Edgeworth Equilibria in Production Economies*

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An Edgeworth equilibrium is an allocation that belongs to the core of every $n$-fold replica of the economy. In this work we study the properties of Edgeworth equilibria for economies with production and infinite-dimensional commodity spaces. Under some relatively mild conditions we establish (among other things) that (1) Edgeworth equilibria exist, (2) every Edgeworth equilibrium is a quasi-equilibrium, and (3) an allocation is an Edgeworth equilibrium if and only if it can be “decentralized” by a price system. Journal of Economic Literature Classification Numbers 021, 022.

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1. INTRODUCTION

The problem of the existence of a competitive equilibrium in pure exchange economies with infinite-dimensional commodity spaces has been extensively investigated in recent years; see [1, 2, 7, 11, 16, 20, 21, 27, 29]. Consequently, when all consumption sets coincide with the positive cone

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252
this problem is well understood in contrast to the problem of existence of competitive equilibrium in production economies; see [10, 12, 15, 17, 23, 30].

The seminal paper in this area is due to Bewley [6], where the commodity space is $L_\infty$. An essential feature of the model examined by Bewley (which extends the classical Arrow–Debreu finite-dimensional model [5]) is that the positive cone has a nonempty interior with respect to the norm topology. The latter property does not hold true for many important commodity spaces which are currently under investigation, e.g., $L_1$, $L_2$ or the space $ca(\Omega)$ of all countably additive measures on a compact Hausdorff topological space $\Omega$. Consequently, in these spaces some additional restrictions must be placed on preferences and technologies to bound the marginal rates of substitution and the marginal rates of transformation in production.

Zame’s paper [30] gives several examples of non-existence of equilibria where either the marginal rate of substitution or marginal rate of transformation is unbounded. In addition, Zame proves the existence of competitive equilibria in production economies for a rich class of normed vector lattices. The idea of Zame’s proof is the same as that in Bewley, i.e., to prove existence on a suitable class of subeconomies and then go to the limit; where Bewley uses economies based on finite-dimensional subspaces, Zame uses economies based on principal ideals. It is a clever argument and uses the lattice-theoretic structure of the commodity space in a nontrivial fashion Bewley makes no use of the lattice-theoretic structure of $L_\infty$.

Mas-Colell [17], stimulated by the works of Negishi [18] and Zame [30] and building on his previous contribution to the literature on exchange economies with infinite-dimensional commodity spaces, has investigated the existence and supportability of Pareto optima in production economies with infinite-dimensional commodity spaces where the positive cone has an empty interior. This extends the work of Debreu [8] who assumed that the cone had a nonempty interior. Mas-Colell’s approach is to extend his notion of uniform properness of preferences to uniform properness of technologies. There are two intuitions one should have about uniform properness: (1) it bounds the marginal rates of substitution and transformation, and (2) for a utility function (production function) which is uniformly proper on the positive cone, the utility function (production function) can be viewed as the restriction of a function defined on a “neighborhood” of the positive cone; see Richard [22] and Richard and Zame [24].

Kahn and Vohra [13] proved the existence of approximate competitive equilibria in production economies where the commodity space was an ordered space with a semi-normed predual. They did not assume that preferences were uniformly proper or that the positive cone had a non-
empty interior. Finally, we mention that after this paper was submitted for publication we learned of the work of Richard [23], where he established the existence of equilibria in production economies with a Riesz space of commodities.

In all of the work on production economies, with the exception of Bewley, the authors have assumed that each agent's consumption set is the positive cone. This is clearly an undesirable assumption, but one that we shall also be forced to make. In this paper, we continue our investigation of Edgeworth equilibria, see [2]. Our first major result (Proposition 44) proves the existence of a core allocation for compact economies. Compact economies satisfy quite weak conditions comparable to those in Bewley, but for economies modeled on Riesz dual systems. In particular, we do not assume that the positive cone of the commodity space has a nonempty interior. We also mention that Yannelis [28] recently established the existence of core allocations in exchange economies without ordered preferences.

To demonstrate the existence of Edgeworth equilibria, we must assume that each agent's consumption set is the positive cone and that preferences are strongly monotone. These assumptions are needed to show that in every replica there exists a core allocation with the equal treatment property. This is Theorem 4.7 in the paper.

Using Mas-Colell's notion of a uniformly proper economy, our next major result (Theorem 5.9) is that in a uniformly proper economy every Edgeworth equilibrium is a quasiequilibrium. In addition, we show for uniformly proper compact economies that Walrasian equilibria exist. The rest of the paper considers a special but important model of production economy, i.e., the model where the aggregate production set is a cone. In section six of the paper we give an existence theorem (Theorem 6.10) for approximate Walrasian equilibria for this class of production economies.

In sum, our paper is concerned with the existence and relationship of the following equilibria notions in a production economy: Edgeworth equilibria, quasiequilibria, Walrasian equilibria; and approximate Walrasian equilibria. Our research is most closely related to the work of Zame, Mas-Colell, and Khan and Vohra and we have benefited a great deal from seeing their unpublished research in this area.

2 Mathematical Preliminaries

This work will be based upon the mathematical framework of Riesz spaces and Banach lattices. For extensive treatments of Riesz spaces and Banach lattices we refer the reader to [3, 4, 14, 25].

Recall that a partially ordered vector space $E$ is said to be a Riesz space (or a vector lattice) whenever for each $x, y \in E$ the least upper bound of the
set \{x, y\} (denoted by \(x \land y\)) and the greatest lower bound of \{x, y\} (denoted by \(x \land y\)) both exist in \(E\). For an element \(x\) in a Riesz space, we put
\[
x^+ = x \lor 0, \quad x^- = (-x) \lor 0, \quad \text{and} \quad |x| = x \lor (-x).
\]
If \(E\) is a partially ordered vector space, then the set \(E^+ = \{x \in E : x \geq 0\}\) is referred to as the positive cone of \(E\) and its elements are called positive elements.

The following useful property, known as the **Riesz Decomposition Property**, will be employed quite often in our proofs. It asserts that if in a Riesz space three positive elements \(x, y, z\) satisfy \(0 \leq x \leq y + z\), then there exist positive elements \(x_1\) and \(x_2\) with \(x = x_1 + x_2\) such that \(0 \leq x_1 \leq y\) and \(0 \leq x_2 \leq z\).

Let \(E\) be a Riesz space. A subset \(A\) of \(E\) is said to be a **solid set** whenever \(|x| \leq |y|\) and \(y \in A\) imply \(x \in A\). Every subset \(A\) of \(E\) is contained in a smallest solid set, called the solid hull of \(A\) and is denoted by \(\text{sol}(A)\).

Clearly, \(\text{sol}(A) = \{x \in E : \exists y \in A \text{ with } |x| \leq |y|\}\). A solid vector subspace of \(E\) is referred to as an **ideal**.

An **order interval** is any set of the form \([a, b]\) = \(\{x \in E : a \leq x \leq b\}\). A subset of a Riesz space \(E\) is **order bounded** if it is contained in an order interval. A linear functional \(f : E \to \mathbb{R}\) is said to be order bounded whenever it carries order-bounded subsets of \(E\) onto bounded subsets of \(\mathbb{R}\). The vector space of all order-bounded linear functionals of \(E\) is called the **order dual** of \(E\) and is denoted by \(E^\sim\). Under the ordering \(f \leq g\) whenever \(f(x) \leq g(x)\) for all \(x \in E^\sim\) the order dual \(E^\sim\) is a Riesz space.

A Hausdorff locally convex topology \(\tau\) on a Riesz space \(E\) is said to be locally convex-solid (and \((E, \tau)\) is called a locally convex-solid Riesz space) whenever \(\tau\) has a basis at zero consisting of convex and solid sets. The topological dual \(E^\prime\) of a locally convex-solid Riesz space \((E, \tau)\) is always an ideal of the order dual \(E^\sim\).

Regarding locally convex-solid Riesz spaces the following result will play an important role in our study.

**Theorem 2.1** Let \((E, \tau)\) be a locally convex-solid Riesz space and let two nets \(\{x_\alpha\}\) and \(\{y_\alpha\}\) satisfy \(0 \leq x_\alpha \leq y_\alpha\) for all \(\alpha\). If \(y_\alpha \to^\tau y\) holds in \(E\) and the order interval \([0, y]\) is weakly compact, then the net \(\{x_\alpha\}\) has a weakly convergent subnet.

**Proof** From the lattice identity \(a = (a - b)^+ + a \land b\), we see that
\[
0 \leq x_\alpha = (x_\alpha - y)^+ + x_\alpha \land y \leq (y_\alpha - y)^+ + y.
\]
Using the Riesz Decomposition Property, we can write \(x_\alpha = z_\alpha - v_\alpha\) with \(0 \leq z_\alpha \leq (y_\alpha - y)^+\) and \(0 \leq v_\alpha \leq y\). From \((y_\alpha - y)^+ \to^\tau 0\), we get \(z_\alpha \to^\tau 0\) and...
Also, from the weak compactness of \([0, y]\), we see that \(\{v_x\}\) has a weakly convergent subnet, and so from \(x_\ast = z_\ast + e_\ast\), we infer that \(\{x_\ast\}\) has a weakly convergent subnet. 

In our economic model the basic concept describing the commodity-price duality will be that of a Riesz system. A Riesz dual system is a dual system \(\langle E, E'\rangle\) such that

1. \(E\) is a Riesz space;
2. \(E'\) is an ideal of the order dual \(E^\sim\) that separates the points of \(E\); and
3. the duality of the system is the natural one, i.e.,
\[
\langle x, x' \rangle = x'(x)
\]
holds for all \(x \in E\) and all \(x' \in E'\).

A Riesz dual system \(\langle E, E'\rangle\) is said to be symmetric whenever the order intervals of \(E\) are weakly compact (i.e., \(\sigma(E, E')\)-compact). A Riesz dual system \(\langle E, E'\rangle\) is symmetric if and only if \(E\) is an ideal of \((E')^\sim\) (where \(E\) is identified in the usual manner as a vector subspace of \((E')^\sim\)).

Regarding symmetric Riesz dual systems, the following result will be very important.

**Theorem 2.2.** Assume that \(\langle E, E'\rangle\) is a symmetric Riesz dual system. If \(A\) is a relatively weakly compact subset of \(E^+\), then \(\text{sol}(A)\) (the solid hull of \(A\)) is also a relatively weakly compact subset of \(E\).

*Proof.* See the proof of [4, Theorem 13.8, p. 206].

3. **The Economic Model**

The characteristics of our economic model are described as follows.

A. The Commodity-Price Duality

The commodity-price duality is given by a Riesz dual system \(\langle E, E'\rangle\): \(E\) is the commodity space and \(E'\) is the price space.

B. Consumers

There are \(m\) consumers indexed by \(i\) such that:

1. Each consumer \(i\) has an initial endowment \(\omega_i > 0\) and his consumption set \(X_i\) is a weakly closed convex subset of \(E^+\) with \(\omega_i \in X_i\).
(2) The total endowment of the consumers (or simply the total endowment) will be denoted by \( \omega \), i.e., \( \omega = \omega_1 + \cdots + \omega_m \).

(3) The preference \( \succeq_i \), of each consumer \( i \) is represented by a quasi-concave utility function \( u_i : X_i \to \mathbb{R}^+ \).

(4) There is a locally convex-solid topology \( \tau \) on \( E \) consistent with \( \langle E, E' \rangle \) such that each utility function \( u_i : (X_i, \tau) \to \mathbb{R}^+ \) is continuous.

C. Producers

We assume that there are \( k \) production firms indexed by \( j \). The production of each producer \( j \) is described by its production possibility set \( Y_j \), the elements of which are referred to as the production plans for the \( j \) producer. For a production plan \( y = y^+ - y^- \in Y_j \), the negative part \( y^- \) of \( y \) is interpreted as the input and the positive part \( y^+ \) as the output. The production sets are assumed to satisfy the following properties:

(1) Each \( Y_j \) is a weakly closed convex subset of \( E \) containing zero; and

(2) For each \( j \) we have \( Y_j \cap E^+ = \{0\} \)

The convex set \( Y = Y_1 + \cdots + Y_k \) is known as the aggregate production set of the economy.

D. Private Ownership

Our economy is a private ownership economy. That is, we shall assume that each consumer \( i \) has a share \( \theta_{ij} (0 \leq \theta_{ij} \leq 1) \) of the profit of producer's \( j \) production plan. Of course, \( \sum_{j=1}^{m} \theta_{ij} = 1 \) for each \( j \). In other words, if each producer \( j \) chooses a production plan \( y_j \in Y_j \) and the prevailing price vector is \( p \), then the wealth \( w_i \) of the \( i \)th consumer is

\[
w_i = p \cdot \omega + \sum_{j=1}^{k} \theta_{ij} p \cdot y_j.
\]

Our economy is now defined as follows.

**Definition 3.1.** An economy \( \mathcal{E} \) is a 4-tuple

\[
\mathcal{E} = (\langle E, E' \rangle, \{(X_i, \omega_i, \succeq_i) : i = 1, \ldots, m\}, \{Y_j : j = 1, \ldots, k\}, \{\theta_{ij} : i = 1, \ldots, m ; j = 1, \ldots, k\})
\]

where the agents' characteristics satisfy properties (A), (B), (C), and (D).
4. Edgeworth Equilibria

The objective of this section is to establish that Edgeworth equilibria exist. That is, to establish Theorem 4.7 which asserts that there exist allocations that belong to the core of every replication of the original economy. In order to prove this, we need some preliminary technical results.

