysis was conducted in an n -period model.

¹⁴ See Grant, Kajii, and Polak (1998, Proposition 1(ii)).

¹⁵ See Wakker (1994, Theorem 24).

¹⁶ For betweenness: replace first-stage smoothness or risk aversion by first-stage smoothness or risk loving; and argue using concavity for convexity. For rank dependence: replace first-stage risk aversion by second-stage risk aversion, and use Grant, Kajii, and Polak (1998, Proposition 1(ii)) to argue that second-stage preferences are quasi-convex. These second-stage preferences must then satisfy independence.

agent prefers early or late resolution might depend on the outcomes at stake. For example, there might be some closed set of outcomes B in \mathcal{X} such that an agent prefers early resolution for all two-stage lotteries in $\mathcal{L}_0(\mathcal{L}(B))$. Similarly, there might be some convex subset \mathcal{B} in $\mathcal{L}(\mathcal{X})$ such that an agent prefers early resolution for all two-stage lotteries in $\mathcal{L}_0(\mathcal{B})$. In each case, the above results adapt to preferences over the relevant set of two-stage lotteries.¹⁷ Finally, the proof of the first result above does not require betweenness for \succeq_{er} , and that of the second result does not require rank-dependence for \succeq_{lr} , though these assumptions are natural when we extend to n -stages.

3. PROOF

LEMMA 1: *Suppose that an agent’s preference relation, \succeq_2 , over two-stage lotteries satisfies recursivity and preference for early resolution; the restricted preference relation \succeq_{er} and \succeq_{lr} each respect first-order stochastic dominance; and that \succeq_{er} satisfies risk aversion. Then the certainty-equivalent function $CE_{lr}(\cdot)$ is convex in the probabilities.*

PROOF: Preference for early resolution implies $[X, \alpha; Y, (1 - \alpha)] \succeq_2 [(\alpha X + (1 - \alpha)Y), 1]$. Recursivity implies $[X, \alpha; Y, (1 - \alpha)] \sim_2 [\delta_{CE_{lr}(X)}, \alpha; \delta_{CE_{lr}(Y)}, (1 - \alpha)]$ and $[(\alpha X + (1 - \alpha)Y), 1] \sim_2 [\delta_{CE_{lr}(\alpha X + (1 - \alpha)Y)}, 1]$. And risk aversion of \succeq_{er} implies $[\delta_{\alpha CE_{lr}(X) + (1 - \alpha)CE_{lr}(Y)}, 1] \succeq_{er} [\delta_{CE_{lr}(X)}, \alpha; \delta_{CE_{lr}(Y)}, (1 - \alpha)]$. Combining, and applying first-order stochastic dominance (indeed just monotonicity), implies $\alpha CE_{lr}(X) + (1 - \alpha)CE_{lr}(Y) \geq CE_{lr}(\alpha X + (1 - \alpha)Y)$. Q.E.D.

We are now ready to prove the first part of the proposition.

PROOF OF PART (I) (BETWEENNESS): Suppose that the function $V: \mathcal{L}(\mathcal{X}) \rightarrow \mathbb{R}$ represents the preference relation, \succeq_{lr} . If \succeq_{lr} satisfies betweenness then, for all pairs of lotteries X and Y in $\mathcal{L}(\mathcal{X})$ and all α in $[0, 1]$, if $V(X) = V(Y)$ then $V(\alpha X + (1 - \alpha)Y) = V(X)$. Recall that the certainty-equivalent function $CE_{lr}(\cdot)$ is a representation of \succeq_{lr} . By Lemma 1, if \succeq_{er} is risk averse, this representation is convex in the probabilities. Alternatively, Grant, Kajii and Polak (1998, Proposition 1(iv)) shows that if \succeq_{er} is smooth (Gateaux differentiable), then the representation $CE_{lr}(\cdot)$ is convexifiable. Therefore, it is enough to prove the following lemma.

LEMMA 2: *Suppose that V_{lr} represents a preference relation \succeq_{lr} that satisfies betweenness. If there exists a strictly increasing function h from \mathbb{R} to \mathbb{R} such that $h \circ V_{lr}$ is convex, then the preference relation satisfies expected utility.*

Roughly speaking, if indifference curves are planar, they cannot be represented by a convex utility function unless they are also parallel. So, from the convexity of CE_{lr} , the preference relation \succeq_{lr} must satisfy expected utility.

