A Theorem on the Number of Nash Equilibria in a Bimatrix Game

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Abstract: We show that if y is an odd integer between 1 and $2^n - 1$, there is an $n \times n$ bimatrix game with exactly y Nash equilibria (NE). We conjecture that this $2^n - 1$ is a tight upper bound on the number of NEs in a "nondegenerate" $n \times n$ game.

1 Introduction

In two-person game theory, perhaps the most important model is the so-called "strategic form" or "bimatrix game form" of a game. The game is represented as a two-dimensional matrix, in which the rows represent the (pure) strategies for one player and the columns those for the other. In each cell is placed a pair of numbers representing the payoffs to the two players if the corresponding pair of strategies is chosen.

In the analysis of bimatrix games, perhaps the most basic solution concept is that of Nash Equilibrium (NE). A pair of (mixed) strategies (p^*, q^*) is a NE provided the first player cannot do any better than to play p^* against the second player's q^*, while, likewise, the second player's "best response" against p^* is q^*. In Nash's seminal papers (1950, 1953), he proved that every bimatrix game has a NE in mixed strategies.

Since then, of course, the study of bimatrix game NEs has gone off in many directions, from applications and refinements of the NE concept to mathematical analyses into the structure of the set of NEs for a given game. In this paper we are concerned with the number of NEs in a bimatrix game. Obviously, Nash's result gives a first insight: this number must be at least one. Subsequently, Lemke and Howson (1964), showed that, under a certain "nondegeneracy assumption," the number of NEs must be finite and odd. This "oddness" result, in various guises, has also appeared in Eaves (1971), Harsanyi (1973), Shapley (1974) and Jansen (1981).[1] Next, in Borm-Gusberts-Tijs (1988) the authors proved that a

[1] The "nondegeneracy" assumptions made in these papers, however, are stated slightly differently from one another.

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nondegenerate 2 by n game can have at most n NEs if n is odd, and $n+1$ if n is even. Finally, in a recent paper of Gul, pearce, and Stacchetti (1993), it was shown that if a nondegenerate game has $2y-1$ NEs, at most y of them are pure-strategy NEs.

In Shapley's paper, in a footnote he observes (without providing a proof) that the maximum number of NEs in a "nondegenerate" 3×3 game is seven. In addition, it is a relatively simple exercise to display examples of 3×3 games having one, three, five and seven NEs. However, for $n \times n$ bimatrix games with $n > 3$, the situation is not so clear, and it is an open problem to characterize fully the numbers of NE that can occur. In this paper, we provide a partial answer to the question, by proving:

Theorem: Let n be given, and let y be any odd integer between 1 and $2^n - 1$. Then there is an $n \times n$ bimatrix game with exactly y NEs.

Hence, if one could show that it was impossible to attain more than $2^n - 1$ NEs, one would have a complete answer to the problem. In fact, we conjecture this to be so.

The paper is organized as follows. In the next section, we provide definitions and background on bimatrix games and NEs. In the following section we prove the above Theorem. Finally, in Section 4, we summarize some of our other results concerning the above conjecture.

2 Bimatrix Games and Nash Equilibria

Let there be two players in a game, denoted by I and II. Player I has m *pure strategies* at his disposal, denoted by $I = \{1, \ldots, m\}$, while II has pure strategy set $J = \{1, \ldots, n\}$.[2] A *mixed strategy* for player I is an element p of the $m-1$-dimensional simplex P, in which p_i is interpreted to be the probability that he plays pure strategy i. Similarly, the set of mixed strategies for II is the $n-1$-dimensional simplex Q. Given $p \in P$, define the *support* of p, or *supp(p)*, to be the set $\{i \in I : p_i > 0\}$, and define *supp*(q) for $q \in Q$ similarly. Finally, denote by e^i the mixed strategy in which I plays i with probability 1, and by e^j that in which II plays j with probability 1.

We are also given two $m \times n$ payoff matrices A and B, where a_{ij} and b_{ij} represent the payoffs for players I and II respectively, if I plays mixed strategy e^i and II plays e^j. Hence, if I chooses mixed strategy $p \in P$ and II chooses $q \in Q$,

[2] Often in the study of bimatrix games the strategies for player II would be labelled $\{m+1, \ldots, m+n\}$. This is done to avoid having to specify both a number and a player in order to specify a strategy. Here, we don't do this because in Section 3 we construct a game in which there is a special relationship between Player I's kth strategy and Player II's kth strategy—and we wish to be able to refer to either as "pure strategy k".

the expected payoff for I is $pAq = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j$, while that for II is $pBq = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i b_{ij} q_j$. Since the two payoff matrices are sufficient to define a bimatrix game, we shall use the terminology "bimatrix game (A, B)."

