# EPISTEMIC CONDITIONS FOR NASH EQUILIBRIUM, AND COMMON KNOWLEDGE OF RATIONALITY

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# EPISTEMIC CONDITIONS FOR NASH EQUILIBRIUM, AND COMMON KNOWLEDGE OF RATIONALITY

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## INTRODUCTION

AUMANN AND BRANDENBURGER (1995) show that, if players' payoffs are mutually known, their rationality is mutually known, their beliefs (or "conjectures") about other players' actions are commonly known and they have a common prior, then, for each player $j$, the conjectures of all the other players about $j$'s action agree and the $n$-tuple of such conjectures (one conjecture about each player) form a Nash equilibrium when viewed as a mixed strategy profile. The authors point out a remarkable aspect of the result: this set of sufficient epistemic conditions for Nash equilibrium does not include common knowledge of rationality. Earlier studies (including some by Aumann and by Brandenburger) had left the impression that common knowledge of rationality would be essential.

This note points out that if we strengthen mutual knowledge of payoffs to common knowledge of payoffs, then the Aumann-Brandenburger conditions imply common knowledge of rationality. In fact, we can dispense with one condition: we do not need the common prior. The intuition is quite straightforward. As a lemma below shows, even without strengthening, the Aumann-Brandenburger conditions imply that it is common knowledge that each player's action is among those that maximize her (mutually known) payoffs with respect to her (commonly known) conjecture about the other players' actions. This is not, however, common knowledge of rationality. Each player $i$ knows that each other player $j$ knows that $i$'s action is among those that maximize her payoffs (say, $g_i$), but $i$ does not necessarily know that $j$ knows that $i$'s payoff function is indeed $g_i$. If we add, however, that the payoffs are commonly known, we obtain common knowledge of rationality.

In games of complete information, common knowledge of payoffs is usually taken to be implicit. Indeed, this is often taken to be the definition of complete information. Thus, given complete information, if rationality is not common knowledge then the Aumann-Brandenburger conditions for Nash equilibrium do not apply.[2]

## SET-UP AND NOTATION

The set-up and notation follow that of Aumann and Brandenburger. Let $\{1,\ldots,n\}$ be a finite set of players. For each player $i$, let $A_i$ be $i$'s finite action set (with typical element $a_i$). Let $A := A_1 \times \ldots \times A_n$ be the set of action profiles (with typical element $a$) and let $A^{-i} := A_1 \times \ldots \times A_{i-1} \times A_{i+1} \times \ldots \times A_n$ be the set of action profiles of players other

[1] I thank an editor and two anonymous referees, Adam Brandenburger, Simon Grant, Stephen King, David Pearce, and Hamid Sabourian for their comments; and LSE for their hospitality.

[2] However, Aumann and Brandenburger's weaker equilibrium conditions for the special case of two-player games (mutual knowledge of conjectures and mutual knowledge of rationality) can still pertain.

than player $i$ (with typical element $a^{-i}$). For each $i$, let $S_i$ be the finite set of $i$'s types (with typical element $s_i$). Let $S := S_1 \times \ldots \times S_n$ be the set of possible states (with typical element $s$) and let $S^{-i} := S_1 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_n$ be the set of type profiles of players other than player $i$ (with typical element $s^{-i}$). Each type $s_i$ of player $i$ is associated with: an action $a_i$ in $A_i$; a function $g_i : A \to \mathbb{R}$ representing type $s_i$'s payoffs; and a probability distribution (or 'theory') on $S^{-i}$. Let $g$ denote an $n$-tuple of payoff functions. Throughout, we will assume that each type $s_i$ of each player assigns probability 1 to being of that type.[3] Theories of player $i$ are only defined on $S^{-i}$, but we can easily extend them to $S$. Define $i$'s probability distribution on $S$ at $s_i$, $p(\cdot; s_i)$, as follows: for each event $E \subset S$, $p(E; s_i)$ is the probability that $s_i$'s theory assigns to the event $\{s^{-i} \in S^{-i} : (s_i, s^{-i}) \in E\}$.