¹⁷ For example, let B be a closed subset of outcomes in \mathcal{X} . Suppose that an agent’s preference relation, \succeq_2 , over two-stage lotteries $\mathcal{L}(\mathcal{L}(\mathcal{X}))$ satisfies recursivity, the restricted preference relations, \succeq_{er} and \succeq_{lr} , each respect first-order stochastic dominance and betweenness, and \succeq_{er} either satisfies risk aversion or smoothness. If preferences over all two-stage lotteries in the set $\mathcal{L}_0(\mathcal{L}(B))$ also satisfy preference for early resolution, then the restriction of \succeq_{lr} to the late-resolution lotteries isomorphic to $\mathcal{L}(B)$ satisfies expected utility.

PROOF: Suppose that the preference relation \succeq_{lr} does not satisfy independence. Then, there exist X_1, X_2, Y in $\mathcal{L}(\mathcal{X})$ such that $V_{lr}(X_1) = V_{lr}(X_2)$ but $V_{lr}(\alpha X_1 + (1 - \alpha)Y) > V_{lr}(\alpha X_2 + (1 - \alpha)Y)$ for some α in $(0, 1)$. Consider the convex hull of $\{X_1, X_2, Y\}$. For each x in the set $\Delta = \{(x_1, x_2) \in \mathbb{R}_+^2: x_1 + x_2 \leq 1\}$, let $V(x) := V_{lr}(x_1 X_1 + x_2 X_2 + (1 - x_1 - x_2)Y)$. The function $h \circ V$ is convex on Δ , and its level sets are straight, but not parallel, lines. So we are done if we can show that a function on \mathbb{R}^2 whose level curves are linear but not parallel on a convex set with nonempty interior cannot be convexified.¹⁸

Let f be a convex function on Δ . For each x in Δ , let $\partial f(x)$ be the set of subgradient vectors at x ; that is, $p \in \partial f(x)$ if and only if $p \cdot (x' - x) \leq f(x') - f(x)$ for all x' in Δ . This definition implies monotonicity: if $p \in \partial f(x)$ and $p' \in \partial f(x')$, then $(p' - p) \cdot (x' - x) \geq 0$. Also, Rockafellar (1970 pp. 233–234) shows: (a) if $x \in \text{Int } \Delta$ then $\partial f(x)$ is nonempty and bounded; and if a sequence $\{x^n: n = 1, \dots\}$ in Δ converges to x , there is $y^n \in \partial f(x^n)$ such that a subsequence of $\{y^n\}$ converges and the limit belongs to $\partial f(x)$. And (b), for any v in \mathbb{R}^2 , $\lim_{t \rightarrow +0} (1/t)(f(x + tv) - f(x)) = \sup\{p \cdot v: p \in \partial f(x)\}$. Assume the level sets of f are straight lines. Then all subgradients at x are perpendicular to the corresponding level line, and in particular, by (a), there is a unique largest gradient vector in $\partial f(x)$.

Consider a rectangle in the interior of Δ on which some level sets of f are not parallel. By an appropriate choice of coordinates, we can identify the rectangle with $[0, 1] \times [0, 1]$. We can choose the rectangle so that its base lies along a level set, and such that $f((0, s))$ is strictly increasing in s . For each s in $[0, 1]$, let $g(s)$ be the slope of the level curve through point $(0, s)$. Since f is convex, it is continuous on the relative interior of its domain, so g is a continuous function. By construction, $g(0) = 0$. Since the level sets are not all parallel, $g(s)$ is not identical to 0. Without loss of generality, assume $g(\bar{s}) > 0$ for some $\bar{s} \in (0, 1]$, and let $s^* = \sup\{s \in [0, \bar{s}]: g(s) \leq 0\}$. Since g is continuous, $s^* < \bar{s}$. Thus we can adjust our rectangle so that $s^* = 0$. We have now constructed a rectangle with $g(0) = 0$ and $g(s) > 0$ for every s in $(0, \bar{s}]$. In our new coordinates, if $g(s) \neq g(s')$, then the level curves corresponding to s and s' intersect at

$$r(s, s') := \left(-\frac{s - s'}{g(s) - g(s')}, \frac{s'g(s) - sg(s')}{g(s) - g(s')} \right),$$

which must be outside Δ .

