AN ECONOMY WITH INFINITE DIMENSIONAL COMMODITY SPACE
AND EMPTY CORE

C.D. ALIPRANTIS * and O. BURKINSHAW *

Indiana - Purdue University, Indianapolis, IN 46223, USA

D.J. BROWN **

Yale University, New Haven, CT 06520, USA

Received 8 September 1986

We present an example of a two person pure exchange economy having an infinite dimensional commodity space and without core allocations

From the classical result of Scarf (1967) on the existence of core imputations for an $n$-person game it is well known that if a pure exchange economy on a finite dimensional commodity space has a finite number of agents (whose preferences satisfy the standard properties), then the economy has a non-empty core. The situation is completely different if the commodity space is infinite dimensional. Araujo (1985) presented an example of a two person pure exchange economy on $L_\infty$ whose core was empty. In this example one of the consumers is not impatient, i.e., his utility function is not Mackey continuous. On the other hand, Jones (1984) constructed an example of a two person pure exchange economy with empty core on $C[0,1]$.

In both the Araujo (1985) and Jones (1984) examples at least one consumer does not have a utility function which is both strictly convex and strongly monotone. Hence, if preferences are norm continuous, strictly convex, strongly monotone and uniformly proper and each consumer’s endowment is strictly positive, then one might hope that the economy has a non-empty core. Our example below (which is much stronger than the Araujo and Jones examples) shows that such a conjecture is false. The example appeared without any details for the first time in Aliprantis, Brown and Burkinkshaw (n.d.).

We shall follow the terminology of Aliprantis et al. and $L_p$ ($1 \leq p \leq \infty$) will denote $L_p[0,1]$. Our economy will have two consumers. Its Riesz dual system that describes the commodity–price duality will be either $\langle C[0,1], ca[0,1] \rangle$ or $\langle L_p, L_q \rangle$ ($1 \leq p \leq \infty; 1/p + 1/q = 1$). The consumers’ initial endowments are $\omega_1 = \omega_2 = 1$ (= the constant function one), and their utility functions are given by the formulas

$$u_1(x) = \int_0^1 x(t) \, dt + \frac{1}{2} \int_0^1 x'(t) \, dt$$
and
$$u_2(x) = \frac{1}{2} \int_0^1 x(t) \, dt + \int_0^1 x'(t) \, dt.$$

* Research supported in part by NSF grant DMS 83-19594
** Research supported in part by NSF grant SES 83-19611

0165-1765/87/$3.50 \copyright 1987,$ Elsevier Science Publishers B V (North-Holland)
Clearly, the utility functions are strongly monotone and strongly concave (and uniformly proper with respect to $\langle C[0, 1], ca[0, 1] \rangle$). The order interval $[0, \omega]$ (where $\omega = \omega_1 + \omega_2$) is $\sigma(L_\infty, L_1)$-compact but it is not $\sigma(C[0, 1], ca[0, 1])$-compact.

We claim that the economy with Riesz dual system $\langle C[0, 1], ca[0, 1] \rangle$ has an empty core. The proof of this claim will follow from the discussion below.

1. *The utility functions are continuous for both the $\| \cdot \|_1$-norm and the Mackey topology $\tau(L_\infty, L_1)$. In particular, they are continuous for the $\| \cdot \|_1$-norm and the $L_p$-norms.*

Let a net $(x_a) \subseteq L_\infty$ satisfy $x_a \tau(L_\infty, L_1), x$. Since $[-1, 1]$ is a convex, circled and $\sigma(L_1, L_\infty)$-compact subset of $L_1$, it follows that:

$$V = [-1, 1]^{\circ} = \left\{ x \in L_\infty : \left| \int_0^1 x(t) y(t) \, dt \right| \leq 1 \text{ for all } y \in [-1, 1] \right\}$$

$$= \left\{ x \in L_\infty : \int_0^1 |x(t)| \, dt \leq 1 \right\}$$

is a $\tau(L_\infty, L_1)$-neighborhood of zero. From this, we see that

$$\| x_a - x \|_1 = \int_0^1 |x_a(t) - x(t)| \, dt \to 0.$$

Now assume that $\| x_a - x \|_1 \to 0$ holds in $L_1^*$. If $f = \chi_{[0, \cdot]} + \frac{1}{2}\chi_{(\cdot, 1]}$, then we have

$$\left| u_1(x_a) - u_1(x) \right| \leq \int_0^1 |\sqrt{x_a(t)} - \sqrt{x(t)}| \, dt + \frac{1}{2} \int_0^1 \frac{|x_a(t) - x(t)|}{\sqrt{x(t)}} \, dt$$

$$= \int_0^1 f(t) |\sqrt{x_a(t)} - \sqrt{x(t)}| \, dt \leq \int_0^1 f(t) \sqrt{|x_a(t) - x(t)|} \, dt$$

$$\leq \frac{1}{\sqrt{2}} \cdot \left( \int_0^1 |x_a(t) - x(t)| \, dt \right)^{\frac{1}{2}} \to 0,$$

where the last inequality holds by virtue of Hölder’s inequality. Therefore, $u_1(x_a) \to u_1(x)$, and similarly $u_1(x_a) \to u_2(x)$.