We will use tildes to denote functions defined on $S$.[4] Thus, for example, $[\tilde{a} = a]$ denotes the event that the action profile is $a$; that is, $\{s \in S : \tilde{a}(s) = a\}$. Where there is no ambiguity, we denote this event by $[a]$ or just by $a$. A conjecture $\phi^i$ of $i$ is a probability distribution on $A^{-i}$. Thus, using our convention, $\tilde{\phi}^i(s)(a^{-i}) := p([a^{-i}]; s_i)$ is the probability assigned by $s_i$'s theory to the event $[a^{-i}]$. Let $\phi$ denote an $n$-tuple of conjectures and let $\tilde{\phi}(s)$ denote the $n$-tuple of conjectures at state $s$. We say a player is rational at $s$ if $\tilde{a}_i(s)$ maximizes the expectation of $i$'s payoff $\tilde{g}_i(s)$ with respect to $i$'s conjecture $\tilde{\phi}^i(s)$. Following Aumann and Brandenburger, we say that a player knows an event $E$ at $s$, if, at $s$, she ascribes probability 1 to $E$.[5] Thus, for example, player $i$ knows that player $j$'s conjecture is $\phi^j$ if player $i$ assigns probability 1 to the event $[\tilde{\phi}^j = \phi^j]$. This notion of knowledge is implied by the standard notion of absolute certainty; thus belief with probability 1 is less restrictive as an assumption but weaker as a conclusion. Let $K_i E$ be the event that player $i$ knows $E$. Let $K^1 E := K_1 E \cap \ldots \cap K_n E$ and let $CKE := K^1 E \cap K^1 K^2 E \cap \ldots$. We say that $E$ is mutually known at $s$ if $s \in K^1 E$, and that $E$ is commonly known at $s$ if $s \in CKE$.

## RESULT AND PROOF

We can now formally state the proposition.

PROPOSITION: *Suppose that, at some state $s$, it is mutually known that each player is rational, commonly known that $\tilde{\phi} = \phi$, and commonly known that $\tilde{g} = g$. Then it is commonly known that each player is rational.*

PROOF: The proof proceeds via the following lemma that is of some interest in its own right since it depends only on assumptions made by Aumann and Brandenburger in their Theorem B.

[3] Although this assumption only invokes belief with probability 1, absolute certainty is implied: there is no room here for player $i$ to be wrong. At each $s$, player $i$ has type $s_i$ and assigns probability 1 to that same type $s_i$. Since this is true at every state $s$, all players including $i$ know (in the sense of absolute certainty) that player $i$ assigns probability 1 to her correct type. Thus player $i$ knows (in the sense of absolute certainty) that the $s_i$ to which she assigns probability 1 is her true type.

[4] Aumann and Brandenburger use bold for this.

[5] Other writers use the term "certainty" to mean belief with probability 1 and reserve "knowledge" to mean what Aumann and Brandenburger call absolute certainty.

LEMMA: *Suppose that, at some state s, it is mutually known that each player is rational, commonly known that* $\tilde{\phi} = \phi$, *and mutually known that* $\tilde{g} = g$. *Then it is commonly known that each player j's action belongs to the set of actions that maximize the payoff function* $g_j$ *with respect to the conjecture* $\phi^j$.