An \((m + k)\)-tuple \((x_1, \ldots, x_m, y_1, \ldots, y_k)\), where \(x_i \in X_i\) \((i = 1, \ldots, m)\) and \(y_j \in Y_j\) \((j = 1, \ldots, k)\), is said to be an allocation whenever

\[
\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} \omega_i + \sum_{j=1}^{k} y_j
\]

The set of all allocations will be denoted by \(\mathcal{A}\). That is,

\[
\mathcal{A} = \left\{ (x_1, \ldots, x_m, y_1, \ldots, y_k) : x_i \in X_i, y_j \in Y_j, \text{ and } \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} \omega_i + \sum_{j=1}^{k} y_j \right\}.
\]

It should be noted that the set \(\mathcal{A}\) of all allocations is a weakly closed subset of \(E^{m+k}\).

A production plan \(y \in Y_j\) is said to be feasible for the \(j\)th producer whenever there exists an allocation \((x_1, \ldots, x_m, y_1, \ldots, y_k)\) such that \(y_j = y\).

Similarly, a bundle \(x \in X_i\) is said to be feasible for the \(i\)th consumer whenever there exists an allocation \((x_1, \ldots, x_m, y_1, \ldots, y_k)\) with \(x_i = x\).

The feasible production set \(\hat{Y}_j\) for the \(j\)th producer is the set of all of its feasible production plans, i.e.,

\[
\hat{Y}_j = \{ y \in Y_j : \exists (x_1, \ldots, x_m, y_1, \ldots, y_k) \in \mathcal{A} \text{ with } y_j = y \}.
\]

Similarly, the feasible consumption set \(\hat{X}_i\) of the \(i\)th consumer is the set of all of its feasible consumption bundles, i.e.,

\[
\hat{X}_i = \{ x \in X_i : \exists (x_1, \ldots, x_m, y_1, \ldots, y_k) \in \mathcal{A} \text{ with } x_i = x \}.
\]

Some basic properties of the sets \(\hat{X}_i\) and \(\hat{Y}_j\) are described in the next result.

**Proposition 4.1.** For an economy with a symmetric Riesz dual system and aggregate production set \(Y = Y_1 + \cdots + Y_k\) the following statements hold

1. If all production sets are order bounded from above, then each feasible production set \(\hat{Y}_j\) is weakly compact.

2. If each feasible production set \(\hat{Y}_j\) is weakly compact and \(X_i = E^+\) holds for each \(i\), then \((Y + \omega) \cap E^+\) is a weakly compact set.
(3) If \((Y + \omega) \cap E^+\) is weakly compact, then the feasible consumption sets \(\tilde{X}\) are all weakly compact subsets of \(E^+\).

**Proof.** (1) Pick some \(a \in E^+\) such that \(z \in Y_j\) \((j = 1, \ldots, k)\) implies \(z \leq a\). Let \(y \in \tilde{Y}_j\). Choose an allocation \((x_1, \ldots, x_m, y_1, \ldots, y_k)\) with \(y_j = y\). Then we have

\[
0 \leq y^- = \sum_{j=1}^k y_j^- + \sum_{i=1}^m x_i = \sum_{i=1}^m y_j^+ + \omega \leq ka + \omega = b \in E^+,
\]

and so

\[-b \leq -y^- \leq y^+ - y^- = y \leq a \leq b.
\]

Therefore, \(\widetilde{Y}_j \subseteq [-b, b]\). Since \([-b, b]\) is weakly compact, we infer that \(\tilde{Y}_j\) is relatively weakly compact. Thus, in order to establish that \(\tilde{Y}_j\) is weakly compact, it suffices to show that \(\tilde{Y}_j\) is weakly closed.

To this end, let \(\{y^x\} \subseteq \tilde{Y}_j\) satisfy \(y^x \rightharpoonup y\) in \(E\). For each \(x\) pick an allocation \((x_1, \ldots, x_m, y_1^x, \ldots, y_k^x)\) with \(y_j^x = y^x\). Since \(\tilde{Y}_j\) is relatively weakly compact, by passing to an appropriate subnet, we can assume that \(y_j^x \rightharpoonup y_j \in \tilde{Y}_j\) holds for each \(j\). From

\[
0 \leq x^x = x_1^x + \cdots + x_m^x = \sum_{j=1}^k y_j^x + \omega \in \tilde{Y}_1 + \cdots + \tilde{Y}_k + \omega.
\]

we see that \(x_j^x\) belongs to the relatively weakly compact set \(\text{sol}((\tilde{Y}_1 + \cdots + \tilde{Y}_k + \omega) \cap E^+)\) (Theorem 2.2). Thus, each net \(\{x_j^x\}\) has a weakly convergent subnet, and so (by passing to an appropriate subnet again) we can assume that \(x_j^x \rightharpoonup x_j \in \tilde{X}\) holds for all \(j\). From

\[
\sum_{i=1}^m x_i = \sum_{i=1}^m \omega, + \sum_{j=1}^k y_j^x.
\]

we get \(\sum_{i=1}^m x_i = \sum_{i=1}^m \omega + \sum_{j=1}^k y_j^x\). This implies \(y \in \tilde{Y}_j\), and so \(\tilde{Y}_j\) is a weakly closed set, as desired.

(2) Since \(X_i = E^+\) holds for each \(i\), it is easy to see that

\[(Y + \omega) \cap E^+ \subseteq (\tilde{Y}_1 + \cdots + \tilde{Y}_k + \omega) \cap E^+.
\]

Therefore, \((Y + \omega) \cap E^+\) is a relatively weakly compact set.

Now assume that a net \(\{(y_1^x + \cdots + y_k^x + \omega)\}\) of \((Y + \omega) \cap E^+\) satisfies \(y_1^x + \cdots + y_k^x + \omega \rightharpoonup z\). Since \(\{y_j^x\} \subseteq \tilde{Y}_j\) holds for all \(j\), we can assume that \(y_j^x \rightharpoonup y_j \in \tilde{Y}_j\) holds for all \(j\). This implies \(z = y_1 + \cdots + y_k + \omega \in (Y + \omega) \cap E^+\), and so \((Y + \omega) \cap E^+\) is weakly closed. Hence, \((Y + \omega) \cap E^+\) is weakly compact.
(3) Fix some $i$, and let $x \in \hat{X}_i$. Pick an allocation $(x_1, \ldots, x_m, y_1, \ldots, y_k)$ with $x_i = x$. From

$$0 \leq x \leq x_1 + \cdots + x_m = \omega + \sum_{j=1}^{k} y_j \in (Y + \omega) \cap E^+,$$

we see that $x \in \text{sol}[(Y + \omega) \cap E^+]$, and so $\hat{X}_i \subseteq \text{sol}[(Y + \omega) \cap E^+]$. Since $\text{sol}[(Y + \omega) \cap E^+]$ is a relatively weakly compact subset of $E$ (Theorem 2.2), it follows that each $\hat{X}_i$ is a relatively weakly compact subset of $E^+$.

Next, assume that a net $\{x^2\}$ of $\hat{X}_i$ satisfies $x^2 \rightharpoonup x$. For each $x$ pick an allocation $(x_1, \ldots, x_m, y_1, \ldots, y_k)$ with $x_1 = x^2$. By the preceding conclusion, we can assume (by passing to a subnet) that $x_1 \rightharpoonup x_i$ holds for each $i$. Let $y_i = \sum_{j=1}^{k} y_{ij}$. From $y_i + \omega = \sum_{j=1}^{k} y_{ij} + \omega \in (Y + \omega) \cap E^+$, we can assume (by passing to a subnet again) that $y_i \rightharpoonup z \in Y$ holds. If $z = z_1 + \cdots + z_k \in Y$, then $(x_1, \ldots, x_m, z_1, \ldots, z_k)$ is an allocation, and so $x_1 \in \hat{X}_i$. Thus, $\hat{X}_i$ is weakly closed, and hence each $\hat{X}_i$ is a weakly compact subset of $E^+$.

We now come to the concept of a compact economy.

**Definition 4.2** An economy is said to be a **compact economy** whenever

1. its Riesz dual system is symmetric and
2. if $Y = Y_1 + \cdots + Y_k$ is its aggregate production set, then $(Y + \omega) \cap E^+$ is a weakly compact set

It should be noted that the weak compactness of $(Y + \omega) \cap E^+$ does not imply the weak compactness of the feasible production sets.

Now let $S$ be a coalition of consumers (i.e., let $S$ be a non-empty subset of $\{1, \ldots, m\}$). A subset $\{z_i, i \in S\}$ of $E^+$ is said to be a feasible assignment for the coalition $S$ whenever

(a) $z_i \in X_i$ for each $i \in S$ and

(b) there exist production plans $h_j \in Y_j (j = 1, \ldots, k)$ such that

$$\sum_{i \in S} z_i = \sum_{i \in S} \omega_i + \sum_{j=1}^{k} \left( \sum_{i \in S} \theta_{ij} \right) h_j.$$

A coalition $S$ blocks an allocation $(x_1, \ldots, x_m, y_1, \ldots, y_k)$ whenever there exists a feasible assignment set $\{z_i, i \in S\}$ for $S$ such that $z_i \triangleright_i x$, holds for all $i \in S$.

If $X_i = X_i + E^+$ holds for each $i$ and preferences are monotone (i.e., $x \triangleright y$ in $X_i$ implies $x \triangleright_i y$), then it should be clear that a coalition $S$ blocks an allocation $(x_1, \ldots, x_m, y_1, \ldots, y_k)$ if and only if there exist consumption bundles $z_i \in X_i (i \in S)$ and production plans $h_j \in Y_j (j = 1, \ldots, k)$ such that
(1) \( z_i \succ x_i \) for each \( i \in S \) and
(2) \( \sum_{i \in S} z_i \leq \sum_{i \in S} a_i + \sum_{j=1}^k \left( \sum_{i \in S} \theta_{ij} \right) h_j. \)

A core allocation is any allocation that cannot be blocked by any coalition. The core \( \text{Core}(\mathcal{E}) \) of an economy \( \mathcal{E} \) is the set of all core allocations.

**Lemma 4.3.** If the Riesz dual system of an economy is symmetric and each production set is order bounded from above, then its core is non-empty.

**Proof.** Fix some \( a \in E^* \) such that \( y \in Y_i \) \( (j = 1, ..., k) \) implies \( y \preceq a \). For each coalition \( S \) of consumers define the set

\[
V(S) = \{ (z_1, ..., z_m) \in \mathbb{R}^m; \exists a \text{ feasible assignment} \}
\]

\[
\{ x_i : i \in S \} \text{ with } u_i(x_i) \geq z_i, \forall i \in S \}
\]

The subsets \( V(S) \) of \( \mathbb{R}^m \) have the following properties.

(1) For each coalition \( S \), the set \( V(S) \) is bounded from above with respect to \( \mathbb{R}^S \). In particular, the non-empty set \( V(S) \cap \mathbb{R}^S_+ \) is bounded relative to \( \mathbb{R}^S \).

To see that \( V(S) \) is bounded from above in \( \mathbb{R}^S \), assume by way of contradiction that there exists a sequence \( \{ (z_1^n, ..., z_m^n) \} \) of \( V(S) \) and some \( r \in S \) such that \( z_j^n \geq n \) holds for all \( n \). Pick \( x_i^n \in X_i \) \( (i \in S) \) and \( y_j^n \in Y_j \) \( (j = 1, ..., k) \) such that

\[
\sum_{i \in S} x_i^n = \sum_{i \in S} \omega_i + \sum_{j=1}^k \left( \sum_{i \in S} \theta_{ij} \right) y_j^n \quad \text{and} \quad u_i(x_i^n) \geq z_i^n \geq n \quad \text{for all } n.
\]

Clearly, \( \{ x_i^n \} \subseteq X \), holds. Since \( X_i \subseteq [0, \omega + k a] \), the sequence \( \{ x_i^n \} \) has a weak accumulation point, say \( x \). Now for each natural number \( l \), the element \( x \) belongs to the weak closure of the set \( \text{co} \{ x_i^n : n \geq l \} \), and hence \( x \) belongs to the \( \tau \)-closure of \( \text{co} \{ x_i^n : n \geq l \} \). Thus, by the \( \tau \)-continuity of \( u_\ast \), there exists a convex combination \( \sum_{i=1}^{l+\mu} \lambda_i x_i^n \) with

\[
\left| u_\ast(x) - u_\ast \left( \sum_{i=l}^{l+\mu} \lambda_i x_i^n \right) \right| < 1.
\]

If \( n \) is an integer among \( \{ l, ..., l+\mu \} \) satisfying

\[
\min \{ u_\ast(x_i^n) : i = l, ..., l+\mu \},
\]

then by the quasi-concavity of \( u_\ast \), we see that

\[
l \leq n \leq u_\ast(x_i^n) \leq u_\ast \left( \sum_{i=l}^{l+\mu} \lambda_i x_i^n \right) \leq u_\ast(\tau) + 1 < \infty.
\]
Since \( I \) is arbitrary, the latter inequality is impossible. Therefore, \( V(S) \) is bounded from above relative to \( \mathbb{R}^5 \).

(2) Each \( V(S) \) is a non-empty proper closed subset of \( \mathbb{R}^m \).

Let \( S \) be a coalition of consumers. Since \( (u_1(\omega_1), \ldots, u_m(\omega_m)) \in V(S) \), we see that \( V(S) \neq \emptyset \). By part (1), we know that \( V(S) \) is bounded from above relative to \( \mathbb{R}^5 \), and this implies that \( V(S) \) is a proper subset of \( \mathbb{R}^m \).

To see that \( V(S) \) is closed, assume that a net \( \{ (z^*_1, \ldots, z^*_m) \} \) of \( V(S) \) satisfies \( (z^*_1, \ldots, z^*_m) \to (z_1, \ldots, z_m) \) in \( \mathbb{R}^m \). For each \( \alpha \) pick \( x^*_\alpha \in X_i \) (\( i \in S \)) and \( y^*_\alpha \in Y_j \) (\( j = 1, \ldots, k \)) such that

\[
\sum_{i \in S} x^*_i = \sum_{i \in S} \omega_i + \sum_{j=1}^k \left( \sum_{i \in S} \theta_i \right) y^*_j \quad (*)
\]

and

\[
z^*_i \leq u_i(x^*_i) \quad \text{for all} \quad i \in S.
\]

In case \( \sum_{i \in S} \theta_i = 0 \), we can assume without loss of generality that \( y^*_i = 0 \).

