

CONSUMPTION, LIQUIDITY CONSTRAINTS AND ASSET  
ACCUMULATION IN THE PRESENCE OF RANDOM  
INCOME FLUCTUATIONS\*

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1. INTRODUCTION

Recent empirical research (Flavin [1981], Hayashi [1982]) has rejected the certainty-equivalent formulation of permanent income hypothesis (Hall [1978]). These findings are often attributed to households' inability to borrow completely against expected future labor income. This paper is a theoretical investigation of optimal consumption behavior under risk aversion, random income fluctuations, and borrowing restrictions. The principle objective is to establish the existence and to investigate the properties of the stationary probability distribution which characterizes the behavior of consumption under these conditions.

Schechtman [1976] proves that optimal consumption converges almost surely to mean income if the rate of interest on savings is zero, the (infinitely lived) household does not discount future utility, borrowing is prohibited, and income is an i.i.d. random variable. Bewley [1976] generalizes Schechtman's result to the case in which income is a stationary stochastic process. In both analyses, asset holdings converge almost surely to infinity so that, asymptotically, the household is able to self-insure completely. Foley and Hellwig [1975] and Schechtman Escudero [1977] generalize Schechtman [1976] by establishing the existence of a limiting distribution for wealth if future utility is discounted at a positive rate. Schechtman and Escudero also prove that the wealth accumulation process is bounded so long as the rate of time preference strictly exceeds the constant rate of interest on savings and the elasticity of marginal utility is itself bounded. Our contribution to this literature is to establish the existence and a number of properties of the stationary probability distribution which characterizes the behavior of consumption in a version of Schechtman and Escudero's [1977] model which allows for lending as well as borrowing in amounts which can be repaid with probability one at a constant, positive rate of interest.

In Section 2, we establish the existence of a unique stationary probability distribution which characterizes the behavior of consumption, relaxing Foley and Hellwig's [1975] and Schechtman and Escudero's [1977] assumptions that the rate of interest is zero and that borrowing is not allowed.

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In Section 3, we derive some basic properties of this distribution. We begin by comparing the expected consumption of two individuals who differ in their pure rates of time preference. We prove that the expected consumption of the low time preference individual is greater than or equal to that of the high time preference individual, even though the former consumes less at any given level of wealth. We conclude this section by comparing the consumption and accumulation behavior of two individuals who confront probability distributions for labor income which differ by a location parameter. If the individuals are able to borrow completely against certain future labor income, we establish that the stationary probability distributions which characterize their consumption in the stochastic steady-state are identical. However, the stationary probability distributions which characterize the asset accumulation of two such individuals are shown to differ by a location parameter equal to the annuity value of the difference in the certain component of their labor incomes. This implies that differences in certain future labor income are, on average in the stochastic steady-state, offset by differences in asset accumulation. The variance and higher central moments of the stationary distribution for asset holdings are thus invariant to differences in certain future labor income.

## 2. AN "INCOME FLUCTUATIONS" PROBLEM

We consider the following "income fluctuations" problem. An individual with an infinite planning horizon must decide at the beginning of each period how much to consume and how many financial claims to purchase or sell. A financial claim costs one consumption good and entitles its owner to  $\rho \equiv 1+r$  consumption goods next period, where  $r$  is the constant, non-negative rate of interest on consumption loans. Total resources available for consumption and saving in period  $t$ ,  $w_t$ , are

$$(1) \quad w_t = \rho a_t + \varepsilon_t + \phi,$$

where  $a_t$  is the stock of financial claims purchased at the beginning of period  $t-1$ ,  $\varepsilon_t$  is the wage received at the beginning of period  $t$  (the inelastic supply of labor being normalized to unity), and  $\phi$  is a nonnegative, finite limit on borrowing. Consumption of  $c_t \leq w_t$  goods in period  $t$  yields utility of  $\beta^t u(c_t)$  where  $\beta \equiv 1/(1+\delta)$ ,  $\delta$  the positive rate of time preference. The transition equations for assets and total resources are given by

$$(2) \quad a_{t+1} = \rho a_t + \varepsilon_t - c_t,$$

$$(3) \quad w_{t+1} = \rho(w_t - c_t) + \varepsilon_{t+1} - r\phi.$$

Formally, the individual's problem is

$$\max E \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to  $c_t + a_{t+1} = \rho a_t + \varepsilon_t$ ,

