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AXIOMS OF REVEALED PREFERENCE FOR NONLINEAR CHOICE SETS

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1. INTRODUCTION

MANY CHOICE MODELS frequently encountered in economics involve choice sets that are nonlinear and objective functions that are concave and monotone increasing. Examples of models that typically fall into this category are models in which consumers or firms possess monopsony power, models of consumers facing either regressive or progressive taxes, models of households consuming commodities produced according to either a convex or a concave production function, and models in which a social planner faces the production possibilities set of an economy. Despite the popularity of these models, no revealed preference axioms characterizing choice data generated by the maximization of a concave function subject to nonlinear choice sets seems to exist in the current literature.[2] In this note, I introduce such axioms.

The axioms can be employed to test the basic hypothesis of optimizing behavior and the concavity of the maximizing function. An appealing property of these tests is that they do not require specification of a parametric structure for the maximizing function. In addition, they can be easily implemented as efficient algorithms, in contradistinction to other known methods.[3]

The choice models are described in the next section. The theorems are presented in Section 3 and the proofs in Section 4.

2. THE CHOICE MODELS

Following Richter (1966), we represent each choice by a pair (B, x) where $x \in B$ is the element chosen from the set of alternatives B, and we call a collection of choices a choice space.[4] The sets of alternatives B that we will consider belong to the class of sets that possess convex and monotone[5] complements and to the class of sets that can be supported by a unique hyperplane. We will call the first type of sets *co-convex* and the second type of sets *supportable*.

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[2] Afriat (1967, 1977, 1981), Chiappori and Rochet (1987), Houthakker (1950), Matzkin and Richter (1990), Richter (1966, 1971), Samuelson (1938), and Varian (1982, 1983) studied axiomatic conditions for demand data when the budget sets are determined by one linear function. Richter (1966) studied axiomatic conditions for choice data generated subject to abstract choice spaces by a not necessarily concave or monotone utility function.

[3] Yatchew's (1985) nonaxiomatic conditions, which can be applied when the choice sets are unions of linear sets, ask whether a system of $(nL)^2 + 2nL + 2n(L-1)K - n$ nonlinear inequalities in $2nL + n(L-1)K$ unknowns has a solution, where n is the number of observations, L is the number of linear sets that determine each choice set, and K is the dimension of the chosen elements. Varian's (1982) nonaxiomatic conditions, which can be applied when the choice sets are intersections of linear sets, ask whether a system of $n^2 + 2n$ linear inequalities in $2n$ unknowns has a solution. In contrast, the axiomatic conditions presented in this paper only require one to create an $n \times n$ matrix of 0's and 1's and to test the existence of symmetries in this matrix.

[4] Similar frameworks were introduced by Uzawa (1956) and Arrow (1959).

[5] A set $A \subset H$ is *monotone in* H if $\forall x \in A$ and $\forall e \geqslant 0$ $x + e \in H$ implies $x + e \in A$; $A \subset H$ is *strictly monotone in* H if $\forall x \in A$ and $\forall e \geqslant 0$ $(e \neq 0)$ $x + e \in \text{int}\, H$ implies $x + e \in \text{int}\, A$.

DEFINITION 1: A *co-convex choice* in $X \subset R^K$ is a pair (B, x) such that (i) $B \subset X$, (ii) B^c is an open, convex, and monotone subset in X, and (iii) for all $e \geqslant 0$ such that $e \neq 0$ and $x + e \in X$, $x + e \in B^c$. A *co-convex choice space* in X is a finite set $\{B^i, x^i\}_{i=1}^n$ such that (B^i, x^i) is a co-convex choice in X $(i = 1, \ldots, n)$.

DEFINITION 2: A *supportable choice* in $X \subset R^K$ is a pair (B, x) such that (i) $B \subset X$ and for which there exists a neighborhood N of x and a unique $s \in R^K_{++}$ such that (ii.1) $B \subset \{z \in X | s \cdot z \leqslant 1\}$, (ii.2) $s \cdot x = 1$, (ii.3) $N \subset X$, (ii.4) $B \cap \text{cl}(N)$ is closed and convex, and (ii.5) $[(B - R^K_+) \cap N] \subset B$. A *supportable choice space* in X is a finite set $\{B^i, x^i\}_{i=1}^n$ such that (B^i, x^i) is a supportable choice $(i = 1, \ldots, n)$.