2. *Let $(x_1, x_2)$ be an allocation with respect to $L_1$ satisfying $x_1 > 0$ and $x_2 > 0$. Then there exist two constants $0 \leq a \leq 2$ and $0 \leq b \leq 2$ with $a \neq b$ such that the allocation $(x_1^*, x_2^*)$, given by

$$x_1^* = ax_1[0, \cdot] + b\chi_{(\cdot, 1]}$$

and

$$x_2^* = (2 - a)\chi_{[0, \cdot]} + (2 - b)\chi_{(\cdot, 1]},$$

satisfies $x_1^* \geq x_1$ and $x_2^* \geq x_2$.*

To see this, let $(x_1, x_2)$ be an allocation with respect to $L_1$ with $x_1 > 0$ and $x_2 > 0$. Put $a = 2\int_0^1 x_1(t) \, dt$, $b = 2\int_1^x x_1(t) \, dt$, and note that $0 \leq a \leq 2$ and $0 \leq b \leq 2$ hold. Let $x_1^*$ and $x_2^*$ be
defined as above. Now using Hölder's inequality, we see that

\[ \int_0^1 \sqrt{\mathbf{x}_1(t)} \, dt \leq \frac{1}{\sqrt{2}} \left[ \int_0^1 \mathbf{x}_1(t) \, dt \right]^{\frac{1}{2}} = \frac{\sqrt{a}}{2}, \quad \text{and} \quad \int_0^1 \sqrt{\mathbf{x}_1(t)} \, dt \leq \frac{\sqrt{b}}{2}. \]

Thus, \( u_1(x_1) = \int_0^1 \sqrt{\mathbf{x}_1(t)} \, dt + \frac{1}{2} \int_0^1 \sqrt{\mathbf{x}_1(t)} \, dt \leq \sqrt{a}/2 + \sqrt{b}/4 = u_1(x^*_1), \) i.e., \( x^*_1 \succ_1 x_1. \) Also,

\[ u_2(x_2) = \frac{1}{2} \int_0^1 \sqrt{2-x_1(t)} \, dt + \int_0^1 \sqrt{2-x_1(t)} \, dt \]

\[ \leq \frac{1}{2\sqrt{2}} \left( \int_0^1 \left[ 2-x_1(t) \right] \, dt \right)^{\frac{1}{2}} + \frac{1}{\sqrt{2}} \left( \int_0^1 \left[ 2-x_1(t) \right] \, dt \right)^{\frac{1}{2}} \]

\[ = \frac{1}{2\sqrt{2}} \left( 1 - \frac{a}{2} \right)^{\frac{1}{2}} + \frac{1}{\sqrt{2}} \left( 1 - \frac{b}{2} \right)^{\frac{1}{2}} = \frac{1}{4\sqrt{2}} - \frac{a}{2} + \frac{1}{\sqrt{2}} - \frac{b}{2} = u_2(x^*_2), \]

and so \( x^*_2 \succ_2 x_2. \)

Next, we must verify that we can choose \( a \) and \( b \) with \( a \neq b. \) To this end, assume that \( a = b. \) In this case we have \( x^*_1 = a \) and \( x^*_2 = 2 - a. \) From \( x^*_1 \succ_1 x_1 > 0 \) and \( x^*_2 \succ_2 x_2 > 0, \) we see that \( 0 < a < 2. \) By the symmetry of the situation, we can also assume that \( 0 < a \leq 1. \) Then we claim that the allocation \((y_1, y_2),(y_1) = \frac{1}{4}a \mathbf{x}_{(0,1]} + \frac{1}{4}a \mathbf{x}_{(1,1]}, \) \( y_2 = (2 - \frac{1}{4}a) \mathbf{x}_{(0,1]} + (2 - \frac{1}{4}a) \mathbf{x}_{(1,1]}, \)

satisfies \( y_1 \succ_1 x^*_1 \) and \( y_2 \succ_2 x^*_2 \) (and hence \( y_1 \succ_1 x_1 \) and \( y_2 \succ_2 x_2 \). Indeed, note first that

\[ u_1(y_1) = \frac{1}{4\sqrt{2}} - \frac{a}{4} \frac{\sqrt{a}}{2} = \frac{1}{4\sqrt{2}} \left( \sqrt{a} - \frac{3}{4}a \right) + \frac{1}{\sqrt{2}} - \frac{a}{2} = u_1(x^*_1). \]

On the other hand, we have

\[ u_2(y_2) = \frac{1}{4\sqrt{2}} - \frac{3}{4}a \frac{\sqrt{a}}{2} + \frac{1}{\sqrt{2}} - \frac{a}{2} = \frac{1}{4\sqrt{2}} \left( \sqrt{a} - 3a \right) + 2\sqrt{a} - a = u_2(x^*_2). \]

and an easy calculation shows that \( u_2(y_2) > \frac{1}{4\sqrt{2}} \sqrt{a} = u_2(x^*_2). \) The proof of part 2 is now complete.