$$c_0 + a_1 = \varepsilon_0,$$

$$c_t \geq 0, \quad a_t + \phi \geq 0.$$

We shall find it useful to make the following standard assumptions

- (A1)  $u(c)$  is strictly increasing, strictly concave, bounded and twice continuously differentiable with  $u'(0) < \infty$  and  $u'(\infty) = 0$ ,  
 (A2) The income received in each period is an independent and identically distributed random variable; let  $G$  denote its cumulative distribution function.  $G(\varepsilon)$  has bounded support  $[\underline{\varepsilon}, \bar{\varepsilon}]$  where  $\underline{\varepsilon} \geq 0$  and has a continuous density on the interval.  
 (A3) The limit on borrowing satisfies

$$0 \leq \phi < \underline{\varepsilon}/r,$$

a restriction which insures solvency with probability one,

- (A4)  $0 \leq r < \delta$ .

Let  $v(w)$  be the value of the objective function of an individual who begins the period with resources  $w$  and behaves optimally. This function must satisfy

$$(4) \quad v(w) = \max_{c \leq w} \{u(c) + \beta \int v(\rho(w-c) + \varepsilon' - r\phi) dG(\varepsilon')\}.$$

Denote the optimal consumption and asset accumulation decision rules by  $c$  and  $g$  respectively. The following proposition establishes the existence and properties of optimal consumption and asset accumulation decisions in the presence of random income fluctuations and a borrowing limit which satisfies (A3).

**PROPOSITION 2.1.** *By Assumptions (A1), (A2), (A3), and (A4), there exists a unique solution to the agent's problem. There is a unique, bounded, continuous, strictly increasing, strictly concave, and once continuously differentiable function  $v$  such that*

$$(5) \quad v(w) = \max_{c \leq w} \{u(c) + \beta \int v(\rho(w-c) + \varepsilon' - r\phi) dG(\varepsilon')\},$$

*optimal consumption is a continuous and strictly increasing function of  $w$ . There exists a unique  $\hat{w}$  which solves*

$$(6) \quad u'(\hat{w}) = \beta \rho \int v'(\varepsilon' - r\phi) dG(\varepsilon'),$$

*and for all  $\underline{\varepsilon} - r\phi \leq w \leq \hat{w}$ , optimal consumption is given by*

$$(7) \quad c(w) = w.$$

*For all  $w > \hat{w}$ , the optimal consumption decision rule satisfies  $\hat{w} < c(w) < w$  and is uniquely defined by*

$$(8) \quad u'(c(w)) = \beta \rho \int v'(\rho(w-c(w)) + \varepsilon' - r\phi) dG(\varepsilon').$$

PROOF. With the exception of the strict concavity of  $v$  and the strict monotonicity of  $c$ , all results are proved in Schechtman and Escudero [1977]. Following Lucas [1980], let  $L$  be the space of continuous, bounded functions  $u: R^+ \rightarrow R$  normed by

$$(9) \quad \|u\| = \sup_w |u(w)|.$$

Define  $T$  as the operator on  $L$  such that (5) reads  $v = Tv$ . Using Berge [1963],  $T: L \rightarrow L$ . Using Blackwell [1965],  $T$  is a contraction so that  $Tv = v$  has a unique solution  $v^* \in L$  and  $\|T^n u - v^*\| \rightarrow 0$  as  $n \rightarrow \infty$  for all  $u \in L$ . Lucas, Prescott and Stokey [1983] prove that  $T$  takes concave functions of  $w$  into strictly concave functions of  $w$  so that  $v^*$  is strictly concave. Q. E. D.

COROLLARY 2.2. *There exists a unique, continuous, and non-decreasing function  $g$  such that*

$$(10) \quad v(w) = u(w - g(w) - \phi) + \beta \int v(\rho(g(w) + \phi) + \varepsilon' - r\phi) dG(\varepsilon'),$$

that is,  $g$  is the optimal asset accumulation decision rule. For all  $\underline{\varepsilon} - r\phi \leq w \leq \hat{w}$ , optimal asset accumulation is given by

$$(11) \quad g(w) = -\phi.$$

For all  $w > \hat{w}$ ,  $g$  is strictly increasing and is uniquely defined by

$$(12) \quad u'(w - g(w) - \phi) = \beta \rho \int v'(\rho(g(w) + \phi) + \varepsilon' - r\phi) dG(\varepsilon').$$

Finally, total resources,  $w_t$ , obey the Markov process

$$(13) \quad w_{t+1} = \rho(w_t - c(w_t)) + \varepsilon_{t+1} - r\phi$$

with state space  $R^+$ .

