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The Set of Nash Equilibria of a Supermodular Game Is a Complete Lattice*

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A Tarski-type fixed point theorem for an ascending correspondence on a complete lattice is proved and then applied to show that the set of Nash equilibria of a supermodular game is a complete lattice. *Journal of Economic Literature* Classification Number: C70, C72.

1. INTRODUCTION

The theory of supermodular games was pioneered by Topkis (1978, 1979) and has since been successfully applied to a variety of economic models (Vives, 1990; Milgrom and Roberts, 1990). It is now well known that the set of Nash equilibria of a supermodular game is nonempty and it has a greatest element and a least one. But the literature does not have a complete answer to the lattice structure of the Nash equilibrium set. In this paper I show that the set of Nash equilibria of a supermodular game is in fact a complete lattice.

Topkis (1978) proved that the set of Nash equilibria of a supermodular game is nonempty and contains a greatest element and a least one. He first established several properties of the "best response" correspondence. He then considered two point-valued selections of the "best response" correspondence and applied Tarski's fixed point theorem for isotone functions on a complete lattice to these two selections to derive the greatest and the least elements of Nash equilibria. Since it was unclear how to make proper point-valued selections to derive other Nash equilibria, Topkis left

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virtually untouched the issue of the lattice structure of the Nash equilibrium set. In a more recent paper, Vives (1990) reconsidered this issue. He derived a partial result by making stronger assumptions under which all Nash equilibria can be obtained as the fixed points of one single selection of the "best response" correspondence.

In this paper I adopt a more natural approach. Instead of considering point-valued selections of the best response correspondence, I consider directly the set-valued best response correspondence itself. This approach has an obvious advantage: the fixed points of the best response correspondence are precisely the set of Nash equilibria of a game, while the fixed points of any of its selections are only a part of the Nash equilibrium set. Key to this approach is, of course, that we have to establish first a Tarski-type fixed point result for set-valued correspondences. This result, together with Topkis' analysis of the best response correspondence of a supermodular game, implies immediately that the set of Nash equilibria of a supermodular game is a complete lattice.

2. A Generalized Tarski's Theorem

A partially ordered set, or poset, $(S, \geq)$ is a set S with a binary relation $\geq$ that is reflexive, antisymmetric, and transitive. We write $x > y$ if $x \geq y$ but $x \neq y$. An interval $[x, y]$ is the set $\{z \in S \mid y \geq z \geq x\}$.

A poset $(S, \geq)$ is a lattice if any two elements x and y of S have a least upper bound, denoted by $x \vee^S y$, and a greater lower bound, denoted by $x \wedge^S y$. A lattice S is complete if any subset T of S has a least upper bound $\wedge^S_{x \in T} x$ and a greatest lower bound $\vee^S_{x \in T} x$.

A subset T of S is a sublattice of S if for any x and y of T, both $x \vee^S y$ and $x \wedge^S y$ belong to T. A sublattice T of S is closed if for any subset U of T, both $\wedge^S_{x \in U} x$ and $\vee^S_{x \in U} x$ belong to T. (The term "closed" is used here in the lattice theoretical sense.)

Let $P(S)$ be the power set of S. A correspondence f from S to S is a function from S to $P(S)$. An element $s \in S$ is a fixed point of a correspondence f is $s \in f(s)$.

A correspondence f is ascending if, for any $x \geq y$, any $s \in f(x)$, and any $t \in f(y)$, it is true that $s \vee^S t \in f(x)$ and $s \wedge^S t \in f(y)$. An obvious corollary is: If f is ascending, then for any $x \geq y$ and any $s \in f(x)$ there is $t \in f(y)$ with $s \geq t$, and for any $b \in f(y)$ there is $a \in f(x)$ with $a \geq b$.

We now prove a fixed point theorem for an ascending correspondence on a complete lattice. It generalizes Tarski's fixed point theorem (1955) for an isotone function in the same fashion that Kakutani's theorem generalized Brouwer's fixed point theorem for a continuous function on a simplex.

THEOREM 1. *Let S be a complete lattice, $f(\cdot)$ a correspondence from S to S, and E the set of fixed points of f. If $f(s)$ is a nonempty closed sublattice of S for every $s \in S$, and f is ascending in s, then E is a nonempty complete lattice.*

Proof. We follow closely the original argument of Tarski.

(i) Let us first show that $\bigwedge^S_{e \in E} e \in E$. Consider the set

$$C = \{c \in S \mid \exists\ x_c \in f(c) \text{ s.t. } x_c \leq c\}.$$

C is nonempty since $1 \in C$, where 1 is the greatest element of S. Let $a = \bigwedge^S_{c \in C} c$. It is obvious that $E \subset C$. Hence, if $a \in E$, then $a = \bigwedge^S_{e \in E} e$. We now prove that $a \in E$ is indeed true.