DEFINITION 3: A *mixed choice space* in $X \subset R^K$ is a finite set $\{B^i, x^i\}_{i=1}^n$ such that (B^i, x^i) is either a co-convex or a supportable choice in X $(i = 1, \ldots, n)$.

Suppose for example that (B, x) is a choice with $B = \{z \in X | g(z) \leqslant 0\}$ where g: $X \to R$ satisfies $g(x) = 0$. Then, if g is a monotone increasing, continuous, and quasi-concave function, (B, x) is a co-convex choice; and if g is monotone increasing, convex, and differentiable at x, (B, x) is a supportable choice.

If all the choices in a choice space are generated by the maximization of a common objective function, we call this function a rationalization. Formally, we have the following definition.

DEFINITION 4: A function V: $X \to R$ is a *rationalization* for a choice space $\{B^i, x^i\}_{i=1}^n$ in X if for all $i = 1, \ldots, n$ and all $y \in B^i$ such that $y \neq x^i$, $V(x^i) > V(y)$.

DEFINITION 5: The function V: $X \to R$ will be called a *regular rationalization* if V is a continuous, strictly concave, and strictly monotone increasing rationalization.

The following definitions are well known (cf. Samuelson (1938), Houthakker (1950), Richter (1966, 1971)):

DEFINITION 6: xSy iff (B, x) is a choice, $y \in B$, and $x \neq y$; xHy iff for some (possibly empty) sequence $w^1, \ldots, w^r \in X$, $xSw^1S \cdots Sw^rSy$.

DEFINITION 7: A budget space $\{B^i, x^i\}_{i=1}^n$ is said to satisfy the *SARP* if H is asymmetric on $\{x^1, \ldots, x^n\}$.

3. THEOREMS

The following results extend the axiomatic theory of revealed preference to apply to co-convex, supportable, and mixed choice spaces.

THEOREM 1: *Suppose that $\{B^i, x^i\}_{i=1}^n (n < \infty)$ is a co-convex choice space in a convex and bounded subset X of R^K. Then, $\{B^i, x^i\}_{i=1}^n$ satisfies the SARP if and only if there exists a regular rationalization for $\{B^i, x^i\}_{i=1}^n$.*

Theorem 1 provides a nonparametric test for the consistency of co-convex choices with the existence of a regular rationalization. The proof of Theorem 1, which is presented in the next section, includes an explicit derivation of a regular rationalization.

The result of Theorem 1 together with the results in Richter (1966) imply that, for finite sets of observations on co-convex choices, the existence of a regular rationalization is observationally equivalent to the existence of *some* rationalization.

Figure 1 in Chiappori and Rochet (1987) as well as Remark 1 in Matzkin and Richter (1990) show that, without additional assumptions, it is not always possible to obtain a *differentiable* regular rationalization for a co-convex choice space that satisfies the SARP.

The characterization of supportable choice sets for which there exists a regular rationalization employs supporting choices, which are defined next:

DEFINITION 8: Let $\{B^i, x^i\}_{i=1}^n$ be a supportable choice space in $X \subset R^K$. For each $i = 1, \ldots, n$, let $s^i \in R^K_{++}$ be such that (i) $s^i \cdot x^i = 1$ and (ii) for all $y \in B^i$, $s^i \cdot y \leqslant 1$. Let $C^i = \{y \in X | s^i \cdot y \leqslant 1\}$. Then, the *supporting choice space* of $\{B^i, x^i\}_{i=1}^n$ is $\{C^i, x^i\}_{i=1}^n$.

THEOREM 2: *Suppose that $\{B^i, x^i\}_{i=1}^n$ $(n < \infty)$ is a supportable choice space in a convex and compact subset X of R^K. Let $\{C^i, x^i\}_{i=1}^n$ be the supporting choice space of $\{B^i, x^i\}_{i=1}^n$. Then, there exists a regular rationalization for $\{B^i, x^i\}_{i=1}^n$ if and only if $\{C^i, x^i\}_{i=1}^n$ satisfies the SARP.*

Figure 1 shows an example of a supportable choice space that satisfies the SARP, while its supporting choice space does not satisfy the SARP. Hence, although the choice space possesses a rationalization, no rationalization for it can be regular. Thus, Theorem 2 provides a method of testing the strict concavity and strict monotonicity of the rationalization for a supportable choice space, when a rationalization is known to exist. Note that to apply this test it is not necessary to observe the choice set in its entirety; it suffices to observe the chosen elements and their supporting choice sets.