Since for each \( j \) we have \( (\sum_{i \in S} \theta_i) y^*_i \in \hat{Y}_j \) and \( \hat{Y}_j \) is weakly compact (Proposition 4.1), it follows (by passing to a subnet if necessary) that \( y^*_i \to y_j \in Y_j \) holds for all \( j = 1, \ldots, k \). Also, from (1) we see that \( x^*_i \in \mathbb{X}_j \) holds for all \( i \in S \), and so from Proposition 4.1(3) (by passing to a subnet again) we can assume that \( x^*_i \to x_i \in X_i \) holds for all \( i \in S \). From (1), we infer that

\[
\sum_{i \in S} x_i = \sum_{i \in S} \omega_i + \sum_{j=1}^k \left( \sum_{i \in S} \theta_i \right) y_j.
\]

To complete the proof of part (2), it suffices to show that \( z_i \leq u_i(x_i) \) holds for each \( i \in S \). To this end, fix \( i \in S \) and let \( \varepsilon > 0 \). Pick some \( \beta \) with

\[
z_i - \varepsilon < z^*_i \quad \text{for all} \quad x \geq \beta.
\]

Since \( x_i \) is in the weak closure of the set \( \text{co} \{ x^*_i : x \geq \beta \} \), it follows that \( x_i \) is also in the \( \tau \)-closure of \( \text{co} \{ x^*_i : x \geq \beta \} \). Thus, by the \( \tau \)-continuity of \( u_\gamma \) there exists a convex combination \( \sum_{i=1}^{t_i} \lambda_i x^*_i \) with \( x_i \geq \beta \) such that

\[
u(\sum_{i=1}^{t_i} \lambda_i x^*_i) < u_i(x_i) + \varepsilon.
\]

If \( \gamma \) is an index among \( \{ \alpha_1, \ldots, \alpha_{t_i} \} \) with

\[
u(x^*_i) = \min \{ u_i(x^*_i) : s = 1, \ldots, t_i \}.
\]
then \( \gamma \geq \beta \) holds, and by the quasi-concavity of \( u_i \), we see that

\[
\psi_i - \varepsilon < \psi_i \leq u_i(x_i) \leq u_i \left( \sum_{j=1}^{l} \psi_j x_j^i \right) < u_i(x_i) + \varepsilon.
\]

Thus, \( \psi_i < u_i(x_i) + 2\varepsilon \) holds for all \( \varepsilon > 0 \), from which it follows that \( \psi_i \leq u_i(x_i) \) holds for all \( i \in S \), as desired.

(3) Each \( V(S) \) is comprehensive, i.e., \( (x_1, \ldots, x_m) \in V(S) \) and \( (z_1, \ldots, z_m) \leq (x_1, \ldots, x_m) \) imply \( (z_1, \ldots, z_m) \in V(S) \).

(4) If \( (x_1, \ldots, x_m) \in V(S) \) and \((z_1, \ldots, z_m)\) satisfies \( z_i = x_i \) for all \( i \in S \), then \((z_1, \ldots, z_m)\in V(S)\).

(5) The market game derived from the economy is balanced.

To see this, consider a balanced family \( \mathcal{B} \) of coalitions with weights \( \{ w_S : S \in \mathcal{B} \} \). That is, \( \sum_{S \in \mathcal{B}} w_S = 1 \) holds for all \( i \), where as usual \( \mathcal{B} = \{ S : i \in S \} \). Now let \((z_1, \ldots, z_m) \in \bigcap_{S \in \mathcal{B}} V(S)\). We have to show that \((z_1, \ldots, z_m) \in V(1, \ldots, m)\).

Let \( S \in \mathcal{B} \). Since \((z_1, \ldots, z_m) \in V(S)\), there exist \( x_i^S \in X_i \) (\(i \in S\)) and \( y_j^S \in Y_j \) (\(j = 1, \ldots, k\)) with

\[
\sum_{i \in S} x_i^S = \sum_{i \in S} \omega_i + \sum_{j=1}^{k} \left( \sum_{i \in S} \theta_{ij} \right) y_j^S,
\]

and \( u_i(x_i^S) \geq z_i \) for all \( i \in S \). Now put

\[
x_i = \sum_{S \in \mathcal{B}} w_S x_i^S \in X_i, \quad i = 1, \ldots, m
\]

and

\[
y_j = \sum_{S \in \mathcal{B}} \sum_{i \in S} w_S \theta_{ij} y_j^S = \sum_{i=1}^{m} \theta_{ij} \left( \sum_{S \in \mathcal{B}} w_S y_j^S \right) \in Y_j, \quad j = 1, \ldots, k
\]

Since each \( x_i \) is a convex combination, it follows from the quasi-concavity of \( u_i \) that \( z_i \leq u_i(x_i) \) holds for all \( i = 1, \ldots, m \). Moreover, we have

\[
\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} \sum_{S \in \mathcal{B}} w_S x_i^S = \sum_{S \in \mathcal{B}} w_S \left( \sum_{i \in S} x_i^S \right)
\]

\[
= \sum_{S \in \mathcal{B}} w_S \left[ \sum_{i \in S} \omega_i + \sum_{j=1}^{k} \left( \sum_{i \in S} \theta_{ij} \right) y_j^S \right]
\]

\[
= \sum_{i=1}^{m} \sum_{j=1}^{k} w_i \omega_j + \sum_{j=1}^{k} \sum_{i=1}^{m} \sum_{S \in \mathcal{B}} w_S \theta_{ij} y_j^S = \sum_{i=1}^{m} \omega_i + \sum_{j=1}^{k} y_j,
\]

which proves that \((z_1, \ldots, z_m) \in V(1, \ldots, m)\), as desired.
Next, by Scarf’s classical result [26], the market game derived from the economy has a non-empty core (i.e., the set \( V(\{1, \ldots, m\}) \cup \bigcup_{S \in \mathcal{K}} \text{Int} \; V(S) \) is non-empty, where \( \mathcal{K} \) denotes the set of all coalitions). Let \((z_1, \ldots, z_m)\) be a core vector. Pick \( x_i \in X_i \) \((i = 1, \ldots, m)\) and \( y_j \in Y_j \) \((j = 1, \ldots, k)\) such that

\[
\begin{align*}
(\text{a}) & \quad \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i + \sum_{j=1}^{k} y_j \\
(\text{b}) & \quad u_i(x_i) \geq z_i \quad \text{for } i = 1, \ldots, m.
\end{align*}
\]

Clearly, \((x_1, \ldots, x_m, y_1, \ldots, y_k)\) is an allocation, and we claim that it is a core allocation. To see the latter, assume by way of contradiction that there exists a coalition \( S \) and a feasible assignment \( \{h_i : i \in S\} \) satisfying \( h_i \succ x_i \) for all \( i \in S \). Then \( u_i(h_i) > u_i(x_i) \geq z_i \) holds for all \( i \in S \), and from this we see that \((z_1, \ldots, z_m) \in \text{Int} \; V(S)\), which is a contradiction. Hence \((x_1, \ldots, x_m, y_1, \ldots, y_k)\) is a core allocation, and therefore the economy has a non-empty core.

We are now in the position to establish that a compact economy has always a non-empty core.

**Proposition 4.4.** If the economy is compact, then its core is a non-empty weakly closed subset of the set of all allocations.

**Proof.** Put \( \hat{Y} = [(Y + \omega) \cap E^+] - \omega \), where \( Y = Y_1 + \cdots + Y_k \), and note that \( \hat{Y} \) is a weakly compact set. Also, we shall consider the set

\[
\mathcal{A} = \left\{(x_1, \ldots, x_m, y) \in E^{m+1} : y = \sum_{j=1}^{k} y_j \text{ with } (x_1, \ldots, x_m, y_1, \ldots, y_k) \in \mathcal{A}\right\}
\]

Clearly, \( \mathcal{A} \subseteq \hat{X}_1 \times \cdots \times \hat{X}_m \times \hat{Y} \), and from this and Proposition 4.1 it is easy to see that \( \mathcal{A} \) is a weakly compact subset of \( E^{m+1} \). The proof of the theorem has two steps.

1. The core is non-empty.

For each \( a \in E^+ \) we shall denote by \( \delta_a \) the economy which comes from our original economy \( \delta \) by replacing each \( Y_i \) by \( Y_i^a = \{y \in Y_i : y \leq a\} \). By Lemma 4.3, we know that \( \text{Core}(\delta_a) \neq \emptyset \). For each \( a \in E^- \) pick some \((x_1^a, \ldots, x_m^a, y_1^a, \ldots, y_k^a)\) in the core of \( \delta_a \) and let \( y_i^a = \sum_{i=1}^{k} y_i^a \) for each \( a \in E^+ \). Since \( \mathcal{A} \) is weakly compact, the net \( \{(x_1^a, \ldots, x_m^a, y^a) : a \in E^+\} \) has a weak accumulation point in \( \mathcal{A} \), say \((x_1, \ldots, x_m, y)\). Then \( x_1 + \cdots + x_m = \omega + y \) and \( y \in Y \). Pick \( y_j \in Y_j \) \((j = 1, \ldots, k)\) with \( y = y_1 + \cdots + y_k \), and we claim that the allocation \((x_1, \ldots, x_m, y_1, \ldots, y_k)\) is a core allocation for our original economy.

To see this, assume by way of contradiction that there exist a coalition \( S \) of consumers, consumption bundles \( h_i \in X_i \) \((i \in S)\), and production plans \( z_j \in Y_j \) \((j = 1, \ldots, k)\) such that...
(a) \( h_i \succ_i x_i \) for all \( i \in S \), and

(b) \( \sum_{i \in S} h_i = \sum_{i \in S} \omega_i + \sum_{j=1}^{k} \left( \sum_{i \in S} \theta_{ij} \right) z_j \).

Now note that for each \( i \in S \) the set

\[ V_i = \left\{ (f_1, \ldots, f_m, g) \in \mathcal{A} : f_i \succ_i h_i \right\} \]

is a weakly closed subset of \( E^{m+1} \), and so \( V = \bigcup_{i \in S} V_i \) is also weakly closed. Thus, its complement \( V^c \) is weakly open. Since \( (x_1, \ldots, x_m, y) \in V^c \) and \( (x_1, \ldots, x_m, y') \) is a weak accumulation point of the net \( \{(x_1^a, \ldots, x_m^a, y^a) : a \in E^+\} \), there exists some \( a \geq |z_1| + \ldots + |z_k| \) such that \( (x_1^a, \ldots, x_m^a, y^a) \in V^c \). Clearly, \( z_j \in Y_j^a \) for each \( j \). Also, \( h_i \succ_i x_i^a \) holds for all \( i \in S \), and so in view of (b) we have \( (x_1^a, \ldots, x_m^a, y_1^a, \ldots, y_k^a) \notin \text{Core}(e_i) \), a contradiction. Therefore, \( (x_1, \ldots, x_m, y_1, \ldots, y_k) \) is a core allocation for our original economy.

(II) The core is a weakly closed set.

Denote by \( C \) the (non-empty) set of all core allocations, and let \( (x_1, \ldots, x_m, y_1, \ldots, y_k) \) be an allocation lying in the weak closure of \( C \). Assume by way of contradiction that there exist a coalition \( S \), consumption bundles \( z_i \in X_i \) (\( i \in S \)) and production plans \( y_j \in Y_j \) (\( j = 1, \ldots, k \)) such that

\[ z_i \succ_i x_i \quad \text{for all} \ i \in S \quad \text{and} \quad \sum_{i \in S} z_i = \sum_{i \in S} \omega_i + \sum_{j=1}^{k} \left( \sum_{i \in S} \theta_{ij} \right) y_j. \]

For each \( i \in S \) the set of allocations

\[ W_i = \left\{ (h_1, \ldots, h_m, g_1, \ldots, g_k) \in \mathcal{A} : h_i \succ_i z_i \right\} \]

is a weakly closed subset subset of \( E^{m+k} \). Thus the set \( W = \bigcup_{i \in S} W_i \) is weakly closed in \( E^{m+k} \), and so its complement \( W^c \) is weakly open. From \( (x_1, \ldots, x_m, y_1, \ldots, y_k) \in W^c \), we infer that \( W^c \cap C \neq \emptyset \). If \( (h_1, \ldots, h_m, g_1, \ldots, g_k) \in W^c \cap C \), then we have

\[ z_i \succ_i h_i \quad \text{for all} \ i \in S \quad \text{and} \quad \sum_{i \in S} z_i = \sum_{i \in S} \omega_i + \sum_{j=1}^{k} \left( \sum_{i \in S} \theta_{ij} \right) y_j, \]

which contradicts the fact that \( (h_1, \ldots, h_m, g_1, \ldots, g_k) \) is a core allocation. Hence, \( (x_1, \ldots, x_m, y_1, \ldots, y_k) \in C \), and so \( C \) is a weakly closed set.

Next, let us briefly recall the replication concept of an economy with production as it was introduced by H. Nikaido in [19, p. 288]. If \( n \) is a natural number, then the \( n \)-fold replica of the economy is a new economy with the following characteristics.