Given an initial distribution for total resources  $F_0(w)$ , the distribution  $G(\varepsilon)$  and the difference equation (13) together determine the sequence of distributions  $F_1(w), F_2(\quad), \dots$  which prevail at dates  $t = 1, 2, \dots$ . To say that a sequence  $w_0, w_1, \dots$  is subject to the transition probabilities  $K$  means that  $K(w, w')$  is the conditional probability of the event  $\{w_1 \leq w'\}$  given that  $w_0 = w$ . Equation (13) and the definition of  $G$  imply that

$$(14) \quad K(w, w') = G(w' - \rho h(w) + r\phi),$$

where  $h(w) \equiv w - c(w)$ . If the probability distribution of  $w_0$  is  $F_0$ , the distribution of  $w_1$  is given by

$$(15) \quad F_1(w') = \int_{\kappa} K(w, w') F_0(dw).$$

where  $\kappa$  is the set  $[\underline{\varepsilon} - r\phi, \infty)$ .

DEFINITION. *The distribution  $F$  is a stationary distribution for  $K$  if  $HF = F$ ,*

where the operator  $H$  is defined by

$$(16) \quad HF(w') = \int_{\kappa} K(w, w')F(dw).$$

Schechtman and Escudero [1977] provide the following restriction on preferences which insure that the accumulation process is bounded.

(A5) *The elasticity of  $u'(c)$  is uniformly bounded; i.e., there exists a  $\bar{c}$  s.t. for all  $c > \bar{c}$ :*

$$-cu''(c)/u'(c) \leq M < \infty.$$

Schechtman and Escudero [1977] go on to show that (A1), (A5), and the additional assumptions that the income received in each period is a countable random variable, that borrowing is prohibited and that the rate of interest is zero are sufficient to prove the existence of a limiting distribution for total wealth.

The first task shall be to relax these latter three assumptions and to prove the existence of a unique, continuous distribution for  $w_t$  under assumptions (A1) through (A5). Armed with these results and the properties of  $c$  and  $g$ , we then establish the existence of unique, stationary cumulative distribution functions which characterize consumption and asset accumulation behavior in a stochastic steady-state. Following the approach suggested by Mendelssohn and Sobel [1980] and Danthine and Donaldson [1981] as adapted from Feller [1971], Donaldson and Mehra [1983], and Rosenblatt [1967], we now prove the following theorem.

**THEOREM 2.1.** *By Assumptions (A1) through (A5), there exists a continuous stationary distribution function for total resources, denoted  $F(w)$ , which is the unique solution to the functional equation  $HF = F$ :*

$$(17) \quad F(w') = \int_{\kappa} G(w' - \rho h(w) + r\phi) dF(w).$$

For any  $F_0$ ,

$$(18) \quad \lim_{n \rightarrow \infty} H^n F_0 = F.$$

Furthermore,  $F$  possesses a continuous density function which is positive on the compact subset  $\Omega = [\underline{\varepsilon} - r\phi, \bar{w}]$  where

$$(19) \quad \bar{w} = \inf [w: \rho h(w) + \bar{\varepsilon} - r\phi = w].$$

**PROOF.** By Assumption (A2), the stochastic kernel

$$(20) \quad K(w, w') \equiv G(w' - \rho h(w) + r\phi)$$

has a continuous density, denoted  $k(w; w')$ . We first show that  $K$  is regular in the sense of Feller [1971, p. 272]. That is, we must show that the family of transforms  $\mu_t(\cdot)$  defined by

$$(21) \quad \mu_t(w') = \int k(w, w') \mu_{t-1}(w) dw, \quad \mu_0 = 0$$

for  $\mu_t$  continuous and bounded is equicontinuous whenever  $\mu_0$  is uniformly continuous on any closed interval  $[\underline{w}, \bar{w}]$ .

Let  $M = \max_{w \in [\underline{w}, \bar{w}]} |\mu_0(w)|$ . By the recursive definition of  $\mu_t(\cdot)$ ,  $\forall t |\mu_t(w)| \leq M$ .

