

A Weak Monotonicity Property of the Nucleolus

By L. Zhou¹

Abstract: It is shown that the nucleolus of a TU game and many of its derivative concepts satisfy a weak monotonicity property.

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The nucleolus of a transferable utility game (Schmeidler (1969)) has many interesting properties as a solution concept (see Maschler et al (1979)). The most important one is that it always selects a core allocation whenever the core is nonempty. But it is exactly because of this property that it fails a monotonicity property (Young (1985)). This has been viewed as a major disadvantage of the nucleolus when compared to other solution concepts such as the Shapley value, etc. In response to this point, I show in this note that the nucleolus of a TU game and many of its derivative concepts do satisfy a weaker yet reasonable monotonicity property.

A TU game is a pair (N, ν) in which $N = \{1, \dots, n\}$ is a finite set of players and ν is a real-valued function on the set of non-empty coalitions of players. For each coalition S , $\nu(S)$ represents the maximum payoff to players in coalition S if they cooperate.

The excess of a coalition S with respect to a payoff vector x is

$$e^\nu(S, x) = \nu(S) - \sum_{i \in S} x_i .$$

Let $\theta^\nu(x)$ be the $(2^n - 1)$ -dimensional vector whose entries are the excesses arranged in a decreasing order, i.e.,

$$\begin{aligned} \theta_i^\nu(x) &= e^\nu(S_i, x), \quad \text{and} \\ \theta_i^\nu(x) &\geq \theta_j^\nu(x), \text{ for } 1 \leq i \leq j \leq 2^n - 1 . \end{aligned}$$

¹ Lin Zhou, Cowles Foundation, Yale University, P.O. Box 2125, Yale Station, New Haven, CT 06520, USA.

Definition 1: The nucleolus of a TU game v is the set of lexicographical minimizers of the vector of excesses over the set of payoff vectors

$$X^v = \{x \in R^n \mid \sum_{i \in N} x_i \leq v(N)\}.$$

In other words, a payoff vector x that is feasible for the grand coalition is in the nucleolus if for any other feasible payoff vector y , there exists some j ($1 \leq j \leq 2^n - 1$) such that $\theta_i^v(x) = \theta_i^v(y)$ for all $i < j$, and $\theta_j^v(x) < \theta_j^v(y)$.² Schmeidler (1969) proved the following result.

Theorem 1 (Schmeidler): The nucleolus for any TU game contains a unique efficient payoff vector.

We now consider the issue of monotonicity. A value operator ϕ chooses for each game v an efficient payoff vector x . Formally, it is a function $\phi: R^{2^n - 1} \rightarrow R^n$ such that $\sum_{i \in N} \phi_i(v) = v(N)$.

Definition 2: A value operator ϕ is coalitionally monotonic if for all coalitions S and all players $i \in S$, we have

$$\text{for all } v, w \in R^{2^n - 1}: \\ \{v(S) \leq w(S) \text{ and } v(T) = w(T) \text{ for all } T \neq S\} \Rightarrow \{\phi_i(v) \leq \phi_i(w)\}.$$

Coalitional monotonicity requires that when a game v changes to w so that the wealth of a particular coalition S increases while those of other coalitions remain the same, every player in S does not lose as a result of this change. It is obviously a desirable property for a value operator to possess. But Megiddo (1974) proved that the nucleolus is not monotonic, which turns out to be a corollary of a more general result of Young (1985). A value operator is called core-selecting if it selects a core allocation whenever the core is nonempty.

Theorem 2 (Young): No core-selecting value operator ϕ is coalitionally monotonic.

Theorem 2 tells that one cannot expect from a value operator both the core-selecting property and the coalitional monotonicity property. If one wants to keep the more intuitive former, then one has to weaken the latter. The most natural candidate for such a weakening is the following one.

² Notice that in the definition of the nucleolus we do not impose individual rationality. Some authors call it the prenucleolus instead as compared to the original definition of the nucleolus in which individual rationality is required. It, however, has received more attention in recent literature partly because it has a nice characterization (Sobolev (1975)), which is not true if individual rationality is required. Our main result here is also not true in the latter case. This point was made clear by the referee.

Definition 3: A value operator ϕ is weakly coalitionally monotonic if for all coalitions S , we have

$$\text{for all } v, w \in R^{2^n-1}: \\ \{v(S) \leq w(S) \text{ and } v(T) = w(T) \text{ for all } T \neq S\} \Rightarrow \left\{ \sum_{i \in S} \phi_i(v) \leq \sum_{i \in S} \phi_i(w) \right\}.$$

Weakly coalitional monotonicity requires that when a game v changes to w so that the wealth of a particular coalition S increases, *ceteris paribus*, the coalition S as a whole instead of every player in it should not lose. So if a value operator ϕ satisfies weakly coalitional monotonicity, then when the wealth of a coalition S increases, it is always possible to increase the welfare of every player in S through redistribution. This is exactly the idea of the classical Kaldor compensation principle (Kaldor (1939)). We now prove the main result of this note.