Given $q \in Q$, p^* is a *best response* for I against q if $p^*Aq \geq pAq \forall p \in P$. Similarly, q^* is a best response for II against p if $pBq^* \geq pBq \forall q \in Q$. Denote by $BR_I(q)$ the set of all best responses for I against q, and by $BR_{II}(p)$ the set of all best responses for II against p. A *Nash Equilibrium* (NE) is a pair $(p^*, q^*) \in P \times Q$ where $p^* \in BR_I(q^*)$ and $q^* \in BR_{II}(p^*)$.

To aid us in finding NEs, let us define the sets $\mathscr{R}_I(q)$ and $\mathscr{R}_{II}(p)$ as follows: $\mathscr{R}_I(q) = \{i \in I : e^i Aq \geq e^k Aq \forall k \in I\}$ and $\mathscr{R}_{II}(p) = \{j \in J : pBe^j \geq pB\varepsilon^k \forall k \in J\}$. In words. $\mathscr{R}_I(q)$ is the set of best pure strategy responses for I against q, while a similar interpretation holds for $\mathscr{R}_{II}(p)$. The following Lemma is then readily apparent (see. e.g., Shapley (1974) or Jansen (1981)):

Lemma 2.1: A mixed strategy pair (p, q) is a NE of bimatrix game (A, B) iff $supp(p) \subseteq \mathscr{R}_I(q)$ and $supp(q) \subseteq \mathscr{R}_{II}(p)$.

Finally, we define a nondegeneracy assumption, which is equivalent to that of Shapley (1974):

Nondegeneracy Assumption (NA): If $p \in P$ satisfies $|supp(p)| = k$, then there are no more than k pure strategy best responses for II against p. Similarly, if $|supp(q)| = k$, we have $|\mathscr{R}_I(q)| \leq k$.[3]

The following are immediate consequences for games satisfying the NA (see Shapley 1974):

Lemma 2.2: Supose bimatrix game (A, B) satisfies the NA. Then

a) The game has a finite and odd number of NEs;
b) For any NE (p, q) of the game, $|supp(p)| = |supp(q)|$;
c) Suppose $S \subseteq \{1, \ldots, m\}$ and $T \subseteq \{1, \ldots, n\}$. Then the game has at most one NE (p, q) in which $supp(p) = S$ and $supp(q) = T$.

3 A Theorem on Possible Numbers of NEs in an $n \times n$ Bimatrix Game

In this section we state and prove the Theorem stated in the Introduction, i.e.,

Theorem 3.1: Let n be given, and let y be any odd integer between 1 and $2^n - 1$. Then there is an $n \times n$ bimatrix game with exactly y NEs.

Before proving this, we make two remarks:

[3] Here $|S|$ denotes the cardinality of set S.

Remark 1: Because one may judiciously append dominated strategies, we may generalize the Theorem as follows:

Corollary 3.2: Let m and n be given, set $M = min(m, n)$ and let y be any odd integer between 1 and $2^M - 1$. Then there is an $m \times n$ bimatrix game with exactly y NEs.

Remark 2: We should note that the proof of Theorem 3.1 is constructive, i.e., it gives an actual procedure for finding a bimatrix game with the desired number of NEs.

To prove Theorem 3.1, we first make a digression into cooperative game theory. Let $N = \{1, \ldots, n\}$ be the players of an n-person cooperative game, and consider the class of *weighted majority games*, i.e., games in which each player i is assigned a nonnegative *weight* w_i, there is a nonnegative *quota* q, and the characteristic function V is given by:

$$V(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i > q; \\ 0 & \text{otherwise.} \end{cases}$$

An S for which $V(S) = 1$ is called *winning*; otherwise coalition S is *losing*.

Lemma 3.3: Given n and an integer $z \in 1, \ldots, 2^n$, there exists an n-player weighted majority game

a) with exactly z losing coalitions,
b) for which $\sum_{i \in S} w_i \neq q \forall S \subseteq N$, and
c) for which $q = 1$.