(1) The new economy has the same Riesz dual system \( \langle E, E' \rangle \)
(2) There are $mn$ consumers indexed by $(i, s)$ ($i = 1, ..., m; s = 1, ..., n$) such that the consumers $(i, s)$ ($s = 1, ..., n$) are of the "same type" as the consumer $i$ of the original economy. That is, each consumer $(i, s)$ has

(a) $X_i$, as his consumption set, i.e., $X_{is} = X_i$,

(b) an initial endowment $\omega_0$, equal to $\omega_i$, i.e., $\omega_{is} = \omega_i$ (and so the total endowment of the new economy is $\sum_{i=1}^{m} \sum_{s=1}^{n} \omega_{is} = m \omega_i$), and

(c) a utility function $u_{is}$ equal to $u_i$, i.e., $u_{is} = u_i$.

(3) There are $kn$ producers indexed by $(j, t)$ ($j = 1, ..., k; t = 1, ..., n$) with the following properties,

(i) The production possibility set of the $(j, t)$ producer is $Y_j$, i.e., $Y_{jt} = Y_j$; and

(ii) The share $\theta_{is}$ of the $(i, s)$ consumer to the profit of the $(j, t)$ producer is given by

$$\theta_{is} = \begin{cases} 0 & \text{if } s \neq t \\ \theta_j & \text{if } s = t. \end{cases}$$

The proof of the next proposition should be immediate.

**Proposition 4.5.** Every replication of a compact economy is itself compact.

Now let $(x_1, ..., x_m, y_1, ..., y_k)$ be an allocation of the original economy. If $n$ is a natural number, then by assigning the consumption bundle $x_i$ to each consumer $(i, s)$ (i.e., $x_{is} = x_i$ for $s = 1, ..., n$) and the production plan $y_j$ to each producer $(j, t)$ (i.e., $y_{jt} = y_j$ for $t = 1, ..., n$), it is easy to see that this assignment defines an allocation for the $n$-fold replica economy. Thus, every allocation of the original economy can be considered (in the above manner) as an allocation for every $n$-fold replica of the original economy.

**Definition 4.6.** An allocation of an economy is said to be an *Edgeworth Equilibrium* whenever it belongs to the core of every $n$-fold replica of the economy.

Do Edgeworth equilibria exist? Before presenting an affirmative answer, let us review a few facts about preferences. Recall that a preference $\succeq$ on a convex set $X$ is said to be

(a) *strongly monotone*, whenever $x, y \in X$ and $x \succ y$ imply $x \succ y$, and

(b) *convex*, whenever $x \succ y$ in $X$ implies $zx + (1 - z)y \succ y$ for all $0 < z < 1$.

The following two basic properties about preferences will be employed in the proof of the next theorem.
(1) If a preference \( \succeq \) defined on \( E^+ \) is weakly convex (i.e., \( \{ x \in E^+. x \succeq y \} \) is convex for all \( y \in E^+ \)), continuous for some linear topology on \( E \) and strongly monotone, then \( \succeq \) is also convex.

To see this, assume \( x \succ y \) and let \( 0 < z < 1 \). Then \( x \succ 0 \), and since \( \lim_{\epsilon \to 0}\epsilon x = x \) holds for every linear topology on \( E \), it follows that there exists some \( 0 < \epsilon < 1 \) with \( \epsilon x \succ y \). By the weak convexity, we have \( x(\epsilon x) + (1 - \epsilon) y \succeq y \). On the other hand, from \( zx + (1 - z)y \succ z(\epsilon x) + (1 - \epsilon) y \) and the strong monotonicity of \( \succeq \), we infer that \( zx + (1 - z)y \succ z(\epsilon x) + (1 - \epsilon) y \). Thus, \( zx + (1 - z)y \succ y \).

(2) If \( X_i = E^+ \) holds for each consumer \( i \) and each preference \( \succeq_i \), is in addition strongly monotone, then a coalition \( S \) blocks an allocation \( (x_1, \ldots, x_n, y_1, \ldots, y_k) \) if and only if there exists a feasible assignment \( \{ h_i : i \in S \} \) for \( S \) such that

(i) \( h_i \succeq_i x_i \) for all \( i \in S \) and

(ii) \( h_i \succeq_i x_i \) holds for at least one \( i \in S \).

To see this, assume that (i) and (ii) are true. Fix some \( r \in S \) with \( h_r \succeq_r x_r \). Since \( \tau \)-lim \( x \tau h_r = h_r \), it follows from \( X_r = E^+ \) and the \( \tau \)-continuity of \( \succeq_r \), that there exists some \( 0 < \epsilon < 1 \) with \( \epsilon h_r \succeq_r x_r \). If \( l > 1 \) is the number of elements of \( S \), then put \( f_i = \epsilon h_i \) and \( f_r = h_r + [(1 - \epsilon)/(l - 1)] h_r \in E^+ \) for \( i \in S \) and \( i \neq r \). From the strong monotonicity of preferences, we infer that \( f_i \succeq_i x_i \) for all \( i \in S \) and moreover \( \sum_{i \in S} f_i = \sum_{i \in S} h_i \). The above show that \( S \) blocks the allocation \( (x_1, \ldots, x_n, y_1, \ldots, y_k) \).

We are now in the position to present an existence theorem for Edgeworth equilibria. It is at this theorem that the strong monotonicity of preferences plays a crucial role. As we shall see, this guarantees an equal treatment property for the core allocations in each replica of the original economy.

**Theorem 4.7.** If the economy is compact, preferences are in addition strongly monotone and \( X_i = E^+ \) holds for all \( i \), then the set of all Edgeworth equilibria is a non-empty weakly closed subset of \( E^{n+k} \).

**Proof.** Let \( \delta_n \) denote the \( n \)-fold replica of our original economy. For each \( n \), let

\[ C_n = \mathcal{F} \cap \text{Core}(\delta_n) \]

It should be clear that the set of all Edgeworth equilibria is precisely the set \( \bigcap_{n \geq 1} C_n \). The proof will be based upon the following properties of the sets \( C_n \).

(1) Each \( C_n \) is non-empty.
Note first that (by Proposition 4.5) the economy $\delta_n$ is a compact economy. By Proposition 4.4, we know that $\text{Core}(\delta_n) \neq \emptyset$. Let
\[
(x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, \ldots, x_{mn},
\quad y_{11}, \ldots, y_{1n}, y_{21}, \ldots, y_{2n}, \ldots, y_{kn})
\]
be a core allocation for $\delta_n$. Then we claim that
\[
x_{ir} \succcurlyeq x_{is} \quad \text{for} \quad r, s = 1, \ldots, n \quad \text{and} \quad i = 1, \ldots, m,
\]
\(\text{i.e., no consumer prefers his bundle to that of another consumer of the same type.}
\)
To see this, note first that (by rearranging the consumers of each type), we can suppose that $x_{ir} \succeq x_{is}$ holds for all $i$ and $r$. Put
\[
z_i = \frac{1}{n} \sum_{r=1}^{n} x_{ir} \geq 0, \quad i = 1, \ldots, m
\]
and
\[
y_j = \frac{1}{n} \sum_{i=1}^{n} y_{ij} \in Y_j, \quad j = 1, \ldots, k.
\]
Then we have
\[
\sum_{i=1}^{m} z_i = \frac{1}{n} \sum_{i=1}^{n} \sum_{r=1}^{n} x_{ir} = \frac{1}{n} \left( \sum_{r=1}^{m} \sum_{r=1}^{n} \omega_{rr} + \sum_{j=1}^{k} \sum_{i=1}^{n} y_{ij} \right) = \omega + \sum_{j=1}^{k} y_j,
\]
and so $(z_1, \ldots, z_m, y_1, \ldots, y_k) \in \mathcal{A}$. Also, by the quasi-concavity of the utility functions, we have $z_i \succeq x_{ii}$ for each $i = 1, \ldots, m$. Now assume by way of contradiction that there exists some $(i, r)$ such that $x_{ir} \succ x_{is}$. The latter, in view of the convexity of $\succeq$, implies $z_i \succ x_{ii}$. Now if each consumer $(i, 1)$ is assigned the bundle $z_i$ and each producer $(j, t)$ chooses the production plan $y_j$ (i.e., $y_{jt} = y_j$), then it is easy to see that $\{z_i: i = 1, \ldots, m\}$ is a feasible assignment for the coalition $\{(i, 1): i = 1, \ldots, m\}$ that blocks the original core allocation, which is impossible. This contradiction establishes the validity of our claim.

Next, note that by the quasi-concavity of the utility functions we have $z_r \succeq x_r$ for $r = 1, \ldots, n$ and $i = 1, \ldots, m$. An easy argument now shows that $(z_1, \ldots, z_m, y_1, \ldots, y_k) \in C_n$, and thus $C_n$ is non-empty.

(2) For each $n$ we have $C_{n+1} \subseteq C_n$.

This follows easily from the fact that if a coalition $S$ of consumers of $\delta_n$ blocks an allocation of $\mathcal{A}$, then $S$ also blocks the same allocation in $\delta_{n+1}$.

(3) Each $C_n$ is weakly closed.
The proof of this claim is straightforward

(4) The set of all Edgeworth equilibria is weakly closed.

This follows from (3) by observing that the set of all Edgeworth equilibria is precisely the set \( \bigcap_{n=1}^{\infty} C_n \).

(5) The economy has an Edgeworth equilibrium.

For each \( n \) let

\[
\hat{C}_n = \left\{ (x_1, \ldots, x_m, y) \in E^{m+1}; y = \sum_{j=1}^{\hat{c}} y_j \text{ with } (x_1, \ldots, x_m, y_1, \ldots, y_k) \in C_n \right\}.
\]

Since each \( C_n \) is non-empty, we see that each \( \hat{C}_n \) is likewise non-empty.

From \( C_{n+1} \subseteq C_n \), it follows that \( \hat{C}_{n+1} \subseteq \hat{C}_n \). In addition, we claim that each \( \hat{C}_n \) is a weakly compact subset of \( E^{m-1} \). To see the latter, note first that

\[
\hat{C}_n \subseteq \hat{X}_1 \times \cdots \times \hat{X}_m \times [(Y + \omega) \cap E^+ - \omega]
\]

and Proposition 4.1(3) we see that each \( \hat{C}_n \) is a relatively weakly compact subset of \( E^{m-1} \). Now let \( \{(x_1^n, \ldots, x_m^n, y^n)\} \) be a net of some \( \hat{C}_n \) satisfying \( (x_1^n, \ldots, x_m^n, y^n) \to (x_1, \ldots, x_m, y) \). Pick \( y_j^n \in Y_j \) (\( j = 1, \ldots, k \)) with \( y^n = \sum_{j=1}^{\hat{c}} y_j^n \) and \( (x_1^n, \ldots, x_m^n, y_1^n, \ldots, y_k^n) \in C_n \). An easy argument shows that there exist \( y_j \in Y_j \) (\( j = 1, \ldots, k \)) such that \( y = \sum_{j=1}^{\hat{c}} y_j \) and \( (x_1, \ldots, x_m, y_1, \ldots, y_k) \in \alpha \). If \( (x_1, \ldots, x_m, y_1, \ldots, y_k) \notin C_n \), then some coalition \( S \) of the \( n \)-fold replica economy \( \delta_n \) blocks \( (x_1, \ldots, x_m, y_1, \ldots, y_k) \) in \( \delta_n \). Since

\[
(x_1^n, \ldots, x_m^n, y^n) \to (x_1, \ldots, x_m, y)
\]

and each set \( \{z \in E^+ : z \succ x_i\} \) is weakly open relative to \( E^+ \), it is easy to see that \( S \) blocks \( (x_1^n, \ldots, x_m^n, y_1^n, \ldots, y_k^n) \) in \( \delta_n \) for some \( z \), which is a contradiction. Hence, \( (x_1, \ldots, x_m, y) \in \hat{C}_n \). This implies that \( \hat{C}_n \) is weakly closed, and hence weakly compact.

Now from the finite intersection property we have \( \bigcap_{n=1}^{\infty} \hat{C}_n \neq \emptyset \). Fix some \( (x_1, \ldots, x_m, y) \in \bigcap_{n=1}^{\infty} \hat{C}_n \), and then pick \( y_j \in Y_j \) (\( j = 1, \ldots, k \)) with \( y = \sum_{j=1}^{\hat{c}} y_j \). We claim that \( (x_1, \ldots, x_m, y_1, \ldots, y_k) \) is an Edgeworth equilibrium.

To see this, assume by way of contradiction that \( (x_1, \ldots, x_m, y_1, \ldots, y_k) \) can be blocked by a coalition \( S \) in the \( r \)-fold replica of the economy. Since \( (x_1, \ldots, x_m, y_1, \ldots, y_k) \in \hat{C}_n \), there exist \( z_j \in Y_j \) (\( j = 1, \ldots, k \)) such that \( (x_1, \ldots, x_m, z_1, \ldots, z_k) \in C_n \), and an easy argument shows that \( (x_1, \ldots, x_m, z_1, \ldots, z_k) \) can be blocked by the coalition \( S \) in the \( r \)-fold replica of the economy, which is impossible. The proof of the theorem is now complete.

It should be noted that when \( Y_j = \{0\} \) for each \( j \), then our production economy reduces to the pure exchange case.
5. Quasiequilibria and Edgeworth Equilibria

In this section we shall study the relationships between Edgeworth equilibria and quasiequilibria. The major result, which is Theorem 5.9, asserts that in a uniformly proper economy every Edgeworth equilibrium is a quasiequilibrium.