Then

$$(22) \quad |\mu_t(w) - \mu_t(w')| = \left| \int k(v, w) \mu_{t-1}(v) dv - \int k(v, w') \mu_{t-1}(v) dv \right| \\ \leq \int |k(v, w) - k(v, w')| |\mu_{t-1}(v)| dv \\ < e$$

for some  $\delta$  sufficiently small to ensure  $|w - w'| < \delta \rightarrow |k(v, w) - k(v, w')| < e/M, \forall v$ . Such a  $\delta$  is possible as  $k$  is uniformly continuous on  $[\underline{w}, \bar{w}]$ .

We next show that, once the process (13) has entered  $\Omega$ , there is zero probability that it will depart from it. Schechtman and Escudero [1977, Theorem 3.9, p. 162] show that Assumptions (A4) and (A5) are sufficient to ensure that there exists a  $\bar{w} < \infty$  such that

$$(23) \quad \rho(w - c(w)) + \bar{\varepsilon} - r\phi < \bar{w}, \quad w > \bar{w}.$$

From the continuity of  $c(w)$ , there exists at least one  $w$  which solves  $w = \rho h(w) + \bar{\varepsilon} - r\phi$ . Let  $\bar{w}$  be the infimum of the set  $[w: w = \rho h(w) + \bar{\varepsilon} - r\phi]$  and recall that  $\Omega \equiv [\underline{\varepsilon} - r\phi, \bar{w}]$ . As illustrated in Figure 1,  $\bar{w} \in [w: w = \rho h(w) + \bar{\varepsilon} - r\phi]$ . From the above arguments, it follows that

$$(24) \quad \rho(w - c(w)) + \bar{\varepsilon} - r\phi \leq \bar{w}, \quad w \leq \bar{w}.$$

Observe that for  $w > \hat{w}$ ,  $u'(c(w)) = \beta \rho E u'(c(\rho h(w) + \varepsilon' - r\phi))$  by the envelope theorem. This implies that  $u'(c(w)) < \beta \rho u'(c(\rho h(w) + \underline{\varepsilon} - r\phi))$  so that  $w > \rho h(w) + \underline{\varepsilon} - r\phi$ . It follows that, for  $w \in \Omega$

$$(25) \quad \underline{\varepsilon} - r\phi \leq \rho h(w) + \varepsilon' - r\phi \leq \bar{w}.$$

That is, for  $W$  not in  $\Omega$  and  $w \in \Omega$

$$(26) \quad k(w, W) = dG(W - \rho h(w) + r\phi) = 0.$$

We now show that any interval disjoint from  $\Omega$  is a transient set. There are two types of intervals to consider, depicted as  $[\bar{w}, w_1]$  and  $[w_1, w_2]$  in Figure 2. Suppose first that, for some time  $t$ ,  $w_t = w_0 \in [\bar{w}, w_1]$ ; we must show that the process passes out of  $[\bar{w}, w_1]$ . Consider any sequence  $\{\varepsilon_{t+n}\}$ ,  $n = 1, 2, \dots$  and form the sequences  $w_{t+n}$  and  $w_{t+n}(\bar{\varepsilon})$  defined, respectively, by

$$(27) \quad w_{t+n} = \rho h(w_{t+n-1}) + \varepsilon_{t+n},$$

$$(28) \quad w_{t+n}(\bar{\varepsilon}) = \rho h(w_{t+n-1}(\bar{\varepsilon})) + \bar{\varepsilon}, \quad w_t = w_t(\bar{\varepsilon}) = w_0.$$