3. Assume that \((x_1, x_2)\) is an allocation with \( 0 < x_i \in C[0,1] (i = 1, 2). \) Then there exists an allocation \((y_1, y_2)\) with \( y_i \in C[0,1] (i = 1, 2) \) satisfying \( y_i \succ_1 x_i \) for \( i = 1, 2. \) In other words, the economy with Riesz dual system \( (C[0,1], ca[0,1]) \) has no core allocations.

To see this, let \((x_1, x_2)\) be an allocation with \( 0 < x_i \in C[0,1] (i = 1, 2). \) By part 2 there exist two constants \( 0 \leq a \leq 2 \) and \( 0 \leq b \leq 2 \) with \( a \neq b \) such that the allocation \((x^*_1, x^*_2), \) given by

\[ x^*_1 = a \mathbf{x}_{(0,1]} + b \mathbf{x}_{(1,1]} \] and \[ x^*_2 = (2 - a) \mathbf{x}_{(0,1]} + (2 - b) \mathbf{x}_{(1,1]}, \]

satisfies \( x^*_1 \succ_1 x_1 \) and \( x^*_2 \succ_2 x_2. \) Since \( x^*_1 \) and \( x^*_2 \) are not continuous functions, we see that \( x^*_1 \neq x_1 \) and \( x^*_2 \neq x_2. \) Thus, by the strong concavity of the utility functions, we infer that the allocation
(θ₁, θ₂) given by θ₁ = 1/2(x₁ + x₁*) and θ₂ = 1/2(x₂ + x₂*) satisfies θ₁ ≥ x₁ and θ₂ ≥ x₂.

Now since C[0, 1] is ||·||₁ dense in L₁, there exists a sequence {zₙ} ⊂ C[0, 1] satisfying 0 ≤ zₙ ≤ 2 for all n and lim ||θ₁ - zₙ||₁ = 0 (and, of course, lim ||θ₂ - (2 - zₙ)||₁ = 0). By virtue of the ||·||₁-continuity of the utility functions (part 1), there exists some n so that the allocation (zₙ, 2 - zₙ) satisfies zₙ ≥ x₁ and 2 - zₙ ≥ x₂. This completes the proof of part 3.

By Aliprantis et al. (example 5.2) we know that the economy with Riesz dual system ⟨Lₘ, ...⟩ Walrasian equilibria. Part 2 tells us that these Walrasian equilibria must be of a very special type. Next, we exhibit a Walrasian equilibrium for the economy with Riesz dual system ⟨Lₚ, Lₐ⟩.

4. The allocation (x₁, x₂), given by

\[ x₁ = \frac{2}{3}x_{[0, \gamma]} + \frac{1}{3}x_{[\gamma, 1]} \quad \text{and} \quad x₂ = \frac{2}{3}x_{[0, \gamma]} + \frac{1}{3}x_{[\gamma, 1]}, \]

is a Walrasian equilibrium for an economy whose Riesz dual system is ⟨Lₚ, Lₐ⟩; 1 ≤ p, q ≤ ∞, 1/p + 1/q = 1. Moreover, in this case, the Lebesgue integral is a price that supports the allocation (x₁, x₂).

To see this, note first that \( u₁(x₁) = u₂(x₂) = \sqrt{\frac{1}{3}} \). Also, if \( f = x_{[0, \gamma]} + \frac{1}{3}x_{[\gamma, 1]} \), then we have \( u₁(x) / x(t) dt \). Therefore, if \( x ≪ x₁ \), then by using Hölder's inequality, we see that

\[ \sqrt{\frac{1}{3}} = u₁(x₁) ≤ u₁(x) = \int₀¹ f(t) x(t) dt \]

\[ ≤ \left( \int₀¹ [f(t)]^2 dt \right)^{1/2} \left( \int₀¹ x(t) dt \right)^{1/2} = \sqrt{\frac{1}{3}} \left( \int₀¹ x(t) dt \right)^{1/2}, \]

and so \( \int₀¹ x(t) dt \geq 1 \), i.e.,

\[ \int₀¹ x(t) dt \geq 1 = \int₀¹ \omega₁(t) dt. \]

Similarly, \( x ≻ x₂ \) implies \( \int₀¹ x(t) dt \geq \int₀¹ \omega₂(t) dt \). In other words, the Lebesgue integral is a price that supports (x₁, x₂), and so (x₁, x₂) is a Walrasian equilibrium.

References