**Definition 5.1.** An allocation \((x_1, \ldots, x_m, y_1, \ldots, y_k)\) is said to be a **Walrasian** (or a competitive) **equilibrium** whenever there exists a price \(p \neq 0\) such that:

(a) For each consumer \(i\) the bundle \(x_i\) is a maximal element in the budget set

\[
\mathcal{B}(p) = \left\{ x \in X_i : p \cdot x \leq p \cdot \omega_i + \sum_{j=1}^{k} \theta_j \cdot p \cdot y_j \right\},
\]

i.e., \(x_i \in \mathcal{B}(p)\) and \(x_i \geq x\) holds for all \(x \in \mathcal{B}(p)\), and

(b) For each \(j\) the production plan \(y_j\) maximizes profit at prices \(p\) over \(Y_j\), i.e.,

\[
p \cdot y_j = \max \{ p \cdot z : z \in Y_j \}, \quad j = 1, \ldots, k.
\]

It should be clear that an allocation \((x_1, \ldots, x_m, y_1, \ldots, y_k)\) is a Walrasian equilibrium if and only if there exists a price \(p \neq 0\) such that

1. \(p \cdot x_i \leq p \cdot \omega_i + \sum_{j=1}^{k} \theta_j p \cdot y_j,\)
2. \(x_i \succ x\), \(x_i \in X_i\), implies \(x \succ p \cdot \omega_i + \sum_{j=1}^{k} \theta_j p \cdot y_j\); and
3. \(p \cdot y_j = \max \{ p \cdot z : z \in Y_j \} \) for \(j = 1, \ldots, k\).

**Definition 5.2.** An allocation \((x_1, \ldots, x_m, y_1, \ldots, y_k)\) is said to be a **quasiequilibrium** whenever there exists a price \(p \neq 0\) such that:

(a) \(p \cdot x_i \leq p \cdot \omega_i + \sum_{j=1}^{k} \theta_j p \cdot y_j,\)

(b) \(x_i \succ x\), \(x_i \in X_i\), implies \(x \geq p \cdot \omega_i + \sum_{j=1}^{k} \theta_j p \cdot y_j\); and

(c) \(p \cdot y_j = \max \{ p \cdot z : z \in Y_j \} \) for \(j = 1, \ldots, k\).

Any price \(p\) that satisfies the properties of Definition 5.1 or 5.2 is known as a price supporting the allocation.

Clearly, a Walrasian equilibrium is a quasiequilibrium. Also, the next result tells us that a competitive equilibrium is always an Edgeworth equilibrium. Its proof is similar to that of the finite-dimensional case and is omitted.
PROPOSITION 5.3. Every Walrasian equilibrium is an Edgeworth equilibrium.

In the pure exchange case, Mas-Colell [16] proved that quasiequilibria exist, and the authors generalized this result in [2] by proving that every Edgeworth equilibrium is a quasiequilibrium. In the infinite dimensional setting, Zame [30] was the first to establish the existence of quasiequilibria in economies with production.

Our next objective is to show that under certain conditions an Edgeworth equilibrium is a quasiequilibrium. To do this, we need some preliminary discussion.

We start by introducing some useful convex sets. For each consumer \( i \) we define his "share set" by

\[
Z_i = \left\{ \sum_{j=1}^{k} \theta_j z_j \in Y_i \right\} = \sum_{j=1}^{k} \theta_j Y_i.
\]

Now consider \( m \) consumption bundles \( x_i \in X_i \) (\( i = 1, \ldots, m \)). For each \( i \), we shall denote by \( F_i^* \) the "strictly better" set of \( x_i \), i.e., \( F_i^* \) is the convex set defined by

\[
F_i^* = \{ x \in X_i \; | \; x \succ x_i \}.
\]

With the above convex sets, we shall also associate the important convex set

\[
H^* = \text{co}\left[ \bigcup_{i=1}^{m} (F_i^* - Z_i - \omega_i) \right]
\]

\[
= \left\{ \sum_{i=1}^{m} \lambda_i (x_i - z_i - \omega_i) \; | \; \lambda_i \geq 0, i, \succ x_i, z_i \in Z_i, \text{ and } \sum_{i=1}^{m} \lambda_i = 1 \right\}.
\]

In order to ensure that the sets \( F_i^* \) are non-empty, we shall assume for the rest of this section that each preference relation satisfies the following non-satiation property:

\( x \in X_i \), then there exists some \( z \in X_i \) with \( z \succ x \).

An important property of the convex set \( H^* \) is described in the next Proposition.

PROPOSITION 5.4 Assume that \( X_i = E^+ \) holds for all \( i \). If \( (x_1, \ldots, x_m, y_1, \ldots, y_k) \) is an Edgeworth equilibrium, then for each \( h \geq 0 \) we have \( 0 \notin h + H^* \).
Proof. Let \( h \geq 0 \), and assume by way of contradiction that \( 0 \in h + H^* \).

Thus, there exist \( v_i \in F_i^*, z_i \in Z_i \), and \( \lambda_i \geq 0 \), with \( \sum_{i=1}^m \lambda_i = 1 \) such that

\[
\sum_{i=1}^m \lambda_i (v_i - z_i - \omega_i) = 0,
\]

and so

\[
\sum_{i \in S} \sum_{i=1}^m \lambda_i (v_i - z_i - \omega_i) = 0 \tag{\ast}
\]

Next, let \( S = \{ i : \lambda_i > 0 \} \), and note that from (\ast) it follows that

\[
\sum_{i \in S} \lambda_i v_i \leq \sum_{i \in S} \lambda_i z_i + \sum_{i \in S} \lambda_i \omega_i, \tag{\ast\ast}
\]

Now if \( n \) is a positive integer and \( i \in S \), let \( n_i \) be the smallest integer greater or equal than \( n\lambda_i \) (i.e., \( 0 \leq n_i - n\lambda_i \leq 1 \)). Since \( v_i \succeq x_i \) and \( \lim_{n \to \infty} n\lambda_i/n_i = 1 \) for each \( i \in S \), we can choose (by the continuity of the utility functions) \( n \) large enough so that

\[
f_i = (n\lambda_i/n_i) v_i \succeq x_i \quad \text{for all} \quad i \in S. \tag{\ast\ast\ast}
\]

(Here we use the fact that \( X_i = \mathbb{R}^+ \) so that \( f_i \in X_i \).) Taking into account (\ast\ast), we infer that

\[
\sum_{i \in S} n_i f_i = \sum_{i \in S} n_i \lambda_i v_i \leq \sum_{i \in S} n_i \lambda_i z_i + \sum_{i \in S} n_i \lambda_i \omega_i,
\]

\[
\leq \sum_{i \in S} n_i (n\lambda_i/n_i) z_i + \sum_{i \in S} n_i \omega_i.
\]

Since \( 0 \leq n\lambda_i/n_i \leq 1 \), we see that \( h_i = (n\lambda_i/n_i) z_i \in Z_i \), and so from the preceding inequality, we conclude that

\[
\sum_{i \in S} n_i f_i \leq \sum_{i \in S} n_i h_i + \sum_{i \in S} n_i \omega_i,
\]

By rearranging the consumers, we can also assume that \( S = \{ 1, \ldots, l \} \), where \( 1 \leq l \leq m \). For each \( i \in S \) pick \( h_j \in Y_j \) (\( j = 1, \ldots, k \)) such that

\[
h_i = \sum_{j=1}^k \theta_{ij} h_j
\]

Let \( n = n_1 + \cdots + n_l \), and let \( \delta_n \) denote the \( n \)-fold replica of our economy. For each \( i \in S \), let \( T_i \) be the set of consumers of \( \delta_n \) defined by

\[
T_i = \{ (i, s) : n_0 + n_1 + \cdots + n_{i-1} + 1 \leq s \leq n_1 + \cdots + n_l \},
\]

where \( n_0 = 0 \). Clearly, \( T_i \cap T_j = \emptyset \) for \( i \neq j \). Now consider the coalition \( T \) of
\( \delta \) given by \( T = \bigcup_{i \in S} T_i \). Next, for each consumer \((i, s) \in T\), we assign the bundle

\[ \xi_{is} = f_i, \]

and to each producer \((j, t) \ (j = 1, \ldots, k; \quad n_0 + n_1 + \cdots + n_{i-1} + 1 \leq t \leq n_1 + \cdots + n_i)\) we assign the production plan

\[ \xi_{jt} = h_{yt}. \]

Now note that

\[ \xi_{ys} \succ_{t, i, x_{is}} x_{tj} \quad \text{for all} \quad (i, s) \in T, \]

and moreover,

\[
\sum_{(i, s) \in T} \xi_{is} = \sum_{i \in S} n_i f_i \\
\leq \sum_{i \in S} n_i \omega_i + \sum_{i \in S} n_i h_i \\
= \sum_{(i, s) \in T} \omega_{is} + \sum_{i \in S} n_i \sum_{j = 1}^k \theta_{ij} h_{ij} \\
= \sum_{(i, s) \in T} \omega_{ys} + \sum_{j = 1}^k \sum_{i \in S} \left( \sum_{i \in T} \theta_{ij} \right) \xi_{jt}.
\]

The above show that the coalition \( T \) blocks \((x_1, \ldots, x_m, y_1, \ldots, y_k)\) in the \( n \)-fold replica of the economy, which is impossible. Hence, \( 0 \notin h + H^* \) must hold as desired.

To continue our discussion we need the concept of uniform properness for preferences and production sets as it was introduced by Mas–Colell in [16] and [17]. The uniform properness for preferences is defined as follows.

**Definition 5.5 (Mas–Colell).** A preference relation \( \succ \) on a convex set \( X \) is said to be uniformly proper whenever there exist a vector \( a > 0 \) and some \( \tau \)-neighborhood \( V \) of zero such that \( x - \lambda a + z \not\succ x \) in \( X \) with \( x > 0 \) imply \( z \not\in xV \).

Recall that if a preference relation \( \succ \) is defined on \( E^+ \), then a commodity bundle \( x > 0 \) is said to be strongly desirable whenever \( x + \lambda x > x \) holds for all \( x \in E^+ \) and all \( \lambda > 0 \). In case \( \succ \) is a uniformly proper preference relation defined on \( E^+ \), then the vector \( a > 0 \) in the definition of properness is a strongly desirable commodity bundle. Indeed, in this case, if \( x \in E^+ \) and \( x > 0 \) satisfy \( x \succ x + \lambda a \), then from \( (x + \lambda a) - \lambda a + x = \).
$x \geq x + za$ and the uniform properness, it follows that $0 \notin zV$, which is impossible. Hence, $x + za > x$ holds for all $x \in E^+$ and all $z > 0$.

In [17] Mas-Colell also introduced the concept of properness for production sets. Following A. Mas-Colell's ideas in [17], we shall say that a set $T$ is a pre-technology set for a production set $Z$ whenever

1. $Z \subseteq T$;
2. $x \in T$ implies $x^+ = x \lor 0 \in T$; and
3. $T - E^+ = T$, i.e., $T$ satisfies the free disposal condition.

Now the corresponding notion of uniform properness for production sets is as follows.

**Definition 5.6.** (Mas-Colell). A production set $Z$ is said to be uniformly proper whenever there exist a pre-technology set $T$ for $Z$ a vector $b > 0$ and a $r$-neighborhood $V$ of zero such that $y \in T \setminus Z$ and $y + zb + z \in Z$ with $z > 0$ imply $z \notin zV$.

Recently, Richard [23] formulated a notion of uniform properness for production sets without using pre-technology sets. Concerning uniformly proper production sets we have the following useful property.

**Lemma 5.7.** Suppose that a production set $Z$ is uniformly proper, and let $T$, $b > 0$ and $V$ be as in Definition 5.6. If $z \in Z$, $y \in E^+$ and $z > 0$ satisfy $y \in zV$ and $y \leq z^+ + zb$, then $z - zb + y \in Z$.

**Proof.** Assume that $z \in Z$, $y \in E^+$ and $z > 0$ satisfy $y \in zV$ and $y \leq z^+ + zb$. We can also assume that $V$ is a symmetric neighborhood. Put $x = z - zb + y$, and note that

$$x = z^+ - (z^+ + zb - y) \leq z^+. \quad (*)$$

Since $z \in Z \subseteq T$, it follows that $z^+ \in T$ and so from $(*)$ we see that $x = z - zb + y \in T$.

Now assume by way of contradiction that $x \notin Z$. Then we have $z = x + zb - y \in Z$. From $x \in T \setminus Z$ and the properness condition, we infer that $-y \notin zV$, i.e., $y \notin zV$, which is impossible. The proof of the lemma is now complete.

We now come to the concept of a proper economy.

**Definition 5.8.** An economy is called a uniformly proper economy whenever all preferences and all production sets are uniformly proper. That is, an economy is uniformly proper whenever there exist $a_i > 0$ ($i = 1, \ldots, m$).
$b_j > 0$ ($j = 1, \ldots, k$), pre-technology sets $T_j$ for $Y_j$ ($j = 1, \ldots, k$) and an open convex solid $\varepsilon$-neighborhood $V$ of zero such that

1. $x - xa + z \geq x$ in $X$, and $x > 0$ imply $z \notin xV$; and
2. $y + xb + z \in Y_j$ with $y \in T_j \setminus Y_j$, and $z > 0$ imply $z \notin xV$.

Mas-Colell [17] used uniform properness on preferences and production sets to show that any Pareto optimal allocation can be supported by a non-zero price. Next, we use uniform properness to show that every Edgeworth equilibrium is a quasiequilibrium.

**Theorem 5.9.** If $X_i = E^+$ holds for each $i$ and the economy is uniformly proper, then every Edgeworth equilibrium is a quasiequilibrium.

**Proof.** This proof is an adaptation of Mas–Colell’s proof of Theorem 1 in [17].

Let $a_i$ ($i = 1, \ldots, m$), $b_j$ ($j = 1, \ldots, k$) and $V$ be as in Definition 5.8 of a proper economy. Put $a = a_1 + \cdots + a_m + b_1 + \cdots + b_k$, and let $\Gamma$ be the (non-empty) open convex cone generated by $-a + (1/m)V$, i.e.,

$$
\Gamma = \left\{ x \left( -a \pm \frac{1}{m} v \right) \mid x > 0 \text{ and } v \in V \right\}.
$$

Now let $(x_1, \ldots, x_m, y_1, \ldots, y_k)$ be an Edgeworth equilibrium. If $H^*$ is the convex set associated with $(x_1, \ldots, x_m)$ as in Proposition 5.4, then we claim that $H^* \cap \Gamma \neq \emptyset$.