For mixed choice spaces, the proofs of Theorems 1 and 2 given in the next section immediately imply the following corollary.

COROLLARY: *Suppose that $\{B^i, x^i\}_{i=1}^n$ $(n < \infty)$ is a mixed choice space in a convex and compact subset X of R^K. For each $i = 1, \ldots, n$, let (D^i, x^i) be equal to (B^i, x^i) if (B^i, x^i) is a co-convex choice in X, and let (D^i, x^i) be equal to the supporting choice of (B^i, x^i) if (B^i, x^i) is a supportable choice in X. Then, there exists a regular rationalization for $\{B^i, x^i\}_{i=1}^n$ if and only if $\{D^i, x^i\}_{i=1}^n$ satisfies the SARP.*

4. PROOFS OF THE THEOREMS

PROOF OF THEOREM 1: It is well known that the existence of a rationalization implies that $\{B^i, x^i\}_{i=1}^n$ satisfies the SARP (cf. Richter (1966, 1971)). Hence, we only need to show that if $\{B^i, x^i\}_{i=1}^n$ satisfies the SARP, there exists a regular rationalization for $\{B^i, x^i\}_{i=1}^n$. We prove this in three steps. In Step 1 we find "indifference classes" $C(1), \ldots, C(T)$ and convex and monotone "upper-contour sets" $Z(1), \ldots, Z(T)$ for $C(1), \ldots, C(T)$. In Step 2 we obtain for each of the extreme points of $Z(t)$ a "perturbation" of $Z(t)$ that contains $Z(t)$. In Step 3 we find "utility values" V^t and "constraint multipliers" $\lambda^t > 0$ for each $t = 1, \ldots, T$, and we employ these values to obtain a regular rationalization.

Step 1: Let $D(0) = \{x^i | i = 1, \ldots, n\}$ and for $t = 1, 2, \ldots$ define

$$(1.1) \qquad C(t) = \{x \in D(t-1) | \sim [(xHy) \text{ for some } y \in D(t-1)]\} \quad \text{and}$$

$$(1.2) \qquad D(t) = D(0) \setminus \bigcup_{s=1}^{t} C(s).$$

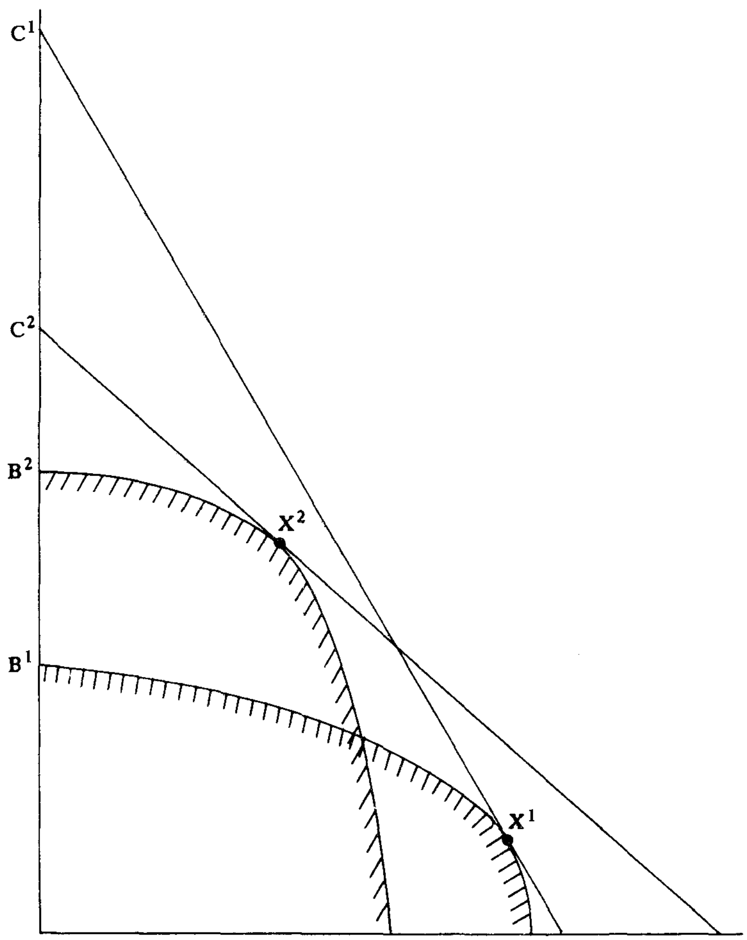

FIGURE 1

Note that $C(t) \neq \varnothing$ if $D(t-1) \neq \varnothing$, since otherwise there would exist a finite sequence $\{x^i, x^r, \ldots, x^j\} \subset D(t-1)$ with $x^i H x^r H \cdots H x^j H x^i$, contradicting the SARP. Hence, there exists a first T such that $D(T) \neq \varnothing$; by (1.2), $C(1), \ldots, C(T)$ is a partition of $D(0)$.