Proof: To show this, we will show how to define the vector w. First, define the weight vector $\hat{w} \in \Re^n$ by $\hat{w}_i = 2^{i-1}$ for $i = 1, \ldots, n$. Given these weights, it is easy to see that, for $x = 0, 1, 2, \ldots, 2^n - 1$, there is exactly one coalition S with $\sum_{i \in S} \hat{w}_i = x$. So, if the quota were $z - \varepsilon$, there would be exactly z losing coalitions. Now, [in order for c) of the Lemma to hold] we normalize, setting $w_i = (\hat{w}_i / z - \varepsilon)$ for $i = 1, \ldots, n$. QED.

Now, given n and y as in the hypothesis of the Theorem, let $z = (y + 1/2)$. It is clear that since y is an odd integer between 1 and $2^n - 1$, z is an integer between 1 and 2^{n-1}. So, let $w_1, \ldots, w_{n-1}$ be the weights in an $n-1$-player weighted majority game having z losing coalitions and satisfying the conclusions of Lemma 3.3. Now consider the $n \times n$ bimatrix game given by

$$(A, B) = \begin{pmatrix} (1,1) & (0,0) & (0,0) & \cdots & (0,0) & (0,0) \\ (0,0) & (1,1) & (0,0) & \cdots & (0,0) & (0,0) \\ (0,0) & (0,0) & (1,1) & \cdots & (0,0) & (0,0) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (0,0) & (0,0) & (0,0) & \cdots & (1,1) & (0,0) \\ (w_1,0) & (w_2,0) & (w_3,0) & \cdots & (w_{n-1},0) & (1,1) \end{pmatrix}.$$

Lemma 3.4: The game defined above satisfies the Nondegeneracy Assumption.

Proof: First, we note that for any $p \in P$, $\mathscr{R}_{II}(p) \subseteq supp(p)$.[4] Hence $|R_{II}(p)| \leq |supp(p)|$. Next, consider any $q \in Q$. This time, it is clear that $\mathscr{R}_I(q) \subseteq supp(q) \cup \{n\}$. Hence, the only way that $|\mathscr{R}_I(q)| \leq |supp(q)|$ could possibly be false is if a) $n \notin supp(q)$, b) all of the strategies in $supp(q)$ are best responses for I against q, and c) strategy n is also a best response for I against q. But, a) and b) above together imply that $q_j = (1/|supp(q)|)$ for $j \in supp(q)$. In this case, the payoff of any strategy in $supp(q)$ for I would be $(1/|supp(q)|)$. But if the payoff of his nth strategy were also this amount, this would imply $\sum_{j \in supp(q)} w_j = 1$. This contradicts c) of lemma 3.2.

Lemma 3.5: Let (p, q) be any NE of (A, B) defined above. Then $supp(p) = supp(q)$.

Proof: Suppose $supp(p)$ is known, and consider any $j \notin supp(p)$. Then, since $pBe^j = 0$, we know that $j \notin supp(q)$ (Lemma 2.1). But, since (A, B) satisfies the NA, we have $|supp(q)| = |supp(p)|$ (Lemma 2.2b). Hence $supp(p) = supp(q)$.

Lemma 3.6: Given (A, B) as defined above, and suppose $S \subseteq N \equiv \{1, \ldots, n\}$ satisfies $S \neq \varnothing$ and $S \not\ni n$. Then,

1) If $\sum_{i \in S} w_i < 1$, there is exactly 1 NE (p, q) in which $supp(p) = S$.
2) If $\sum_{i \in S} w_i > 1$, there are no NEs (p, q) in which $supp(p) = S$.
3) If $\sum_{i \in S} w_i < 1$, there is exactly 1 NE (p, q) in which $supp(p) = S \cup \{n\}$.
4) If $\sum_{i \in S} w_i > 1$, there are no NEs (p, q) in which $supp(q) = S \cup \{n\}$.

Proof: First we note that (because of Lemmas 2.2c and 3.5) given S, there can be at most 1 NE (p, q) in which the support for the Player I's strategy is S. and there can be at most 1 in which that support is $S \cup \{n\}$.

First suppose (p, q) is to be a NE in which $supp(p) = S$. By lemma 3.5, it is necessarily true that $supp(q) = S$ also. In fact, in order for $supp(p)$ to be a subset of $\mathscr{R}_I(q)$, and for $supp(q)$ to be a subset of $\mathscr{R}_{II}(p)$, it must be that $p_i = (1/|S|) \forall i \in supp(p)$ and $q_j = (1/|S|) \forall j \in supp(q)$. This gives expected payoff $(1/|S|)$ for both players, and will clearly be a NE iff Player I has no incentive to deviate to pure strategy n. But the expected payoff for I of playing e^n is $(1/|S|) \sum_{i \in S} w_i$, so we have proven 1) and 2).