To see this, assume by way of contradiction that $H^* \cap \Gamma \neq \emptyset$ Then there exist $f_j \geq 0$ with $f_i > x_i$, $\hat{\lambda}_i \geq 0$ with $\hat{\lambda}_1 + \cdots + \hat{\lambda}_m = 1$, $y_i \in Y_j$ ($i = 1, \ldots, m; j = 1, \ldots, k$) and some $\varepsilon > 0$ such that

$$
\sum_{i=1}^{m} \hat{\lambda}_i \left( f_i - \sum_{j=1}^{k} \theta_{ij} y_j - \omega_i \right) + \varepsilon a \in \frac{\varepsilon}{m} V. \quad (1)
$$

Note that the set $S = \{ i, \hat{\lambda}_i > 0 \}$ is non-empty.

Now consider the positive elements

$$
\nu = \sum_{i=1}^{m} \hat{\lambda}_i \left[ \omega_i + \sum_{j=1}^{k} \theta_{ij}(y_j)^+ \right]
$$

and

$$
\begin{align*}
\nu &= \sum_{i=1}^{m} \hat{\lambda}_i \left[ f_i + \sum_{j=1}^{k} \theta_{ij}(y_j)^- \right] + \varepsilon a \\
&= \sum_{i=1}^{m} (\hat{\lambda}_i f_i + \varepsilon a) + \sum_{i=1}^{m} \sum_{j=1}^{k} \left[ \hat{\lambda}_i \theta_{ij}(y_j)^- + \frac{\varepsilon}{m} b_j \right].
\end{align*}
$$
From (1), we see that
\[ z - y = \sum_{i=1}^{m} \lambda_i \left( f_i - \sum_{j=1}^{k} \theta_{ij} y_{ij} - \omega_i \right) + \varepsilon a \in \frac{e}{m} V \]  
(2)

Moreover, we have
\[ 0 \leq (z - y)^+ \leq z = \sum_{i=1}^{m} (\lambda_i f_i + \varepsilon a_i) + \sum_{i=1}^{m} \sum_{j=1}^{k} \left[ \lambda_i \theta_{ij} (y_{ij})^- + \frac{e}{m} b_i \right] . \]  
(3)

From \( z - y \in (e/m) V \) and the solidness of \( V \) we see that
\[ (z - y)^+ \in \frac{e}{m} V. \]  
(4)

Applying the Riesz Decomposition Property to (3), we can write
\[ (z - y)^+ = s + t, \]  
(5)

where,
\[ 0 \leq s \leq \sum_{i=1}^{m} (\lambda_i f_i + \varepsilon a_i) \]  
(6)

and
\[ 0 \leq t \leq \sum_{i=1}^{m} \sum_{j=1}^{k} \left[ \lambda_i \theta_{ij} (y_{ij})^- + \frac{e}{m} b_i \right] . \]  
(7)

Now applying the Riesz Decomposition Property to (6), we can write \( s = \sum_{i=1}^{m} s_i \) with \( 0 \leq s_i \leq \lambda_i f_i + \varepsilon a_i \) for each \( i \). From \( 0 \leq s_i \leq (z - y)^+ \in (e/m) V \) and the solidness of \( V \), we see that
\[ s_i \in \frac{e}{m} V, \quad i = 1, \ldots, m \]  
(8)

Let
\[ g_i = \begin{cases} f_i & \text{if } i \notin S \\ f_i + \frac{e}{\lambda_i} a_i - \frac{1}{\lambda_i} s_i & \text{if } i \in S \end{cases} \]

Clearly, \( g_i \geq f_i \) holds for all \( i \notin S \), and we claim that \( g_i \geq f_i \) for each \( i \in S \). Indeed, if in the latter case we have
\[ f_i = g_i - \frac{e}{\lambda_i} a_i + \frac{1}{\lambda_i} s_i \geq g_i. \]
then by the properness we must have \((1/\lambda_i) s_i \notin (\epsilon/\lambda_i) V_i\), i.e., \(s_i \notin \epsilon V\), which contradicts (8).

Next, using (7) and the Riesz Decomposition Property we can write
\[ t = \sum_{i=1}^{\kappa} \sum_{j=1}^{k} t_{ij} \]
with \(0 \leq t_{ij}, \lambda_j, \theta_{ij}, (y_{ij})^+ + (\epsilon/m) b_j \). Let \( T = \{(i, j) : \lambda_j, \theta_{ij} > 0\} \), and define
\[ z_{ij} = \begin{cases} y_{ij} - \epsilon(m \lambda_j, \theta_{ij})^{-1} b_j + (\lambda_j, \theta_{ij})^{-1} t_{ij} & \text{if } (i, j) \in T \\ 0 & \text{if } (i, j) \notin T. \end{cases} \]

Fix \((t, j) \in T\). From \(0 \leq t_{ij} \leq t \leq (z - y)^+ \in \epsilon(m) V\) and the solidness of \(V\), we infer that \(t_{ij} \in (\epsilon/m) V\), and so \((\lambda_j, \theta_{ij})^{-1} t_{ij} \in (m \lambda_j, \theta_{ij})^{-1} V\). Now from the inequality
\[ 0 \leq (\lambda_j, \theta_{ij})^{-1} t_{ij} - (y_{ij})^- - \epsilon(m \lambda_j, \theta_{ij})^{-1} b_j, \]
and Lemma 5.7, we conclude that \(z_{ij} \in Y_j\). Hence, \(z_{ij} \in Y_j\) holds for all \((i, j)\).

Now for \(\lambda_i = 0\), we have \(s \leq \epsilon a_i\), and so \(\epsilon a_i - s \geq 0\) for all \(i \in S\). Similarly, for \(\lambda_j \theta_{ij} = 0\), we have \(t_{ij} \leq (\epsilon/m) b_j\), and so \((\epsilon/m) b_j - t_{ij} \geq 0\) for all \((i, j) \notin T\).

Taking into account these observations, we see that
\[
\sum_{i=1}^{m} \lambda_i \left( g_i - \sum_{j=1}^{k} \theta_{ij} y_{ij} - \omega_i \right)
= \sum_{i=1}^{m} \lambda_i \left( f_i - \sum_{j=1}^{k} \theta_{ij} y_{ij} - \omega_i \right) + \epsilon \sum_{i \in S} a_i - \sum_{i \in S} s_i
+ \frac{\epsilon}{m} \sum_{(i, j) \in T} b_j - \sum_{(i, j) \in T} t_{ij}
\leq \sum_{i=1}^{m} \lambda_i \left( f_i - \sum_{j=1}^{k} \theta_{ij} y_{ij} - \omega_i \right) + \epsilon \sum_{i \in S} a_i - \sum_{i \in S} s_i
+ \sum_{i \notin S} (\epsilon a_i - s_i) + \frac{\epsilon}{m} \sum_{(i, j) \in T} b_j + \sum_{(i, j) \notin T} t_{ij} + \sum_{(i, j) \notin T} \left( \frac{\epsilon}{m} b_j - t_{ij} \right)
= \sum_{i=1}^{m} \lambda_i \left( f_i - \sum_{j=1}^{k} \theta_{ij} y_{ij} - \omega_i \right) + \epsilon \sum_{i=1}^{m} a_i - \sum_{i=1}^{m} s_i
+ \frac{\epsilon}{m} \sum_{i=1}^{m} \sum_{j=1}^{k} b_j - \sum_{i=1}^{m} \sum_{j=1}^{k} t_{ij}
= \sum_{i=1}^{m} \lambda_i \left( f_i - \sum_{j=1}^{k} \theta_{ij} y_{ij} - \omega_i \right) + \epsilon a - (s + t)
= z + (s + t) = z - (z - y)^+ = - (z - y)^- \leq 0
\]
Clearly, the element
\[ g = \sum_{i=1}^{m} \lambda_i \left( g_i - \sum_{j=1}^{k} \theta_{ij} z_{ij} - \omega_i \right) \in H^* \]
satisfies \( g \leq 0 \). Now let \( h = -g \geq 0 \). Then from \( h + g = 0 \), we see that \( 0 \in h + H^* \), which contradicts Proposition 5.4. Thus \( H^* \cap \Gamma = \emptyset \) holds, as claimed.

Finally, by the classical separation theorem there exist a non-zero price \( p \) and some constant \( c \) such that
\[ p \cdot h \geq c \geq p \cdot g \]
holds for all \( h \in H^* \) and all \( g \in \Gamma \). Since \( \Gamma \) is a cone, we see that \( c \geq 0 \). Now if \( y \succ x \) holds in \( E^+ \), then \( x - \sum_{j=1}^{k} \theta_{ij} y_{ij} - \omega_i \in H^* \), and so \( p \cdot y \geq p \cdot (\omega_i + \sum_{j=1}^{k} \theta_{ij} p \cdot y_{ij} \). On the other hand, we know that each \( a_i > 0 \) is a strongly desirable commodity for \( y \succ x \). If \( z \in Y_r \), then put \( v = a_i + \cdots + a_m, y_{ij} = 1 \), for \( j \neq i \) and \( z_r = z \), and note that
\[ \frac{y_r - z + \alpha}{m} v = \sum_{r=1}^{n} \frac{1}{m} \left( (x_r + \alpha v) - \sum_{j=1}^{k} \theta_{ij} z_{ij} - \omega_i \right) \in H^* \]
holds for all \( x > 0 \). Hence, \( p \cdot y_r - p \cdot z + (\alpha/m) p \cdot v \geq 0 \) holds for all \( x > 0 \), and so \( p \cdot y_r \geq p \cdot z \) for all \( z \in Y_r \). Therefore, \( p \cdot y_r = \max \{ p \cdot z : z \in Y_r \} \) holds for all \( r = 1, \ldots, k \), and this completes the proof of the theorem.

The works of Mas-Colell [17] and Zame [30] on economies with production involve different assumptions on the production technology and use different methods. A natural way to measure the efficiency of the production process is to compare the size of the input with the size of the output. With this idea, Zame [30] introduced the condition that the marginal efficiency of production is weakly bounded. A natural interpretation of this requirement is that a small loss of input only gives rise to a small loss of output. This condition is different from requiring that the production sets are uniformly proper, but it achieves a similar aim. Although it may be possible to combine the two approaches, it is not clear that our results on uniformly proper economies hold in W Zame’s model [30].

Recall that a positive element \( x > 0 \) is said to be strictly positive (in symbols, \( x \succ 0 \)) whenever \( p \cdot x > 0 \) holds for all \( 0 < p \in E^+ \). If \( \omega \succ 0 \) and preferences are strongly monotone, then it is easy to see that the concepts of quasiequilibrium and Walrasian equilibrium coincide. Therefore, the next theorem that generalizes the classical theorem of Debreu and Scarf [9] is an immediate consequence of the preceding result.
Corollary 5.10. Assume that each consumption set satisfies \( X_i = E^- \) and that preferences are in addition strongly monotone. If the economy is uniformly proper and \( \alpha \geq 0 \), then an allocation is an Edgeworth equilibrium if and only if it is a Walrasian equilibrium.

In particular, in this case, if the economy is also compact, then Walrasian equilibria exist.

6. Decentralizing Equilibria

In the preceding section we saw that in a uniformly proper economy every Edgeworth equilibrium is a quasiequilibrium, and hence it can be decentralized by a price. In the absence of uniform properness, the best we can expect is that an Edgeworth equilibrium can be decentralized "approximately" by a price. In Theorem 6.3 we establish that an Edgeworth equilibrium can be approximately price supported in the sense that expenditures are approximately minimized and profits are approximately maximized. On the other hand, economies having a cone as the aggregate production set are studied. For these economies, Theorem 6.10 asserts that approximately equilibria exist.

Recall that the “share set” of each consumer \( i \) is defined by

\[
Z_i = \left\{ \sum_{j=1}^{k} \theta_{ij} z_j : z_j \in Y_j \right\} = \sum_{j=1}^{k} \theta_{ij} Y_j.
\]

Also, for each fixed \( a \in E^- \) and each consumer \( i \), we define the convex set

\[
Z_i^a = \left\{ \sum_{j=1}^{k} \theta_{ij} z_j : z_j \in Y_j \text{ and } z_j \leq a \right\}.
\]

In case the Riesz dual system for an economy is symmetric, the convex sets \( Z_i^a \) are weakly closed. The details follow.

Lemma 6.1. If the Riesz dual system for the economy is symmetric, then for each \( i \) and each \( a \in E^- \) the convex set \( Z_i^a \) is weakly closed.

Proof. Fix \( i \) and \( a \in E^- \), and let \( f \) be an element in the weak closure of \( Z_i^a \). Then \( f \) is also in the \( \tau \)-closure of \( Z_i^a \). Pick a net \( \{ f_s \} \) of \( Z_i^a \) with \( f_s \to f \).

For each \( z \) choose \( y_z^s \in Y_z \) with \( y_z^s \leq a \) and \( f_s = \sum_{z=1}^{k} \theta_{ij} z^s \). (In case \( \theta_{ij} = 0 \), we shall assume that \( y_z^s = 0 \).)