Suppose that for all $s \in (0, \bar{s}]$, there is $s' \in (0, s)$ such that $-(s/g(s)) \leq -(s'/g(s'))$. Then we can find a decreasing sequence $s^n, n = 1, 2, \dots$, with $s^n \rightarrow 0$, such that the intersections $r(s^n, 0)$ of the level curves corresponding to s^n and 0 stay bounded as s^n approaches 0. Write $x^0 = (0, 0)$, $x^n = (0, s^n)$, $z^0 = (1, 0)$, $z^n = (1, s^n + g(s^n))$, and $r^n = r(s^n, 0)$. By construction, $f(x^n) = f(z^n)$ for all n , and $x^n \rightarrow x^0$ and $z^n \rightarrow z^0$ as $n \rightarrow \infty$. Now by Rockafellar's fact (a) above, we can find a sequence of subgradient vectors $q^n \in \partial f(z^n)$ whose subsequence converges to a vector $q^0 \in \partial f(z^0)$. And by fact (b) above, if we set $p^0 \in \partial f(x^0)$ to be the largest subgradient, then $\lim_{t \rightarrow +0} (1/t)(f(x^0 + tv) - f(x^0)) = \|p^0\|$, where $v = (1/\|p^0\|)p^0$. Since $g(s) > 0$ for all $s \in (0, \bar{s}]$, $f(x^0 + tv) - f(x^0) > f(z^0 + tv) - f(z^0)$ for any $t > 0$. Since $q^0 \in \partial f(z^0)$ is colinear to p^0 , $f(z^0 + tv) - f(z^0) \geq q^0 \cdot (z^0 + tv - z^0) = t\|q^0\|$. Thus $(1/t)(f(x^0 + tv) - f(x^0)) > \|q^0\|$ for any $t > 0$, and so $\|p^0\| \geq \|q^0\|$.

¹⁸ This can be shown by applying Kannai (1977)'s general result but we give a direct argument here. For intuition, see Aumann (1975, p. 629).

On the other hand, by monotonicity, $q^n \cdot (z^n - x^0) \geq p^0 \cdot (z^n - x^0) (> 0)$. Since q^n and p^0 are perpendicular to $(z^n - r^n)$ and $(r^n - x^0)$ respectively, this becomes

$$1 \geq \frac{p^0 \cdot (z^n - r^n)}{q^n \cdot (r^n - x^0)}.$$

And, since the angles between the vectors in the numerator and the denominator are the same, we get

$$1 \geq \frac{\|p^0\| \|z^n - r^n\|}{\|q^n\| \|x^0 - r^n\|}$$

for every n . But this is a contradiction since

$$\frac{\|p^0\|}{\|q^n\|} \rightarrow \frac{\|p^0\|}{\|q^0\|} \geq 1, \quad \text{and} \quad \frac{\|z^n - r^n\|}{\|x^0 - r^n\|} > 1$$

and is bounded away from 1 since r^n is bounded.

Thus, there exists an $\hat{s} \in (0, \bar{s}]$, such that for all $s \in (0, \hat{s})$, $-(s/g(s)) < -(\hat{s}/g(\hat{s}))$. This implies that $g(\hat{s}) > g(s)$, and

$$-\frac{\hat{s} - s}{g(\hat{s}) - g(s)} > -\frac{\hat{s}}{g(\hat{s})}.$$

Hence $r(\hat{s}, s)$ exists for all $s \in (0, \hat{s})$ and the set $\{r(\hat{s}, s) : s \in (0, \hat{s})\}$ is bounded. Then the situation is symmetric to that above, with the level curve through $(0, \hat{s})$ playing the role of the base line. Choosing an increasing sequence $\{s^n : n = 1, 2, \dots\}$ with $s^n \rightarrow \hat{s}$, a symmetric argument leads to a contradiction. *Q.E.D.*

PROOF OF PART (II) (RANK-DEPENDENCE): The following lemma extends a result of Chew, Karni, and Safra (1987) eschewing the need for Gateaux differentiability.