Now suppose (p, q) is to be a NE in which $supp(p) = S \cup \{n\} = supp(q)$. Since $supp(q) \subseteq \mathscr{R}_{II}(p)$, we must have $p_i = (1/(|S| + 1) \forall i \in supp(p)$. Also, since $i \in \mathscr{R}_I(q)$ for $i \in S$, there must be a constant c for which $q_j = c$ for $j \in S$, and so $q_n = 1 - c|S|$. This gives expected payoff c for I if he plays e^i, for any $i \in S$. Hence, in order for (p, q) to be a NE, I's payoff from playing e^n must also be c. Hence, $1 * (1 - c|S|) + c * \sum_{i \in S} w_i$ must be equal to c, which gives $c = (1/|S| + 1 - \sum_{i \in S} w_i)$. If $\sum_{i \in S} w_i < 1$, we have $c < (1/|S|)$, the vector q is in the simplex Q, and (p, q) is indeed a NE. If $\sum_{i \in S} w_i > 1$,

[4] This is due to the way we labelled the strategies (see footnote 2) and because II's payoff matrix B is an identity matrix.

we have $c > (1/|S|)$, and $\sum_{j\in S} q_j > 1$. Hence $q \notin Q$, and there is no NE. So we have shown 3) and 4).

Finally, let us count up the NE for game (A, B). We do this by considering each possible $T \subseteq N$, and seeing if there is a NE (p, q) in which $supp(p) = T$. First, if $T \not\ni n$, Lemma 3.5 says a NE exists iff $T \neq \emptyset$ and $\sum_{i\in T} w_i < 1$; by w's definition there are $z - 1$ such T's. On the other hand, if $T \ni n$, let $S = T/n$. If $T \neq \{n\}$, we know a NE exists iff $\sum_{i\in S} w_i < 1$; again this gives $z - 1$ NEs. But there is one more NE, for the case where $T = \{n\}$.[5] Hence, in total we have $2z - 1$ NEs, which is exactly y. QED.

It is interesting to examine the two "endpoint cases" for our construction, i.e., where $y = 1$ and where $y = 2^n - 1$. In the former case, the construction makes pure strategy n a dominant strategy for Player I, i.e., *Player I's pure strategy best response does not depend on what Player II does.* On the other hand, the construction for the latter case calls for very low w_i's, and so Player I's best strategy is to try and play the same pure strategy as II, i.e., *his pure strategy best response is completely dependent on what Player II does.* The fact that the cases $y = 1$ and $y = 2^n - 1$ represent opposite extremes in this way leads us to believe that 1 and $2^n - 1$ represent bounds on the number of NEs one could obtain in a nondegenerate game. Obviously, the lower bound of 1 is known to be true; hence we are left with:

Conjecture: If an $n \times n$ bimatrix game satisfies the NA, one cannot obtain more than $2^n - 1$ NEs.

In the next section we summarize some of our work (appearing elsewhere) regarding this conjecture.

4 On the Maximum Number of NEs in a Nondegenerate Bimatrix Game

Our first result is that we can prove the above conjecture in the special case of the *coordination game*, i.e., where $A = B$:

Theorem (Quint-Shubik, 1995): Consider an $m \times n$ coordination game, and set $M = \min(m, n)$. Then if the game satisfies the NA, there are no more than $2^M - 1$ NEs. Furthermore there is an easy construction which attains this bound.

Our other results concern bimatrix games with a small number of strategies for one of the players. We prove the following using Lemke-Howson diagrams (see Shapley 1974), and, in the case of c) and d) below, also some elementary graph theory. All of the results below are in Quint-Shubik (1994):

[5] That NE is of course given by $(p = e^n, q = e^n)$.

a) If a $1 \times n$ bimatrix game satisfies the NA, there can be no more than 1 NE. [This is a trivial statement, but we include it here for completeness.]

b) If a $2 \times n$ bimatrix game satisfies the NA, there can be no more than n NEs if n is odd, or $n+1$ NEs if n is even. [This result was proved first in Borm-Gusberts-Tijs 1988.] We also provide a simple construction which attains this bound.

c) If a $3 \times n$ bimatrix game satisfies the NA, there can be no more than $2n+1$ NEs.

d) If a $4 \times n$ bimatrix game satisfies the NA, there can be no more than $(n^2 + 5n + 2)/2$ NEs.

We note that a), b) and c) together prove our conjecture for $n \leq 3$, but unfortunately, plugging in $n = 4$ in d) gives a bound of 19 for the 4×4 game, not 15.

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