PROOF OF THE LEMMA: By Aumann and Brandenburger's Lemma 4.1, if it is mutually known that conjectures are $\phi$ then conjectures are indeed $\phi$. Thus, player $i$'s action conjecture at $s$ is indeed $\phi^i$. Let $a_j$ be an action of player $j$ assigned positive probability by player $i$; that is, $p(a_j; s_i) = \phi^i(a_j) > 0$. Since action conjectures are common knowledge at $s$, in particular they are mutual knowledge. And, by Aumann and Brandenburger's Lemma 4.3, a player knows each of several events if and only if she knows that they all obtain. So, at $s$, player $i$ assigns probability 1 to the event that $\tilde{g}_j = g_j$, $j$ is rational, and $\tilde{\phi}^j = \phi^j$. Call this joint event $B$. Thus, we have $p(a_j \mid B; s_i) > 0$. In particular, the event $[a_j] \cap B$ can not be empty: there must exist a state at which player $j$ is rational and chooses $a_j$ while having conjecture $\phi^j$ and payoff function $g_j$. That is, $a_j$ maximizes the payoff function $g_j$ with respect to $j$'s conjecture $\phi^j$. By Aumann and Brandenburger's Lemma 2.6, $i$ believes $\phi^i$ if and only if she knows she believes $\phi^i$. Hence, if $i$ has conjecture $\phi^i$, then $i$ knows that the action profile of the other players $a^{-i}$ is in $\{a^{-i} \in A^{-i} : \phi^i(a^{-i}) > 0\}$. Combining these steps, we conclude that each player $i$ knows that each player $j$'s action belongs to the set of actions that maximize $g_j$ with respect to $\phi^j$. Since action conjectures are common knowledge, the conclusion of the lemma follows. *Q.E.D.*

We can now prove the proposition. Combining the lemma with the extra assumption that the payoff functions, $g$, are common knowledge, we obtain that, at state $s$, it is commonly known that each player $j$'s action belongs to the set of actions that maximize the payoff function $g_j$ with respect to $j$'s conjecture $\phi^j$, commonly known that $\tilde{\phi} = \phi$, and commonly known that $\tilde{g} = g$. Repeated use of Lemma 4.3 (quoted above) then yields that, at state $s$, it is commonly known that each player $j$'s action belongs to the set of actions that maximize $g_j$ with respect to $\phi^j$, that $\tilde{\phi} = \phi$, and that $\tilde{g} = g$; that is, rationality is common knowledge. *Q.E.D.*

## DISCUSSION

This result clarifies how sufficient epistemic conditions for Nash equilibrium relate to those for rationalizability. If payoffs are common knowledge, then the Aumann-Brandenburger conditions (minus a common prior) imply common knowledge of rationality, which in turn implies Brandenburger and Dekel's (1987) "correlated-belief" version of rationalizability. For these games, this set is larger than Bernheim's (1984) and Pearce's (1984) original rationalizable set. Elsewhere, Brandenburger and Dekel (1989) showed that, for the special case of two-player complete information games, given common knowledge of conjectures, Nash equilibrium is equivalent to common knowledge of rationality. Our result shows that for $n$-player complete information games, given common knowledge of conjectures, mutual knowledge of rationality is equivalent to common knowledge of rationality. To get sufficient conditions for Nash equilibrium in $n$-player complete information games, however, we need either to assume a common prior (as do Aumann and Brandenburger) or to assume directly that all other players' conjectures about each individual player's actions coincide and that each player's joint conjecture about the actions of the other players is stochastically independent (as do Brandenburger and Dekel (1989)).

The result may also help explain why the earlier writers thought common knowledge of rationality was essential for Nash equilibrium. Most of these earlier studies (including those by Aumann and Brandenburger) assumed that the game's payoffs were common knowledge. Indeed, it was often implicit in the formulations of the problem. A corresponding assumption is still implicit in Aumann's (1995) formulation of interactive beliefs for extensive form games. The assumption was natural since these studies dealt only with games of complete information. There is a sense, then, in which the earlier view was correct. In a setting with common knowledge of the payoffs, if rationality is not common knowledge then either conjectures about the other players' actions are not common knowledge or rationality is not even mutual knowledge.

A natural-sounding converse of our proposition does not hold. It is not the case that if we strengthen mutual knowledge of rationality to common knowledge of rationality then Aumann and Brandenburger's conditions for Nash equilibrium imply common knowledge of payoffs. The reason is similar to why we cannot discover all of an agent's underlying preferences by observing her revealed choices. Common knowledge of rationality and common knowledge of action conjectures imply that it is common knowledge that each agent's actions maximize whatever are her payoffs with respect to her (commonly known) conjectures. However, the agent does not necessarily know that other agents know her actual payoff function. There could be some other payoff function that is maximized by the same actions.

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