Since the B^i sets are closed, there exists $\eta \in R^K_{++}$ such that $x^i - \eta \subset (B^j)^c$ whenever both $x^i \neq x^j$ and $-(x^j S x^i)$ $(i, j = 1, \ldots, n)$. Here, since X is bounded there exist $w^1, w^2 \in R^K$ such that $(X + \{4\eta\}) \subset \operatorname{int} H^0 \equiv \{x \in R^K | w^1 \leqslant x \leqslant w^2\}$. Let $\eta(t) = (2)^{-t}\eta$ and define, for $t = 1, \ldots, T$, the sets

(1.3) $\quad S(t) = D(t-1) \setminus C(t),$

(1.4) $\quad F(t) = \{y - \eta(t) | y \in S(t)\},$ and

(1.5) $\quad \tilde{Z}(t) = \operatorname{com}^+[C(t) \cup F(t)] \cap H^0,$

where $\operatorname{com}^+(A)$ denotes the convex hull of $\{z + R^K_+ | z \in A\}$. For any convex set A, let $\gamma(A)$ denote the set of all extreme points of A and let $\Psi(A)$ denote the elements w of

$\gamma(A)$ such that for some element y of A and $e \geqslant 0$, $e \neq 0$, $w = y + e$. Then

$$(1.6) \qquad \forall t \in \{1,\ldots,T\} \quad \tilde{Z}(t) \text{ is monotone in } H^0 \text{ and convex,}$$

$$(1.7) \qquad \forall t \in \{1,\ldots,T\} \quad \tilde{Z}(t+1) \subset \operatorname{int} \tilde{Z}(t),$$

$$(1.8) \qquad \forall t \in \{1,\ldots,T\} \quad \forall x^i \in C(t) \quad B^i \cap \tilde{Z}(t) = \{x^i\}, \quad \text{and}$$

$$(1.9) \qquad \forall t \in \{1,\ldots,T\} \quad \forall x^i \in C(t) \quad x^i \in \gamma(\tilde{Z}(t)) \setminus \Psi(\tilde{Z}(t)),$$

where by $\operatorname{int} \tilde{Z}(t)$ we mean the interior relative to H^0. Statement (1.6) follows by (1.5); (1.7) follows by (1.1)–(1.5); (1.8) follows by (1.1)–(1.5), the monotonicity and convexity of $(B^i)^c$, and (iii) in Definition 1; and (1.9) follows by (1.8) and the monotonicity and convexity of $(B^i)^c$.

For any $\alpha \in R^K_{++}$ and any $t \in \{1,\ldots,T\}$ let

$$(1.10) \qquad H^1_\alpha = \{x \in R^K | w^1 \leqslant x \leqslant w^2 - \alpha\} \quad \text{and}$$

$$(1.11) \qquad Z_\alpha(t) = \operatorname{com}^+\{\gamma(\tilde{Z}(t)) \setminus \Psi(\tilde{Z}(t)), \{w - \alpha | w \in \Psi(\tilde{Z}(t))\}\} \cap H^1_\alpha.$$

Then, by (1.5), (1.10), (1.11), (1.8), the closeness of the B^i sets, and (1.7), α can be chosen small enough such that

$$(1.12) \qquad \forall t \in \{1,\ldots,T\} \quad \gamma(Z_\alpha(t)) = (\gamma(\tilde{Z}(t)) \setminus \Psi(\tilde{Z}(t))) \cup \{w - \alpha | w \in \Psi(\tilde{Z}(t))\}.$$

$$(1.13) \qquad \forall t \in \{1,\ldots,T\} \quad \forall w \in \Psi(Z_\alpha(t)) \quad \forall x^i \in C(t) \quad w - \alpha \in (B^i)^c, \quad \text{and}$$

$$(1.14) \qquad \forall t \in \{1,\ldots,T\} \quad \forall w \in \Psi(Z_\alpha(t+1)) \quad w - \alpha \in \operatorname{int} Z_\alpha(t).$$