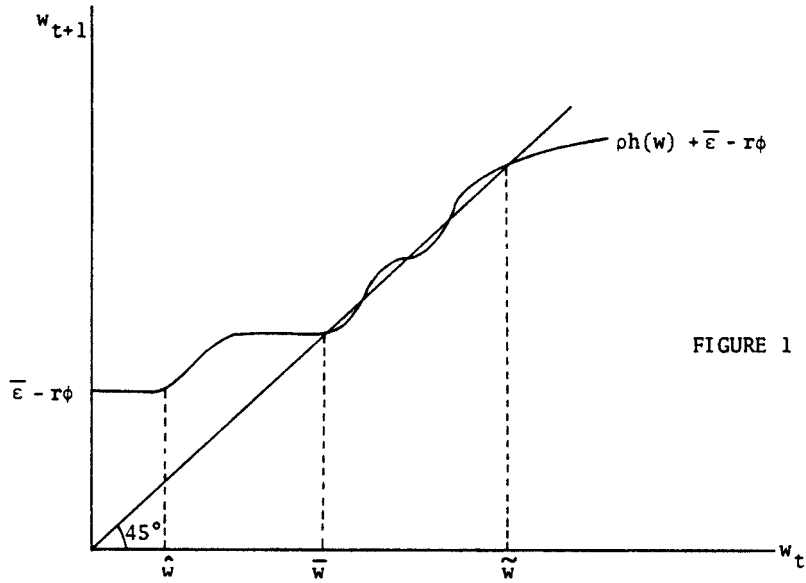


FIGURE 1

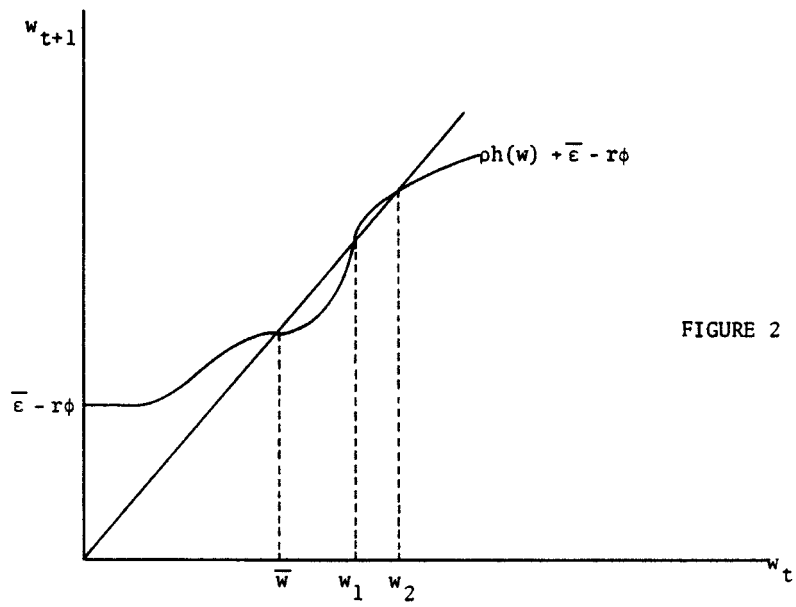


FIGURE 2

Without loss of generality, assume  $\varepsilon_{t+1} < \bar{\varepsilon}$ , then  $\forall n, w_{t+n} < w_{t+n}(\bar{\varepsilon})$ . By construction  $w_{t+n}(\bar{\varepsilon}) \rightarrow \bar{w}$  so that for some  $N, w_{t+n} < \bar{w}$  with probability one. Thus  $[\bar{w}, w_1]$  is a transient set. Suppose now that  $w_t = w_0 \in [w_1, w_2]$ . Since  $w > \rho h(w) + \underline{\varepsilon} - r\phi$ , there exists by continuity an  $\hat{\varepsilon} > \underline{\varepsilon}$  such that, for all  $w \in [w_1, w_2]$ ,  $\rho h(w) + \hat{\varepsilon} < w$ . Consider any sequence  $\{\varepsilon_{t+n} \leq \hat{\varepsilon}\}$ ,  $n = 1, 2, \dots$ . By construction  $w(\hat{\varepsilon})$  converges to some point to the left of  $w_1$ , so that for some  $N, w_{t+n} < w_1$  so long as each  $\varepsilon_{t+n} \leq \hat{\varepsilon}$ . The probability of this occurrence is at least  $\xi^N$  where

$$(29) \quad \xi = \int_{\underline{\varepsilon}}^{\hat{\varepsilon}} dG(\varepsilon) > 0.$$

Thus with probability at least  $\xi^N$ , the process leaves  $[w_1, w_2]$  never to return. The expected number of visits to  $[w_1, w_2]$  is less than

$$(30) \quad \sum_{j=1}^{\infty} (1 - \xi^N)^j < \infty.$$

Hence,  $[w_1, w_2]$  is transient.