For any $c \in C$, there is $x_c \in f(c)$ such that $x_c \leq c$. Since f is ascending and $a \leq c$, there is $y_c \in f(a)$ such that $y_c \leq x_c \leq c$. Let $y = \bigwedge^S_{c \in C} y_c$. Because $f(a)$ is a closed sublattice of S, $y \in f(a)$. Clearly $y \leq a$ since $y = \bigwedge^S_{c \in C} y_c \leq \bigwedge^S_{c \in C} c = a$. Then, since f is ascending, there is $z \in f(y)$ such that $z \leq y \in f(a)$. Hence, $y \in C$. So we also have $a \leq y$ by the definition of a. Therefore, $a = y \in f(a)$, i.e., $a \in E$.

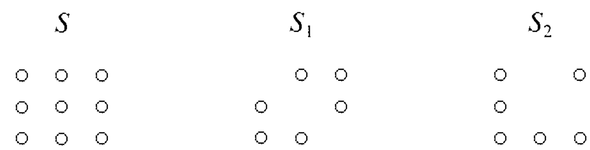
(ii) Similarly, we can show that $\bigvee^S_{e \in E} e \in E$.

(iii) It is already established in (i) that E is nonempty. To show that E is a complete lattice, we have to show that $\bigvee^E_{e \in U} e$ and $\bigwedge^E_{e \in U} e$ exist for any $U \subset E$. Note that $\bigwedge^E_{e \in U} e$ and $\bigvee^E_{e \in U} e$ are respectively the greatest and the least elements of U in E instead of S. Let us deal with the former.

Take $b = \bigwedge^S_{e \in U} e$, the greatest element of U in S. For any $e \in U \subset E$, since $e \in f(e)$ and f is ascending, there is $x_e \in f(b)$ such that $x_e \geq e$. Let $x = \bigwedge^S_{e \in U} x_e$. Clearly $x \geq b$, and $x \in f(b)$ since $f(b)$ is a closed sublattice of S. Because f is ascending, there is $x_c \in f(s)$ with $x_c \geq b$ for every $s \geq b$. Hence, if we let $S' = [b, 1]$, and g from S' to S' defined by $g(s) = f(s) \cap [b, 1]$ for all $s \in S'$, then $g(s)$ is nonempty for every $s \in S'$. Since both $f(s)$ and $[b, 1]$ are closed sublattices of S for every $s \in S'$, $g(s)$ must be a closed sublattice of S'. Also, since both f and h, which assigns each $s \in S'$ the constant interval $[b, 1]$, are ascending on S, $g = f \cap h$ is ascending on S'. Hence, S' and g satisfy the assumptions of the theorem. Therefore, if we let $b' = \bigwedge^{S'}_{e \in E'} e$, in which E' is the set of fixed points of g on S', then $b' \in E'$ according to (i). Since $E' = E \cap [b, 1]$, b' is indeed the least fixed point that is greater than or equal to b, i.e., $b' = \bigwedge^E_{e \in U} e$.

The existence of $\bigwedge^E_{e \in U} e$ can be proved in a similar fashion. Q.E.D.

Remark 1. The conclusion of Theorem 1 that E is a nonempty complete lattice is stronger than the statement that E is a nonempty and $\bigwedge^S_{e \in E} e \in E$ and $\bigvee^S_{e \in E} e \in E$. But it does not imply that E is a sublattice of S. The following example illustrates these two points. It was ascribed to Mas-Colell by Vives (1990).



In the example, S_1 is not a lattice, even though it has both a greatest element and a least one, and S_2 is a complete lattice, yet it is not a sublattice of S.

Remark 2. The condition in Theorem 1 that $f(s)$ is a sublattice of S closed in the lattice theoretical sense is equivalent to the condition that it is a sublattice of S that is topologically closed in the interval topology of S. The interval topology of a lattice $(S, \geq)$ is the topology for which all closed sets are intersections of finite unions of intervals of the following forms: S, $[x, \infty)$ and $(-\infty, y]$, for all $x, y \in S$. Frink proved that a complete lattice is compact in its interval topology, and Birkhoff proved the converse that any lattice compact in its interval topology must be complete; consequently, any sublattice of a complete lattice is closed in the lattice theoretical sense if and only if it is topologically closed in the interval topology (see Birkhoff (1967)).