Let H^1 and $Z(t)$ $(t = 1,\ldots,T)$ denote the sets H^1_α and $Z_\alpha(t)$ for some α satisfying (1.12)–(1.14). Then,

$$(1.15) \qquad \forall t \in \{1,\ldots,T\} \quad Z(t) \text{ is strictly monotone in } H^1 \text{ and convex,}$$

$$(1.16) \qquad \forall t \in \{1,\ldots,T\} \quad Z(t+1) \subset \operatorname{int} Z(t),$$

$$(1.17) \qquad \forall t \in \{1,\ldots,T\} \quad \forall x^i \in C(t) \quad B^i \cap Z(t) = \{x^i\}, \quad \text{and}$$

$$(1.18) \qquad \forall t \in \{1,\ldots,T\} \quad \forall x^i \in C(t) \quad x^i \in \gamma(Z(t)),$$

where by $\operatorname{int} Z(t)$ we mean the interior relative to H^1. Statement (1.15) follows from (1.6) and (1.11); (1.16) follows from (1.7), (1.11), and (1.14); (1.8), (1.11), (1.12), (1.13), the monotonicity and convexity of $(B^i)^c$, and (iii) in Definition 1 imply (1.17); and (1.18) follows from (1.17) and the convexity of $(B^i)^c$.

Step 2: For any $v \in R^K_{++}$, any $t \in \{1,\ldots,T\}$, and any $y \in \gamma(Z(t))$, let

$$(1.19) \qquad H^2_v = \{x \in R^K | w^1 \leqslant x \leqslant w^2 - \alpha - v\} \quad \text{and}$$

$$(1.20) \qquad D_v(y) = \operatorname{com}^+\{y, \{q - v | q \neq y \text{ and } q \in \gamma(Z(t))\}\} \cap H^2_v.$$

Then, by (1.11), (1.19), (1.20), (1.17), the closeness of the B^i sets, and (1.16), v can be chosen small enough so that (1.21) and (1.22) are satisfied, where

$$(1.21) \qquad \forall t \in \{1,\ldots,T\} \quad \forall y \in \gamma(Z(t))$$
$$\gamma(D_v(y)) = \{y, \{q - v | q \neq y \text{ and } q \in \gamma(Z(t))\}\},$$

$$(1.22) \qquad \forall t \in \{1,\ldots,T\} \quad \forall x^i \in C(t) \quad \forall q \in \gamma(Z(t)) \quad q \neq x^i \Rightarrow q - v \in (B^i)^c.$$

Let H^2 and $D(y)$ $(y \in \gamma(Z(t)),\ t \in \{1,\ldots,T\})$ denote the sets H^2_v and $D_v(y)$ for some v

satisfying (1.21) and (1.22). Then,

$$(1.23)\quad \forall t \in \{1,\dots,T\} \quad \forall y \in \gamma(Z(t)) \quad D(y) \text{ is strictly monotone in } H^2 \text{ and convex,}$$

$$(1.24)\quad \forall t \in \{1,\dots,T\} \quad \forall s \geqslant t \quad \forall y \in \gamma(Z(s)) \quad \forall w \in \gamma(Z(t))$$
$$w \neq y \Rightarrow y \in \text{int } D(w),$$

$$(1.25)\quad \forall t \in \{1,\dots,T\} \quad \forall x^i \in \gamma(Z(t)) \quad B^i \cap D(x^i) = \{x^i\}, \quad \text{and}$$

$$(1.26)\quad \forall t \in \{1,\dots,T\} \quad \forall x^i \in \gamma(Z(t)) \quad x^i \in \gamma(D(x^i)).$$

Statement (1.23) follows from (1.15) and (1.20); (1.24) follows by (1.16) and (1.20) when $s > t$ and by (1.20) when $s = t$; (1.20), (1.22), the monotonicity and convexity of $(B^i)^c$, and (iii) in Definition 1 imply (1.25); and (1.21) implies (1.26).