We finally show that any point of  $[\underline{\varepsilon} - r\phi, \bar{w}]$  can be reached from any other point of that interval in a finite number of steps. This insures that  $F$  is strictly positive on  $[\underline{\varepsilon} - r\phi, \bar{w}]$ . Now,  $\forall w \in [\underline{\varepsilon} - r\phi, \bar{w}]$ ,  $\rho h(w) + \bar{\varepsilon} - r\phi > w$ ,  $\rho h(w) + \underline{\varepsilon} - r\phi < w$ , and  $h(w)$  and  $dG(\cdot)$  are continuous. Hence,  $\forall w \in [\underline{\varepsilon} - r\phi, \bar{w}]$  there exists an interval of positive length surrounding  $w$ ,  $l(w)$ , such that  $k(w, l(w)) > 0$ . Since these intervals cover  $[\underline{\varepsilon} - r\phi, \bar{w}]$ ,  $F$  is strictly positive on  $[\underline{\varepsilon} - r\phi, \bar{w}]$ .

Q. E. D.

Consumption and asset holdings also evolve according to Markov processes which are given by

$$(31) \quad c_{t+1} = c(\rho(c^{-1}(c_t) - c_t) + \varepsilon_{t+1} - r\phi),$$

$$(32) \quad a_{t+1} = \rho a_t + \varepsilon_t - c(\rho a_t + \varepsilon_t + \phi).$$

The existence of limiting distributions which characterize the behavior of consumption and asset accumulation in the stochastic-steady state follow immediately from the theorem.

**COROLLARY 2.3.** *There exists a unique, continuous, stationary distribution, denoted  $J(c)$ , which characterizes the behavior of consumption in the stochastic steady-state. The support of  $J(c)$  is the compact interval  $[\underline{\varepsilon} - r\phi, c(\bar{w})]$ .  $J(c)$  is given by*

$$(33) \quad J(c') \equiv \text{Prob}(c_t \leq c') = F(c^{-1}(c')).$$

**COROLLARY 2.4.** *There exists a unique, almost-everywhere continuous, stationary distribution, denoted  $X(a)$ , which characterizes asset accumulation in the stochastic steady-state. The support of  $X(a)$  is the compact interval  $[-\phi, g(\bar{w})]$  of  $R$ .  $X$  has a single mass point at  $a = -\phi$  such that*

$$(34) \quad X(-\phi) \equiv \text{Prob}(a = -\phi) = F(\hat{w}).$$

For  $a > -\phi$ ,  $X(a)$  is defined by

$$(35) \quad X(a') \equiv \text{Prob}(a \leq a') = F(g^{-1}(a')).$$

PROOF.  $g(w) = -\phi$  for  $0 \leq w \leq \hat{w}$  which implies (34).  $g(w)$  is continuous and strictly increasing over the compact interval  $[\hat{w}, \bar{w}]$  so that the inverse  $g^{-1}(a')$  exists and is right continuous at  $a' = -\phi$  which implies (35) and the right continuity of  $X$  at  $-\phi$ . Q. E. D.

### 3. CONSUMPTION IN THE STOCHASTIC STEADY-STATE: THE ROLE OF TIME PREFERENCE AND CERTAIN LABOR INCOME

We now compare the expected consumption of two individuals who differ only in their pure rates of time preferences. We first show that the average propensity to consume is directly related to the rate of time preference. We next show that the expected asset holdings of low time preference individuals are no less than those of high time preference individuals. However, this reasoning does not necessarily apply to the probability distribution of consumption, for, although expected asset holdings are inversely related to time preference and consumption is a strictly increasing function of assets, less is consumed at each level of asset holdings given a lower rate of time preference. We now establish that, in the income fluctuations problem studied here, expected consumption is itself inversely related to the rate of time preference.