Since \( 0 \leq (y_z^s)^+ \leq a \) holds for all \( x \) and \( j \) and the order interval \( [0, a] \) is
weakly compact, we can suppose (by passing to an appropriate subnet) that for each \( j \) we have

\[
(y^*_j)^* \xrightarrow{n} y^*_j.
\]

(\(*\))

Also, from

\[
0 \leq \theta_\alpha(y^*_j) - \leq \sum_{r=1}^{k} \theta_\alpha(y^*_r) - = - \sum_{r=1}^{k} \theta_\alpha(y^*_r) + \sum_{r=1}^{k} \theta_\alpha(y^*_r) + \\
\leq -f_s + a \leq (f - f_s)^* + f^* + a \xrightarrow{n} f^* + a
\]

and Theorem 2.1, it follows (by passing to a subnet again) that for each \( j \) we have

\[
(y^*_j)^* \xrightarrow{n} y^*_j.
\]

(\(*\*)

From (\(*\)) and (\(*\*)\), we infer that

\[
1^*_j = (y^*_j)^* - (y^*_j)^* - \xrightarrow{n} y^*_j = y_j,
\]

for each \( j \) Since each \( Y_j \) is weakly closed, we see that \( y_j \in Y_j \) and moreover, from \( y^*_j \leq a \), we infer that \( y_j \leq a \) Finally, note that

\[
f = n - \lim f_s = n - \lim \sum_{r=1}^{k} \theta_\alpha(y^*_r) = \sum_{r=1}^{k} \theta_\alpha(y^*_r) \in Z^*_i.
\]

and the proof is finished  

Consider \( m \) consumption bundles \( x_i \in X_i \) \((i = 1, \ldots, m)\). For each \( i \) we shall denote by \( F_i \) the "better set" of \( x_i \), i.e., \( F_i \) is the weakly closed convex set defined by

\[
F_i = \{ x \in X_i : x \succeq_i x_i \}.
\]

With the above convex sets we shall associate the important convex set

\[
H = \text{co} \left[ \bigcup_{i=1}^{m} (F_i - Z^*_i - \omega_i) \right]
\]

\[
= \left\{ \sum_{i=1}^{m} \lambda_i(x_i - z_i - \omega_i) : \lambda_i \geq 0, x_i \succeq_i x_i, z_i \in Z^*_i \text{ and } \sum_{i=1}^{m} \lambda_i = 1 \right\}
\]

PROPOSITION 6.2 Assume that the Riesz dual system for the economy is symmetric, that preferences are monotone and that each consumption set
satisfies \( X_i + E^+ = X_i \). If \( x_i \in X_i, (i = 1, \ldots, m) \) are consumption bundles, then for each \( a \in E^+ \) the convex set

\[
H_a = \text{co} \left[ \bigcup_{i=1}^m (F_i - Z_i^a - \omega_i) \right]
\]

is a weakly closed subset of \( E \).

**Proof.** Fix \( a \in E^+ \), and let \( f \) be in the weak closure of \( H_a \). Then \( f \) is in the \( \tau \)-closure of \( H_a \), and so there exists a net \( \{ f_n \} \) of \( H_a \) with \( f_n \to \tau f \). For each \( z \) let \( x_i \geq x_i, z_i \in Z_i^a, \lambda_i \geq 0 \) with \( \sum_{i=1}^m \lambda_i^z = 1 \) such that

\[ f_n = \sum_{i=1}^m \lambda_i^z (x_i - z_i - \omega_i) \]

By passing to a subnet, we can assume that \( \lambda_i^z \to \lambda_i \) holds in \( \mathbb{R} \) for each \( i \). Clearly, \( \lambda_1 + \cdots + \lambda_m = 1 \). Let \( S = \{ i : \lambda_i > 0 \} \), and note that \( S \neq \emptyset \). From

\[
0 \leq \sum_{i=1}^m \lambda_i^z_{x_i} - \sum_{i=1}^m \lambda_i^z(z_i^a)^- = \sum_{i=1}^m \lambda_i^z (x_i - z_i - \omega_i) + \sum_{i=1}^m \lambda_i^z \omega_i + \sum_{i=1}^m \lambda_i^z (z_i^a)^+ \leq f_x + \omega + a,
\]

we see that

\[
0 \leq \lambda_i^z x_i \leq f_x + \omega + a \quad \text{and} \quad 0 \leq \lambda_i^z (z_i^a)^- \leq f_x + \omega + a
\]

hold for all \( i \) and all \( x \). Thus, by Theorem 2.1, we can assume (by passing to a subnet again) that for each \( i \in S \) we have

\[
i_i^z \xrightarrow{\tau} x_i \in F_i \quad \text{and} \quad (z_i^a)^- \xrightarrow{\tau} z_i^a
\]

From \( 0 \leq (z_i^a)^+ \leq a \) and the weak compactness of \([0, a]\), we can assume (by passing to a subnet once more) that for each \( i \) we have \((z_i^a)^- \to \tau z_i^a\). Thus, taking into consideration that \( Z_i^a \) is weakly closed (Lemma 6.1), we see that for each \( i \in S \) we have

\[
z_i^a = (z_i^a)^+ - (z_i^a)^- \xrightarrow{\tau} z_i^a - z_i^a = z_i \in Z_i^a
\]

In addition, from \( 0 \leq \lambda_i^z (z_i^a)^+ \leq \lambda_i^z a \), it follows that \( \lambda_i^z (z_i^a)^+ \to \tau 0 \) for all \( i \notin S \).
Now from the weak closedness of $E^+$ and the inequality

$$f_s = \sum_{i=1}^{m} \lambda_i^*[v_i^+ - (z_i^+) - \omega_i]$$

$$\geq \sum_{i \in S} \lambda_i^*[v_i^+ - \sum_{i=1}^{m} \lambda_i^*[z_i^+] + \sum_{i \in S} \lambda_i^*[z_i^-] - \sum_{i=1}^{m} \lambda_i^*[\omega_i],$$

we infer that

$$f = \lim_{s \to 0} f_s \geq \sum_{i \in S} \lambda_i v_i^+ - \sum_{i \in S} \lambda_i z_i^+ + \sum_{i \in S} \lambda_i^* z_i^- - \sum_{i \in S} \lambda_i^* \omega_i,$$

$$= \sum_{i \in S} \lambda_i (v_i - z_i - \omega_i) = g.$$  

For $i \notin S$, let $v_i = x_i$ and $z_i = 0$. Then $v_i + f - g \geq v_i \geq x_i$, and

$$f = (f - g) + g = \sum_{i=1}^{m} \lambda_i [(v_i + f - g) - z_i - \omega_i] \in H_a,$$

and the proof of the theorem is finished.

When preferences are strongly monotone the Edgeworth equilibria are characterized as follows.

**Theorem 6.3.** Assume that the Riesz dual system for the economy is symmetric, that preferences are strongly monotone and that $x_i \in E^+$ holds for each consumer $i$. Then an allocation $(x_1, \ldots, x_m, y_1, \ldots, y_k)$ is an Edgeworth equilibrium if and only if for each $f > 0$, each $\epsilon > 0$ and each $a \in E^+$ there exists a price $p \in E$ such that:

1. $p \cdot f = 1$;
2. $x \succeq x, x \in E^+$ implies $p \cdot x \geq p \cdot w_i + \sum_{i=1}^{k} \theta_i p \cdot y_i - \epsilon$; and
3. $p \cdot y_j \geq \sup \{ p \cdot z : z \in Y_j \text{ and } z \leq a \} - \epsilon$ for each $j$.

**Proof.** Assume that $(x_1, \ldots, x_m, y_1, \ldots, y_k)$ is an Edgeworth equilibrium. Fix $f > 0, \epsilon > 0$ and $a \in E^+$. We can suppose that $a \succeq y_1 \vee \cdots \vee y_k$. From

$$\frac{\epsilon}{m} f + H_a = \frac{\epsilon}{2m} f + \left( \frac{\epsilon}{2m} f + H_a \right) \leq \frac{\epsilon}{2m} f + H^*,$$

and Proposition 5.4, we see that $0 \notin (\epsilon/m)f + H_a$. Since $H_a$ is weakly closed.
(Proposition 6.2), it follows from the classical separation theorem that there exists some \( p \in E^* \) such that

\[
p \cdot \left( \frac{e}{m} f + g \right) > 0
\]

holds for all \( g \in H_a \). Since \( \theta_{r_1} = \sum_{j=1}^{k} \theta_{j} y_{j} \in Z \), it follows that \( 0 = \sum_{r=1}^{m} \frac{1}{m}(x, -h_{r} - \omega_{r}) \in H_a \), and from (\(*)\) we see that \( p \cdot f > 0 \). Thus, replacing \( p \) by \( p/p \cdot f \), we can assume that \( p \cdot f = 1 \).

Now let \( x \gtrsim x_i \) holds in \( E^* \). Then \( x - \sum_{j=1}^{k} \theta_{j} y_{j} - \omega_{j} \in H_a \), and hence \( p \cdot ((e/m) f + x - \sum_{j=1}^{k} \theta_{j} y_{j} - \omega_{j}) \) holds. This implies

\[
p \cdot x \geq p \cdot \omega_{j} + \sum_{j=1}^{k} \theta_{j} p \cdot y_{j} - \frac{e}{m} p \cdot \omega_{j} + \sum_{j=1}^{k} \theta_{j} p \cdot y_{j} - e.
\]

Next, let \( z \in Y_j \) satisfy \( z \leq a \). Put \( h_{i} = y_{i} \) for \( i \neq j \) and \( h_{j} = z \). From

\[
\frac{1}{m} (y_{j} - z) = \frac{1}{m} \left( \sum_{j=1}^{k} x_{j} - \sum_{i=1}^{m} \omega_{i} - \sum_{i=1}^{k} h_{i} \right) = \frac{1}{m} \left( x - \sum_{i=1}^{k} \theta_{i} h_{i} - \omega_{i} \right) \in H_{a}
\]

and (\(*)\), we see that \( p \cdot ((e/m) f + (1/m)(y_{j} - z)) > 0 \). Therefore, \( p \cdot y_{j} \geq p \cdot z - e \) holds for all \( z \in Y_j \) with \( z \leq a \), from which it follows that

\[
p \cdot y_{j} \geq \sup \{ p \cdot z : \ z \in Y_j \ \text{and} \ z \leq a \} - e.
\]

For the converse, assume that the allocation \( (x_{1}, \ldots, x_{m}, y_{1}, \ldots, y_{k}) \) satisfies (1), (2), and (3). Also, assume by way of contradiction that there exist an \( n \)-fold replica of the economy, a coalition \( S \) of consumers of the \( n \)-fold replica, a subset \( \{ h_{i, s} : (i, s) \in S \} \) of \( E^* \) and production plans \( z_{j} \in Y_{j} = Y_{j} (j = 1, \ldots, k) \) such that

\[
h_{i, s} \gtrsim_{(i, s)} x, \quad \text{for all} \ (i, s) \in S
\]

and

\[
\sum_{(i, s) \in S} h_{i, s} = \sum_{(i, s) \in S} \omega_{i} + \sum_{j=1}^{k} \sum_{(i, s) \in S} \theta_{i, s} z_{j}.
\]

Now let \( f = \sum_{(i, s) \in S} h_{i, s} \) and let \( a = \sum_{i=1}^{m} \mid x_{i} \mid + \sum_{i=1}^{k} \sum_{j=1}^{r} \mid z_{j} \mid \). Then for each \( l \) there exists some \( p_{l} \in E^* \) such that

\[
p_{l} \cdot f = 1,
\]

\[
v \gtrsim x_{i} \ \text{in} \ E^* \ \text{implies} \ p_{l} \cdot x \geq p_{l} \cdot \omega_{i} + \sum_{j=1}^{k} \theta_{j} p_{l} \cdot y_{j} - \frac{1}{l}.
\]
and 

\[ p_l \cdot y_j \geq \sup \{ p_l \cdot y \mid y \in Y_j \quad \text{and} \quad y \leq a \} - \frac{1}{l} \quad (5) \]

Choose \( 0 < \delta < 1 \) such that \( \delta h_n, x_n \geq x, \) holds for all \( (t, s) \in S \) By (4) and (5) for each \( (t, s) \in S \) we have

\[ p_l \cdot (\delta h_n) \geq p_l \cdot \omega_n + p_l \cdot \left( \sum_{j=1}^k \theta_{ij} y_j \right) - \frac{1}{l} \]

\[ = p_l \cdot \omega_n + p_l \cdot \left( \sum_{j=1}^k \sum_{i=1}^n \theta_{ij} y_j \right) - \frac{1}{l} \]

\[ \geq p_l \cdot \omega_n + p_l \cdot \left( \sum_{j=1}^k \sum_{i=1}^n \theta_{ij} z_{ij} \right) - \frac{2}{l}, \quad (6) \]

and so

\[ p_l \cdot \left( \delta \sum_{(l, s) \in S} h_n \right) \]

\[ \geq p_l \cdot \left( \sum_{(l, s) \in S} \omega_n \right) + p_l \cdot \left[ \sum_{j=1}^k \sum_{i=1}^n \left( \sum_{(l, s) \in S} \theta_{ij} \right) z_{ij} \right] - \frac{2mn}{l} \quad (7) \]

Combining (2) and (7), we obtain

\[ \delta = \delta p_l \left( \sum_{(l, s) \in S} h_n \right) \geq p_l \left( \sum_{(l, s) \in S} h_n \right) - \frac{2mn}{l} = 1 - \frac{2mn}{l} \]

for each \( l, \) and so \( \delta \geq 1, \) which is a contradiction. Therefore, the allocation \((x_1, \ldots, x_m, y_1, \ldots, y_k)\) is an Edgeworth equilibrium. \( \Box \)

The rest of the section is devoted to proving the existence of approximate equilibria. To do this, we need some preliminary technical results.

**Definition 6.4** A subset \( Y \) of a vector space \( E \) is said to be continuous for a linear topology \( \xi \) on \( E \) (briefly, \( \xi \)-continuous) whenever \( \{ y_n \} \subseteq Y \) and \( y_n \rightarrow^\xi 0 \) imply \( y_n \rightarrow^\xi 0. \)

The continuity property of the production set conveys the fact that small inputs produce small outputs if the production set is a cone, then the continuity of the production set seems to be a natural condition. The next two results will clarify the situation. Recall that a production set is any weakly closed convex subset \( Y \) of \( E \) satisfying \( Y \cap E^* = \{ 0 \}. \)
**PROPOSITION 6.5.** Assume that a production set \( Y \) is a cone. If \( \langle E, E' \rangle = \langle ca(\Omega), ca'(\Omega) \rangle \) for some Hausdorff compact topological space \( \Omega \) (in particular, if \( E \) is finite dimensional), then \( Y \) is norm continuous.