For all $t \in \{1,\dots,T\}$ and all $y \in \gamma(Z(t))$ let

$$(1.27)\quad P(y) = \{p \in R^K \mid \|p\| = 1 \text{ and for some face } A \text{ of } D(y) \text{ that is adjacent to } y$$
$$A \subset \{z \in R^K | p \cdot z = p \cdot y\}\}.$$

Then, (1.23) implies that

$$(1.28)\quad \forall t \in \{1,\dots,T\} \quad \forall y \in \gamma(Z(t)) \quad \forall p \in P(y) \quad p \in R^K_{++}, \quad \text{and}$$

$$(1.29)\quad \forall t \in \{1,\dots,T\} \quad \forall y \in \gamma(Z(t)) \quad \forall p \in P(y)$$
$$\{z \in R^K | p \cdot z = p \cdot y\} \text{ supports } D(y).$$

For all $t \in \{1,\dots,T\}$ let

$$(1.30)\quad E(t) = \gamma(Z(t)).$$

Then,

$$(1.31)\quad E(t) \text{ is a finite set}$$

and

$$(1.32)\quad C(t) \subset E(t).$$

Step 3: Fix $\delta > 0$. In the same spirit of the algorithm described in Varian (1982) (see also Chiappori and Rochet (1987, Lemma 1), we let $V^T = \lambda^T = 1$, and for $t = T-1, T-2, \dots,$ we define V^t and λ^t by

$$(1.33)\quad V^t = \min_{s>t} \min_{y \in E(s)} \min_{p \in P(y)} \min_{w \in E(t)} \{V^s + \lambda^s p \cdot (w - y) - \delta, V^s - \delta\},$$

$$(1.34)\quad \lambda^t = \max_{s>t} \max_{y \in E(s)} \max_{w \in E(t)} \max_{p \in P(w)} \{(V^s - V^t + \delta)/(p \cdot (y - w)), 1 + \delta\}.$$

Then,

$$(1.35)\quad \lambda^t > 0, \quad \text{and}$$

$$(1.36)\quad \forall t,\ s \in \{1,\dots,T\},\ \forall w \in E(t),\ \forall y \in E(s) \text{ such that } y \neq w, \text{ and } \forall p \in P(y)$$
$$V^t < V^s + \lambda^s p \cdot (w - y).$$

Statement (1.35) follows from (1.34); (1.36) follows from (1.33) when $s > t$ and it follows from (1.23), (1.24), (1.29), and (1.34) when $s \leqslant t$.

Following Matzkin and Richter (1991), we define the function g: $R^K \to R$ by $g(x) = (\sum_{k=1}^{K}(x_k)^2 + T)^{(1/2)} - T^{(1/2)}$, where $T > 0$. Then, (1.31), (1.35), and (1.36) imply that, for all small enough $\varepsilon > 0$,

(1.37) $\quad \forall t,\ s \in \{1,\dots,T\},\ \forall w \in E(t),\ \forall y \in E(s)$ such that $y \neq w$, and $\forall p \in P(y)$

$$V^t < V^s + \lambda^s p(w-y) - \varepsilon g(w-y).$$

Let

(1.38) $$U(x) = \min\{V^t + \lambda^t p\cdot(x-y) - \varepsilon g(x-y) | t = 1,\dots,T,\ y \in E(t), p \in P(y)\}.$$

Then, $U(\cdot)$ is strictly concave and, by (1.28), for $\varepsilon > 0$ small enough, it is strictly monotone (see Matzkin and Richter (1991) for details).

To show that U is a rationalization, we first note that (1.37) and (1.38) imply that

(1.39) $\quad \forall t \in \{1,\dots,T\} \quad \forall y \in E(t) \quad U(y) = V^t.$

Suppose next that $x^i \in C(t), w \in B^i$, and $w \neq x^i$. Then by (1.20), (1.23), and (1.25), there exists $e \geqslant 0$, $e \neq 0$, such that $w + e$ belongs to some phase A of $D(x^i)$. If x^i is one of the extreme points of $D(x^i)$ lying on A, then $A \subset \{z \in R^K | p\cdot z = p\cdot x^i\}$ for some $p \in P(x^i)$. Hence $p\cdot(w+e) = p\cdot x^i$ and by (1.32), (1.38), (1.28), (1.35), and (1.39),

$$\begin{aligned} U(w) &\leqslant V^t + \lambda^t p\cdot(w - x^i) - \varepsilon g(w-x^i) \\ &< V^t + \lambda^t p\cdot(w+e-x^i) = V^t = U(x^i). \end{aligned}$$