**THEOREM 3.1.** *Expected consumption is inversely related to the rate of time preference.*

PROOF. Consider two otherwise identical individuals with rates of time preference  $\delta_1$  and  $\delta_2$  such that  $\delta_2 < \delta_1$ . We shall first show that

$$(36) \quad c(w, \delta_2) = c(w, \delta_1) = w, \quad \text{for } w \leq \hat{w}(\delta_2),$$

$$(37) \quad c(w, \delta_2) < c(w, \delta_1) \leq w, \quad \text{for } w > \hat{w}(\delta_2).$$

From Proposition 2.1 we know that a necessary and sufficient characterization of the optimal consumption decision rule is given by

$$(38) \quad u'(c(w, \delta)) = \beta \rho \int v'(\rho(w - c(w, \delta)) + \varepsilon' - r\phi, \delta) dG(\varepsilon'),$$

for  $w > \hat{w}$ ; otherwise  $c(w) = w$ . We must show that  $v'(w, \delta_2) > v'(w, \delta_1)$ . Following the lead of Danthine and Donaldson [1981], we work recursively with the fact that  $v$  can be viewed as the limit of a sequence of valuation problems. The agent's problem in the next to last period is given by

$$(39) \quad v(w, \delta, 1) = \max_{c \leq w} \{u(c) + \beta E u[\rho(w - c) + \varepsilon' - r\phi]\}.$$

Define the maximand as  $\xi(w, c, \delta, 1)$ . Then, for all  $c$ ,  $\xi_w(w, c, \delta_2, 1) \geq \xi_w(w, c, \delta_1, 1)$ . Since  $v(w, \delta, 1) = \xi(w, \bar{c}, \delta, 1)$  for some  $\bar{c}$ , we have

$$(40) \quad v'(w, \delta_2, 1) \geq v'(w, \delta_1, 1).$$

Proceeding recursively, we can show that

$$(41) \quad v'(w, \delta_2, i) \geq v'(w, \delta_1, i);$$

provided  $v'(w, \delta_2, i-1) \geq v'(w, \delta_1, i-1)$ . It follows directly from Mendelson and Amihud [1982, Lemma p. 696] that  $v'(w, i)$  converges to  $v'(w)$  uniformly on every closed interval so long as  $u'(0) < \infty$ . It follows that

$$(42) \quad v'(w, \delta_2) \geq v'(w, \delta_1).$$

From (38) and (42), we obtain, for  $w > \hat{w}(\delta_1)$ ,

$$(43) \quad u'(c(w, \delta_1)) < \beta_2 \rho \int v'(\rho(w - c(w, \delta_1)) + \varepsilon' - r\phi, \delta_2) dG(\varepsilon').$$

From the strict concavity of  $u$  and  $v$ , we know that consumption must be decreased to restore the equality between the marginal utility of consumption today with the expected marginal valuation of wealth tomorrow. Furthermore, (43) and the definition of  $\hat{w}$  imply  $\hat{w}(\delta_2) < \hat{w}(\delta_1)$ .

We next show that

$$(44) \quad E(a, X(a, \delta_2)) \geq E(a, X(a, \delta_1)).$$

From Theorem 2.1 we know that

$$(45) \quad F(w', \delta_1) = \int G(w' - \rho h(w, \delta_1) + r\phi) dF(w, \delta_1).$$

Associated with  $\delta_2 < \delta_1$  is the function  $h(w, \delta_2)$  which, from (37) and the definition of  $h$  is greater than  $h(w, \delta_1)$  so long as  $w > \hat{w}(\delta_2)$ . Define the mapping  $T_2$  as

$$(46) \quad T_2 F(w) = \int G(w' - \rho h(w, \delta_2) + r\phi) dF(w').$$

From Theorem 2.1, we know that

$$(47) \quad \lim_{n \rightarrow \infty} T_2^n F = F(w, \delta_2).$$

Since  $G$  is a strictly increasing function, we know that

$$(48) \quad G(w' - \rho h(w, \delta_2) + r\phi) \leq G(w' - \rho h(w, \delta_1) + r\phi)$$

with the inequality holding strictly for some  $w \in [\underline{\varepsilon} - r\phi, \bar{w}(\delta_2)]$  if  $\hat{w}(\delta_2) < \bar{\varepsilon} - r\phi$ . This implies that  $T_2 F(w, \delta_1) \leq F(w, \delta_1)$ . It follows that the sequence  $\{T_2^n F(w')\}$  is non-increasing and none of its elements are larger than  $F(w', \delta_1)$ . Thus,  $F(w', \delta_2) \leq F(w', \delta_1)$ .