3. Nash Equilibria of a Supermodular Game

Let us now consider an n-player normal form game G. Each player i's strategy space $(S_i, \geq_i)$ is a lattice that is also compact for a topology τ_i finer than the interval topology. Let $S = \prod_{i \in N} S_i$ be the direct product compact lattice of all S_i. Each player i has a utility function u_i on S that is upper semicontinuous in τ_i on S_i. A game G is supermodular if

(i) each u_i is supermodular in S_i, i.e., for all s_i, $t_i \in S_i$, and $s_{-i} \in \prod_{j \neq i} S_j$,

$$u_i(s_i \vee t_i, s_{-i}) + u_i(s_i \wedge t_i, s_{-i}) \geq u_i(s_i, s_{-i}) + u_i(t_i, s_{-i});$$

and

(ii) each u_i has increasing differences in S_i and $\prod_{j \neq i} S_j$, i.e., for all $s_i \geq_i t_i$, $v_{-i} \geq_{-i} w_{-i}$,

$$u_i(s_i, v_{-i}) - u_i(s_i, w_{-i}) \geq u_i(t_i, v_{-i}) - u_i(t_i, w_{-i}).$$

A strategy profile $s = (s_i)_{i \in N} \in S$ is a Nash equilibrium if $u_i(s_i, s_{-i}) \geq u_i(t_i, s_{-i})$ for all $t_i \in S_i$ and $i \in N$. Hence, if we define the best response correspondence B from S to S by $B(s) = \prod_{i \in N} B_i(s)$, in which $B_i(s) = \text{Argmax}_{s_i \in S_i} u_i(s_i, s_{-i})$, then $s \in S$ is a Nash equilibrium if and only if s is a fixed point of B. Thus we can apply Theorem 1 to study the structure of the Nash equilibrium set of a supermodular game G.

THEOREM 2. *The set of Nash equilibria of a supermodular game G is a complete lattice.*

Proof. Since the set of Nash equilibria of G is exactly the set of fixed points of the best response correspondence B, the theorem is proved if one can show that B satisfies the conditions in Theorem 1. This was done in Topkis (1978), which we recapture here. Because B is the product of B_i's, we only need to consider individual B_i.

First, $B_i(s)$ is compact, hence closed, in the interval topology since it is so in an even finer topology τ_i given that u_i is upper semicontinuous in τ_i.

Second, take $t_i, \hat{t}_i \in B_i(s)$. Since u_i is a supermodular in S, $u_i(t_i \vee \hat{t}_i, s_{-i}) + u_i(t_i \wedge \hat{t}_i, s_{-i}) \geq u_i(t_i, s_{-i}) + u_i(\hat{t}_i, s_{-i})$, so $t_i \vee \hat{t}_i, t_i \wedge \hat{t}_i \in B_i(s)$. Thus, $B_i(s)$ is a sublattice of s_i.

Finally, take any $s_{-i} \geq_{-i} \hat{s}_{-i}$, $t_i \in B_i(s)$, and $\hat{t}_i \in B_i(\hat{s})$. We have

$$\begin{aligned} 0 \geq u_i(t_i \vee \hat{t}_i, s_{-i}) - u_i(t_i, s_{-i}) &\geq u_i(\hat{t}_i, s_{-i}) - u_i(t_i \wedge \hat{t}_i, s_{-i}) \\ &\geq u_i(\hat{t}_i, \hat{s}_{-i}) - u_i(t_i \wedge \hat{t}_i, \hat{\ }_{-i}) \geq 0, \end{aligned}$$

in which the first and the last inequalities hold by the definition of B_i, the second holds because u_i is supermodular, and the third holds because u_i has increasing differences. But then all inequalities must be equations, so $t_i \vee \hat{t}_i \in B_i(s)$ and $t_i \wedge \hat{t}_i \in B_i(\hat{s})$. This means that B_i is ascending.

Q.E.D.

Note, however, that the set of Nash equilibria of a supermodular game generally is not a sublattice of S. Here is an example from Topkis (1979): $N = \{1, 2, 3\}$, $S_i = [0, 1]$ in R^1 for each i, and $u_i(s_1, s_2, s_3) = s_1 s_2 s_3$, for each i. The set of Nash equilibria of this game is not a sublattice because both $(1, 0, 0)$ and $(0, 1, 0)$ are Nash equilibria, but $(1, 1, 0) = (1, 0, 0) \vee^S (0, 1, 0)$ is not.

REFERENCES

BIRKHOFF, G. (1967). *Lattice Theory*, 3rd ed., Amer. Math. Soc. Coll. Publ., Vol. 25, Providence, RI: Amer. Math. Soc.

MILGROM, P., AND ROBERTS, J. (1990). "Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities," *Econometrica* **58,** 1255–1277.

TARSKI, A. (1955). "A Lattice-Theoretical Fixpoint Theorem and Its Applications," *Pacific J. Math.* **5,** 285–309.

TOPKIS, D. (1978). "Minimizing a Submodular Function on a Lattice," *Oper. Res.* **26,** 305–321.

TOPKIS, D. (1979). "Equilibrium Points in Nonzero-Sum n-Person Submodular Games," *SIAM J. Control and Optimum* **17**, 773–787.

VIVES, X. (1990). "Nash Equilibrium with Strategic Complementarities," *J. Math. Econ.* **19,** 305–321.