**Proof.** Let \( Y \) be a production set which is a cone, and let \( \{ z_n \} \subseteq Y \) satisfy \( \| z_n^- \| \to 0 \). Assume by way of contradiction that \( \{ z_n^+ \} \) does not converge in norm to zero. Then, by passing to a subsequence, we can assume that \( \| z_n^+\| \geq \varepsilon > 0 \) holds for all \( n \) and some \( \varepsilon > 0 \). Now let

\[
x_n = z_n^+ / \| z_n^+ \| = z_n^+ / \| z_n^- \| \quad \text{and} \quad (*)
\]

Since \( Y \) is a cone, we have \( x_n \in Y \) for each \( n \). From

\[
\| z_n^- / \| z_n^+ \| \| = \| z_n^- / \| z_n^- \| \leq \| z_n^- \| / \varepsilon \to 0,
\]

we see that \( \lim z_n^- / \| z_n^+ \| = 0 \). On the other hand, we have \( \| z_n^+ / \| z_n^+ \| \| = 1 \) for each \( n \). Since the set \( \{ y \in E^+ : \| y \| = 1 \} \) is weakly compact, it follows that \( \{ z_n^+ / \| z_n^+ \| \} \) has a weak accumulation point \( z > 0 \). From (\( * \)), we conclude that \( z \in Y \cap E^+ = \{ 0 \} \), which is impossible. Hence, \( \| z_n^+ \| \to 0 \) must hold, and the proof of the theorem is finished.

**PROPOSITION 6.6.** Assume that the Riesz dual system for the economy is symmetric, preferences are also strongly monotone, \( X_i = E^+ \) for each \( i \) and that the aggregate production set \( Y = Y_1 + \cdots + Y_k \) is a cone. If the economy has a Walrasian equilibrium, then \( \{ y_n \} \subseteq Y \) and \( y_n^- \to 0 \) imply \( y_n^+ \wedge x \to 0 \) for each \( x \in E^+ \).

In particular, if in this case \( E = \mathbb{R}^n \), then \( \{ y_n \} \subseteq Y \) and \( \| y_n^- \| \to 0 \) imply \( \| y_n^+ \| \to 0 \).

**Proof.** Let \( (x_1, \ldots, x_n, y_1, \ldots, y_k) \) be a Walrasian equilibrium supported by a price \( p \), and let \( y = y_1 + \cdots + y_k \). Since \( Y \) is a cone, we have

\[
\max \{ p \cdot z : z \in Y \} = p \cdot y = 0.
\]

Now let \( \{ y_n \} \subseteq Y \) satisfy \( y_n^- \to 0 \), and let \( x \in E^+ \). From \( p \cdot y_n^+ - p \cdot y_n^- = p \cdot y_n \leq 0 \), we see that \( p \cdot y_n^+ \leq p \cdot y_n^- \), and so in view of \( p \cdot (y_n^+ \wedge x) \leq p \cdot y_n^+ \leq p \cdot y_n^- \to 0 \), we conclude that

\[
p \cdot (y_n^+ \wedge x) \to 0 \quad (**)\]

Since preferences are strongly monotone, we have \( p \gg 0 \), and so the function

\[
\| x \| = p \cdot | x |, \quad x \in E,
\]

EDGEOUGH EQUILIBRIA 285
defines an order continuous norm on $E$. By [3, Theorem 12.9, p. 87], the topology generated by $\| \cdot \|$ and $\tau$ agree on the order interval $[0, x]$, and so in view of (**), we see that $y^*_n \wedge x \to^\tau 0$.  

If $x_i \in X_i$ ($i = 1, \ldots, m$), then we shall denote by $G$ the convex set

$$G = \text{co} \left( \bigcup_{i=1}^m (F_i - \omega_i) \right)$$

$$= \left\{ \sum_{i=1}^m \lambda_i (v_i - \omega_i) \mid v_i \geq x_i, \lambda_i \geq 0 \text{ and } \sum_{i=1}^m \lambda_i = 1 \right\}$$

**Lemma 6.7.** Assume that the Riesz dual system for the economy is symmetric and that each consumption set satisfies $X_i + E^+ = X_i$. If $x_i \in X_i$ ($i = 1, \ldots, m$), then the convex set

$$G = \text{co} \left( \bigcup_{i=1}^m (F_i - \omega_i) \right)$$

is weakly closed.

**Proof.** Apply Proposition 6.2 by taking $Y_j = \{0\}$ ($j = 1, \ldots, k$) and $a = 0$.  

**Proposition 6.8** Assume that for an economy we have:

(a) Its Riesz dual system $\langle E, E' \rangle$ is given by a reflexive Banach lattice, preferences are strongly monotone and $X_i = E^+$ for each $i$.

(b) There is only one producer whose production set $Y$ is a norm continuous cone; and

(c) The share $\theta_i$ of each consumer to the profit of the producer is positive, i.e., $\theta_i > 0$ for each $i = 1, \ldots, m$.

If $(x_1, \ldots, x_m, y)$ is an allocation, then the convex set

$$H = \text{co} \left( \bigcup_{i=1}^m (F_i - \theta_i Y - \omega_i) \right) = G - Y$$

is weakly closed.

**Proof.** Since $\theta_i > 0$ and $Y$ is a cone, we see that $\theta_i Y = Y$ for each $i$, and from this it easily follows that $H = G - Y$. By Lemma 6.7, we know that $G$ is a weakly closed set. To see that $G - Y$ is also weakly closed, let $f$ be in the weak closure of $G - Y$. Since $G - Y$ is convex, $f$ belongs to the norm closure of $G - Y$. So, there exists a sequence $\{ g_n \}$ of $G - Y$ with $\lim \| g_n - y_n - f \| = 0$.  


For each \( n \) write \( g_n = \sum_{i=1}^{m} \lambda_i^n (e_i^n - \omega_i) \), \( z_i^n \geq x_i, \lambda_i^n \geq 0 \) with \( \sum_{i=1}^{m} \lambda_i^n = 1 \) for each \( n \). Then we have

\[
g_n + \omega \geq g_n + \sum_{i=1}^{m} \lambda_i^n \omega_i = \sum_{i=1}^{m} \lambda_i^n z_i^n \geq 0,
\]

and so from

\[
0 \leq y_n^+ = (-y_n)^+ \leq (g_n - y_n + \omega)^+ \to (f + \omega)^+ \text{ (norm)},
\]

we see that \( \{ y_n^- \} \) is a norm-bounded sequence. Next, we claim that \( \{ y_n \} \) is a norm-bounded sequence. Indeed, if this is not the case, then we can assume without loss of generality that \( \lim \| y_n \| = \infty \). Since \( \{ y_n^- \} \) is norm bounded, we see that \( \lim \| y_n^- / \| y_n \| \| = 0 \). In view of \( (y_n / \| y_n \|)^- = y_n^- / \| y_n \| \), the norm continuity of \( Y \) implies

\[
\| (y_n / \| y_n \|)^+ \| = \| y_n^+ / \| y_n \| \| \to 0
\]

and so

\[
1 = \| y_n / \| y_n \| \| \leq \| y_n^+ / \| y_n \| \| + \| y_n^- / \| y_n \| \| \to 0,
\]

which is a contradiction. Hence, \( \{ y_n \} \) is norm bounded.

Since \( E \) is reflexive, \( \{ y_n \} \) has a weakly convergent subsequence. We can assume that \( y_n \rightharpoonup y \in F \). From \( g_n = (g_n - y_n) + y_n \rightarrow^w f + y \) and the closedness of \( G \), we see that \( g = f + y \in G \). Hence, \( f = y - y \in G - Y \), and thus \( G - Y \) is a weakly closed set.

Let \((x_1, \ldots, x_m, y_1, \ldots, y_k)\) be an allocation. Then we shall use the letter \( e \) to designate \( \sum_{i=1}^{m} x_i \), i.e.,

\[
e = \sum_{i=1}^{m} x_i = \omega + \sum_{j=1}^{k} y_j \geq 0.
\]

If \( p \) is a price vector, then \( p \cdot e \) is the total wealth of the consumers with respect to the allocation \((x_1, \ldots, x_m, y_1, \ldots, y_k)\) and the price \( p \).

**Definition 6.9** An allocation \((x_1, \ldots, x_m, y_1, \ldots, y_k)\) is said to be an approximate Walrasian equilibrium whenever for each \( \varepsilon > 0 \) there exists some price \( p \) such that:

(a) \( p \cdot e = 1 \) (where \( e = \sum_{i=1}^{m} x_i \));

(b) \( x \gg_{\varepsilon} x_i \) in \( X_i \), implies \( p \cdot x \geq p \cdot \omega_i + \sum_{j=1}^{k} \theta_j p \cdot y_j - \varepsilon \); and

(c) \( p \cdot y_j \geq \sup \{ p \cdot z : z \in Y_j \} - \varepsilon \) for each \( j \).
Remark. Kahn and Vohra in [13] introduced a notion of approximate equilibrium that is very close to the above definition. They required that the price \( p \) have norm one, i.e., \( \| p \| = 1 \). We require that \( p \cdot e = 1 \) in order to avoid the case \( p \cdot e = 0 \).

Note that if the consumption sets satisfy \( X_i + E^* = X_i \), then any price \( p \) that satisfies property (b) of Definition 6.9 is necessarily a positive price. Indeed, if \( x \geq 0 \), then by the monotonicity of \( \geq_1 \) we have \( x + \delta^{-1} x \geq_1 x_1 \) for all \( \delta > 0 \), and so

\[
p \cdot x \geq \delta \left( p \cdot \omega + \sum_{j=1}^{k} \theta_j p \cdot y_j - p \cdot x_1 - \epsilon \right)
\]

for all \( \delta > 0 \), from which it follows that \( p \cdot x \geq 0 \).

Also, it should be noted that every Walrasian equilibrium is an approximate Walrasian equilibrium.

Finally, we close the paper by presenting an existence theorem for approximate Walrasian equilibria.

**Theorem 6.10** Assume that for a compact economy we have

1. its Riesz dual system \( \langle E, E' \rangle \) is given by a reflexive Banach lattice,
2. preferences are strongly monotone and \( X_i = E^* \) holds for each \( i \), and
3. its aggregate production set \( Y = Y_1 + \cdots + Y_k \) is a norm continuous weakly closed cone.

Then the economy has approximate Walrasian equilibria.

**Proof.** Consider a new economy with Riesz dual system \( \langle E, E' \rangle \) having the same consumers, endowments and preferences but having one producer whose production set is \( Y \). Also, assume that each consumer has the share \( \theta_i = 1/m \) \((i = 1, \ldots, m)\) to the profit of the producer. It is easy to see that this new economy is compact, and so by Theorem 4.7 it has an Edgeworth equilibrium, say \((x_1, \ldots, x_m, y)\). If \( y = y_1 + \cdots + y_k \in Y \), then we claim that \((x_1, \ldots, x_m, y_1, \ldots, y_k)\) is an approximate Walrasian equilibrium.

To see this, let \( \varepsilon > 0 \). Clearly, \( H = G - Y \) holds, and by Proposition 6.8 the convex set \( H \) is weakly closed. Now for \( e = \sum_{n=1}^{m} x, > 0 \) we have

\[
\frac{\varepsilon}{m} + G - Y = \frac{\varepsilon}{2m} + \left( \frac{\varepsilon}{2m} + H \right) \leq \frac{\varepsilon}{2m} + H^*.
\]
and so from Proposition 5.4, we infer that $0 \notin (c/m) e + G - Y$. Then, by the classical separation theorem, there exists some $p \in E^*$ such that

$$p \cdot \left(\frac{c}{m} e + g - y\right) > 0$$

holds for all $g \in G$ and all $y \in Y$. Since $0 = \sum_{i=1}^{m} (1/m)(x_i - \omega_i) - (1/m) y \in G - Y$, we see that $p \cdot e > 0$, and so, replacing $p$ by $p/p \cdot e$, we can assume that $p \cdot e = 1$.

Now assume that $x \succsim_i x$, holds. Then $x - \omega_i \in G$ and \(\sum_{i=1}^{k} \theta_j y_j \in Y_i\). Thus, from (·), we see that

$$p \cdot x \geq p \cdot \omega_i + p \cdot \left(\sum_{j=1}^{k} \theta_j y_j\right) - \frac{c}{m} \geq p \cdot \omega_i + p \cdot \left(\sum_{j=1}^{k} \theta_j y_j\right) - \varepsilon.$$

Next, fix some $j$, and note that

$$\frac{1}{m} \sum_{i=1}^{m} y_i = \sum_{i=1}^{m} \frac{1}{m}(x_i - \omega_i) \in G.$$

For $z \in Y_j$, put $z_i = y_i$ for $i \neq j$ and $z_j = z$, and note that

$$\frac{1}{m}(y_j - z) = \frac{1}{m}(x_j - \omega_j) - \frac{1}{m} \sum_{i=1}^{m} z_i \in G - Y.$$

Therefore, from (·) we see that $p \cdot ((c/m) e + (1/m)(y_j - z)) > 0$, from which it follows that

$$p \cdot y_j \geq p \cdot z - \varepsilon \quad \text{for all} \quad z \in Y_j,$$

and so,

$$p \cdot y_j \geq \sup \{p \cdot z : z \in Y_j\} - \varepsilon \quad \text{for all} \quad j.$$

The above show that $(x_1, ..., x_m, y_1, ..., y_k)$ is an approximate Walrasian equilibrium, and the proof of the theorem is finished.  

REFERENCES