Consider now the relationship between  $X(a', \delta_2)$  and  $X(a', \delta_1)$ . There are three cases to examine. If  $\bar{\varepsilon} - r\phi \leq \hat{w}(\delta_2)$ , then  $\bar{w}(\delta_1) = \bar{w}(\delta_2) = \bar{\varepsilon} - r\phi$  and

$$(49) \quad X(-\phi, \delta_2) = X(-\phi, \delta_1) = 1.$$

If  $\hat{w}(\delta_2) < \bar{\varepsilon} - r\phi \leq \hat{w}(\delta_1)$ , then  $X(a', \delta_1) = 1$  for all  $a' \geq -\phi$ , but

$$(50) \quad X(a', \delta_2) = F(g^{-1}(a', \delta_2), \delta_2) < 1$$

for  $a' < g(\bar{w}(\delta_2))$ . If  $\hat{w}(\delta_1) < \bar{\varepsilon} - r\phi$ , we have for  $a' \in (-\phi, g(\bar{w}(\delta_2)))$

$$(51) \quad X(a', \delta_2) = F(g^{-1}(a', \delta_2), \delta_2) < F(g^{-1}(a', \delta_1), \delta_1) = X(a', \delta_1)$$

which follows directly from the properties of  $g = w - c - \phi$  and the fact that,  $\bar{w}(\delta_2) \geq \bar{w}(\delta_1)$ . Thus, from Hadar and Russell [1971],  $E(a, X(\delta_2)) \geq E(a, X(\delta_1))$ .

Using the transition equation for assets and the existence of a stationary distribution  $X(a, \delta_i)$

$$(52) \quad E(a, X(a, \delta_i)) = (1+r)E(a, X(a, \delta_i)) + E\varepsilon - E(c, J(c, \delta_i)).$$

for  $\delta_i = \delta_1, \delta_2$ . Thus

$$(54) \quad E(c, J(c, \delta_2)) = rE(a, X(a, \delta_2)) + E\varepsilon \\ \geq rE(a, X(a, \delta_1)) + E\varepsilon = E(c, J(c, \delta_1)).$$

Q. E. D.

We conclude with a comparison of the asymptotic consumption and accumulation behavior of two individuals who confront probability distributions for labor income which differ by a location parameter  $\psi$ . We shall assume that the individuals can borrow completely against that part of their future labor income which is known with certainty. The location parameter is thus equal to the difference in the individuals' certain labor income, the lower support of  $G$ , so that  $G(\varepsilon_i, \psi) = G(\varepsilon_i)$ .

**THEOREM 4.2.** *Consider two otherwise identical individuals whose income fluctuations governed by the probability distributions  $G(\varepsilon_i)$  and  $G(\varepsilon_i, \psi)$  respectively. The stationary probability distributions which characterize these individuals' consumption are identical,  $J(c) = J(c, \psi)$ . The stationary probability distributions which characterize these individuals' asset accumulation differ by the location parameter  $\psi/r$ . In particular,*

$$(54) \quad X(a, \psi) = X(a + \psi/r).$$

**PROOF.** The stationary probability distribution which governs total resources is the unique solution to the functional equation

$$(55) \quad F(w') = \int G(w' - \rho h(w) + r\phi) dF(w).$$

The solution to equation (55) depends only on  $G$  and  $h$ . From Proposition 2.1  $h(w) = h(w, \psi)$  when  $\phi(\psi) = \phi + \psi/r$  so long as  $G(\varepsilon_i, \psi) = G(\varepsilon_i)$ . Thus  $J(c_i) = F(c^{-1}(c'))$  is invariant with respect  $\psi$  so long as the individual can borrow completely against the lower support of  $G$ . Using the fact that  $g(w, \psi) = h(w) - \phi(\psi)$ , we obtain,

$$(56) \quad X(a') = F[h^{-1}(a' + \phi)],$$

$$(57) \quad X(a', \psi) = F[h^{-1}(a' + \phi + \psi/r)].$$

It follows directly that

$$(58) \quad X(a', \psi) = X(a' + \psi/r). \quad \text{Q. E. D.}$$

**COROLLARY 4.5.** *Differences in certain labor income are, on average in the stochastic steady-state offset by asset accumulation*

$$(59) \quad E[a, \phi] - E[a, \phi(\psi)] = \psi/r.$$

**COROLLARY 4.6.** *The central moments of the stationary probability distribution which governs the accumulation of assets are invariant to differences in certain labor income:*

$$(60) \quad E[a - E(a, \phi), \phi]^j = E[a - E(a, \phi(\psi)), \phi(\psi)]^j.$$

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