Theorem 3: The nucleolus is weakly coalitionally monotonic.

Proof: Suppose that for a coalition S there are a pair of games v, w such that $v(S) < w(S)$ and $v(T) = w(T)$ for all $T \neq S$. Let x be the nucleolus of v , and y the nucleolus of w . We have to show that $\sum_{i \in S} x_i \leq \sum_{i \in S} y_i$. It is nontrivial only when $x \neq y$.

We now consider $\theta^v(x)$, $\theta^w(x)$, $\theta^v(y)$, and $\theta^w(y)$. Obviously, $\theta^w(x)$ is lexicographically greater than $\theta^v(x)$ and in fact $\theta^w(x)$ is obtained from $\theta^v(x)$ by replacing $e^v(S, x)$ with $e^w(S, x)$ and subsequently moving it to a higher position when necessary. Hence, if we let k denote the position of S in $\theta^w(x)$, i.e., $\theta_k^w(x) = e^w(S, x)$, then we have:

$$\theta_i^v(x) = \theta_i^w(x) \text{ for all } i \leq k - 1. \tag{1}$$

Similarly, if we let l denote the position of S in $\theta^w(y)$, i.e., $\theta_l^w(y) = e^w(S, y)$, then we have:

$$\theta_i^v(y) = \theta_i^w(y) \text{ for all } i \leq l - 1. \tag{2}$$

And since y is the nucleolus of w , $\theta^w(y)$ is lexicographically smaller than $\theta^w(x)$. Let j denote the number such that:

$$\theta_i^w(y) = \theta_i^w(x) \text{ for all } i \leq j - 1, \quad \text{and } \theta_j^w(y) < \theta_j^w(x). \tag{3}$$

Without loss of generality, we assume in (2) that l is so chosen that $\theta_l^w(y) > \theta_{l+1}^w(y)$. This choice has a corollary. Going from w back to v , $e^w(S, y)$ decreases to $e^v(S, y)$. $\theta_l^v(y)$ is then either $e^v(S, y)$ or $\theta_{l+1}^w(y)$, both of which are less than $\theta_l^w(y) = e^w(S, y)$. Hence,

$$\theta_l^v(y) < \theta_l^w(y). \tag{4}$$

We now show that $j \geq k$ and $l \geq k$. Suppose $j < k$. Then (1) and (3) imply that $\theta^w(y)$ is lexicographically less than $\theta^v(x)$. But $\theta^v(y)$ is lexicographically less than $\theta^w(y)$. Thus $\theta^v(y)$ would be lexicographically smaller than $\theta^w(x)$. This contradicts that x is the nucleolus of v . Hence, we must have:

$$j \geq k. \quad (5)$$

If $l < k$, then $l < j$ by (5). These two inequalities, together with (1), (2), (3), and (4), imply that $\theta^v(y)$ is lexicographically smaller than $\theta^v(x)$. This is a contradiction. Therefore,

$$l \geq k. \quad (6)$$

It is now easy to complete the proof by observing

$$\begin{aligned} e^w(S, y) &= \theta_l^w(y) \\ &\leq \theta_k^w(y) \quad \{ \text{by (6)} \} \\ &\leq \theta_k^w(x) \quad \{ \text{by (3) and (5)} \} \\ &= e^w(S, x), \end{aligned}$$

which is equivalent to $\sum_{i \in S} x_i \leq \sum_{i \in S} y_i$. Q.E.D.

Remark 1: The condition needed in the proof is that $e^v(S, x)$ is strictly increasing with respect to $v(S)$, and non-increasing with respect to $\sum_{i \in S} x_i$ for any coalition S .

Hence Theorem 3 holds for a variety of derivative solution concepts of the nucleolus, say the per capita nucleolus, for which the per capita excesses are used in its definition similar to that of the nucleolus.

Remark 2: We now have a brief discussion on NTU games. There are two difficulties here. First, there is not a satisfactory definition of the nucleolus of a NTU game. Second, without the transferable utility hypothesis, there is not a natural measurement of gains or losses to a coalition of two or more players. Hence any possible extension of Theorem 3 to NTU games will not be as clean. Here we present an example. Consider the nucleolus as defined in Example 2 of Kalai (1975). The weakly coalitional monotonicity property is defined as to require that when a game v changes to w so that the set of feasible payoff vectors of a coalition S enlarges, ceteris paribus, players in S should not suffer a Pareto loss. It is easy to see that Theorem 3 extends to NTU games under these modifications.

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