LEMMA 3: *Let $V: \mathcal{L}(\mathcal{X}) \rightarrow \mathbb{R}$ be given by $V(X) = -\int u(x)d[g \circ G_X(x)]$ where G_X is the decumulative function of X , for each X in $\mathcal{L}(\mathcal{X})$, and u and g are functions satisfying the definitions of rank dependence. If the preference relation represented by V is risk averse (loving) then g is convex (concave).*

PROOF: Suppose g is not convex. Then there exists a \hat{q} in $(0, 1)$ and an η in $(0, \min\{\hat{q}, 1 - \hat{q}\})$ such that $\frac{1}{2}g(\hat{q} + \eta) + \frac{1}{2}g(\hat{q} - \eta) < g(\hat{q})$; that is, $[g(\hat{q}) - g(\hat{q} - \eta)] - [g(\hat{q} + \eta) - g(\hat{q})] > 0$. Recall that we take $\mathcal{X} = [0, 1]$. Consider the pair of one-stage lotteries, $Y = [0, (1 - \hat{q} - \eta); x, 2\eta; 1, (\hat{q} - \eta)]$ and $Z = [0, (1 - \hat{q} - \eta); x - \varepsilon, \eta; x + \varepsilon, \eta; 1, (\hat{q} - \eta)]$ in $\mathcal{L}(\mathcal{X})$ where x is in $(0, 1)$ and ε is in $(0, \min\{x, 1 - x\})$. Risk aversion implies that $V(Y) \geq V(Z)$; that is,

$$(1) \quad [g(\hat{q} + \eta) - g(\hat{q})][u(x) - u(x - \varepsilon)] \geq [g(\hat{q}) - g(\hat{q} - \eta)][u(x + \varepsilon) - u(x)].$$

If u is not concave, then we can choose our x and ε such that $[u(x) - u(x - \varepsilon)] < [u(x + \varepsilon) - u(x)]$, yielding a contradiction. Therefore assume u is concave. Since preferences respect strict first order stochastic dominance, the function u is strictly increasing. Since a concave function is absolutely continuous, we conclude that there is an x such that $u'(x)$ exists and is strictly positive (see, for example, Royden (1988, pp. 113–114)).

Therefore, dividing both sides of expression (1) by ε and taking the limit as ε goes to zero, we get $[g(\hat{q} + \eta) - g(\hat{q})]u'(x) \geq [g(\hat{q}) - g(\hat{q} - \eta)]u'(x)$, a contradiction. The proof for risk loving is similar. Q.E.D.

An immediate implication of Lemma 3 is that, for the first stage, g_{er} is convex. We next show that g_{er} is concave. Suppose that g_{er} is not concave. Then there exists a \hat{q} in $(0, 1)$ and an η in $(0, \min\{\hat{q}, 1 - \hat{q}\})$ such that $[g_{er}(\hat{q}) - g_{er}(\hat{q} - \eta)] - [g_{er}(\hat{q} + \eta) - g_{er}(\hat{q})] < 0$. For any r in $[0, 1]$, let $X_r := [1, r; 0, (1 - r)]$ in $\mathcal{L}(\mathcal{X})$ be the one-stage lottery that places weight r on the best outcome and $(1 - r)$ on the worst. Consider the pair of two-stage lotteries, Y^2 and Z^2 , in $\mathcal{L}(\mathcal{L}(\mathcal{X}))$ given by $Y^2 := [X_0, (1 - \hat{q} - \eta); X_{\hat{r}}, 2\eta; X_1, (\hat{q} - \eta)]$ and $Z^2 := [X_0, (1 - \hat{q} - \eta); X_{\hat{r} - \varepsilon}, \eta; X_{\hat{r} + \varepsilon}, \eta; X_1, (\hat{q} - \eta)]$ where \hat{r} is in $(0, 1)$ and ε is in $(0, \min\{\hat{r}, 1 - \hat{r}\})$. Preference for early resolution implies $Z^2 \succeq_2 Y^2$.

Without loss of generality, normalize $u_{lr}(1) = 1$ and $u_{lr}(0) = 0$. For all r in $[0, 1]$, let $c(r) := CE_{lr}(X_r)$. Hence, using recursivity, $Z^2 \succeq_2 Y^2$ implies

$$(2) \quad [g_{er}(\hat{q} + \eta) - g_{er}(\hat{q})][u_{er} \circ c(\hat{r}) - u_{er} \circ c(\hat{r} - \varepsilon)] \\ \leq [g_{er}(\hat{q}) - g_{er}(\hat{q} - \eta)][u_{er} \circ c(\hat{r} + \varepsilon) - u_{er} \circ c(\hat{r})].$$

Expression (2) is analogous to expression (1) with the inequality reversed, the compound function $u_{er} \circ c$ taking the place of u and the function g_{er} taking the place of g . Since our choice of \hat{r} was arbitrary, an argument exactly analogous to that in Lemma 3 generates a contradiction. Since g_{er} is both concave and convex, \succeq_{er} must be expected utility. Q.E.D.

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