If x^i is not one of the exteme points of $D(x^i)$ lying on A, it follows by (1.20), (1.21), and (1.28) that for any $q \in Z(t)$ such that $q - v$ is an extreme point of $D(x^i)$ and for some $p \in P(q)$, $p\cdot(w+e) < p\cdot q$. Hence, by (1.32), (1.38), (1.28), (1.35), and (1.39),

$$\begin{aligned} U(w) &\leqslant V^t + \lambda^t p\cdot(w-q) - \varepsilon g(w-q) \\ &\leqslant V^t + \lambda^t p\cdot(w+e-q) < V^t = U(x^i). \end{aligned}$$

Hence, $U(\cdot)$ is a regular ratonalization for $\{B^i, x^i\}_{i=1}^n$. *Q.E.D.*

A different proof of Theorem 1 can be obtained by first, showing the existence of strictly convex and strictly monotone "upper-contour sets" $Y(1),\dots,Y(T)$ for the "indifference-classes" $C(1),\dots,C(T)$; second, employing the results of Kannai (1974) and Mas-Colell (1974) to show the existence of a concave and monotone increasing function whose level sets are $\partial Y(1),\dots,\partial Y(T)$; and third, obtaining a strictly concave and monotone increasing transformation of the concave function.[6]

Proof of Theorem 2: The existence of a regular rationalization for $\{C^i, x^i\}_{i=1}^n$, when $\{C^i, x^i\}_{i=1}^n$ satisfies the SARP is well known (apply Matzkin and Richter (1991, Theorem 1), modify the arguments in Chiappori and Rochet (1987) to apply to the case in which the demand function is not necessarily invertible, or just employ Theorem 1). Hence, since $B^i \subset C^i$, the existence of a regular rationalization for $\{B^i, x^i\}_{i=1}^n$ is immediate.

Conversely, suppose that V: $X \to R$ is a regular rationalization for $\{B^i, x^i\}_{i=1}^n$ but $\{C^i, x^i\}_{i=1}^n$ does not satisfy the SARP. Then for some sequence $\{r, q, \dots, v, t\} \subset \{1,\dots,m\}$

(2.1) $\quad x^r S x^q S \cdots S x^v S x^t S x^r.$

To show that (2.1) is impossible, we will show that $x^j S x^k$ implies $V(x^j) > V(x^k)$.

[6] This latter approach was taken in Matzkin (1986) to show the existence of a regular rationalization for linear choice spaces satisfying the SARP.

Suppose that $V(x^j) \leqslant V(x^k)$. If $s^j \cdot x^k < 1$, then the definition of supportable choices implies that for some $\lambda \in (0,1)(\lambda x^j + (1-\lambda)x^k) \in B_j$, contradicting, by the strict concavity of V, the hypothesis that V is a rationalization for $\{B^i, x^i\}$. If $s^j \cdot x^k = 1$ then, the strict concavity of V and the convexity of X imply that for some $\alpha \in (0,1)$ $s^j \cdot (\alpha y + (1-\alpha)x^j) = 1$, and $V(\alpha y + (1-\alpha)x^j) > V(x^j)$. The continuity of V implies then that for some $w \ll (\alpha y + (1-\alpha)x^j)$ such that $s^j \cdot w < 1$, $V(w) > V(x^j)$. But we have already shown that this is impossible. Hence, $x^j S x^k$ implies $V(x^j) > V(x^k)$. This implies that (2.1) is impossible and concludes the proof of the theorem. *Q.E.D.*

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Suppose that $V(x^j) \leqslant V(x^k)$. If $s^j \cdot x^k < 1$, then the definition of supportable choices implies that for some $\lambda \in (0,1)(\lambda x^j + (1-\lambda)x^k) \in B_j$, contradicting, by the strict concavity of V, the hypothesis that V is a rationalization for $\{B^i, x^i\}$. If $s^j \cdot x^k = 1$ then, the strict concavity of V and the convexity of X imply that for some $\alpha \in (0,1)$ $s^j \cdot (\alpha y + (1-\alpha)x^j) = 1$, and $V(\alpha y + (1-\alpha)x^j) > V(x^j)$. The continuity of V implies then that for some $w \ll (\alpha y + (1-\alpha)x^j)$ such that $s^j \cdot w < 1$, $V(w) > V(x^j)$. But we have already shown that this is impossible. Hence, $x^j S x^k$ implies $V(x^j) > V(x^k)$. This implies that (2.1) is impossible and concludes the proof of the theorem. *Q.E.D.*

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