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Global Stability in a Class of Markets with Three Commodities and Three Consumers

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The paper gives an estimate of the fraction of initial endowment distributions which result in global stability of a tâtonnement process, for a class of economies which includes the Scarf instability example as a limiting case. The measure of such a fraction for our class of economies is necessarily larger than or equal to the corresponding measure for the Scarf example itself *Journal of Economic Literature* Classification Numbers: 021, 022

1. INTRODUCTION

In this paper we give an estimate of the fraction of initial endowment distributions which result in global stability of the Walrasian tâtonnement process, for a special class of three-commodity, three-consumer economies which includes the Scarf global instability example [4] as a limiting case.

Our general approach to the problem of stability of the tâtonnement process is varying the initial endowment distributions and examining the proportion of such distributions for which the process will be stable. There is some precedence for this problem in the work of Balasko [1] and Keenan [3]. They examine the existence of a set of endowment distributions nearly Pareto optimal for which stability holds, but they are not concerned with measuring the size of this set for given classes of preferences.

In this paper we attempt to provide such a measure for a restricted class of preferences which includes the Scarf preference [4] as a limiting case. Such a measure for the Scarf example itself is exactly given by Hirota [2] and it is quite large. (For a meaning of this kind of approach, see Scarf's

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comment [5].) Our purpose here is to extend this Hirota's result to a more general class of preferences. Our result is that the measure of the set of initial distributions for which global stability holds for an arbitrary preference taken from our preference class, is necessarily larger than or equal to the corresponding measure for Scarf's special preference.

2. Model and Result

Let us consider a class of three-commodity, three-consumer economies that is characterized by

$$\{u^j(x_{ij}, x_{kj}), a^j = (a_{1j}, a_{2j}, a_{3j})\} \qquad \text{for differing } i, j, k,$$

where $u^j(\cdot)$ and a^j denote the jth person's utility function and initial endowment vector.

We put the conventional postulates on utility functions: $u^j(\cdot)$ is defined on the nonnegative orthant of R^2, is continuously differentiable, has positive partial derivatives, and is strictly quasi-concave. To restrict our concern to a special class of economies, we shall, moreover, put the following assumption:

(I) The marginal rate of substitution is unity if the amounts of two commodities are identical, i.e.,

$$(\partial u^j/\partial x_{ij})/(\partial u^j/\partial x_{kj}) = 1 \qquad \text{if } x_{ij} = x_{kj}.$$

The space of all the jth person's utility functions satisfying the above assumptions includes Scarf's utility function $\bar{u}^j(x_{ij}, x_{kj}) = \min[x_{ij}, x_{kj}]$ as a limiting case. For instance, a CES function converges to it as the elasticity of substitution goes to zero. We see, therefore, that the Scarf instability example economy is a limiting one of ours. It is noted that since no homothetic assumption is made except along a ray, the model developed here covers a situation in which there are inferior goods.

Next we restrict a varying range of initial endowment distributions. Let us define the 3×3 endowment matrix $A = [a_{ij}]$ and put a restriction on A as follows; A equals $\theta \bar{A}$, where $\bar{A}$ is a double stochastic matrix, i.e., the sum of each row and each column is unity and θ is an arbitrary positive real number.

The Scarf instability example's $a^1 = (0, 1, 0)$, $a^2 = (0, 0, 1)$, and $a^3 = (1, 0, 0)$ is a special point of our space of initial distributions. If the four elements of A are chosen, the others can be determined by them. This permits us to consider the following parameter space of initial distributions

$$C(\theta)=\{a=(a_{11}, a_{21}, a_{12}, a_{22})\geqslant 0 \mid \theta \geqslant a_{i1}+a_{i2} \quad (i=1,2)$$
$$\theta \geqslant a_{1i}+a_{2i} \quad (i=1,2)$$
$$a_{11}+a_{21}+a_{12}+a_{22}\geqslant \theta\}.$$

We have finished specifying our class of economies. Let us now choose an arbitrary utility function $u^j(\cdot)$ for each person from our utility class. We have, then, each person's demand functions that are denoted by $x_{\tau j}(p_i, p_k; M_j)$s, where p_i and p_k are nominal prices of the ith and kth goods, and M_j the nominal income of the jth person. Of course, the nominal income is determined by an assignment value of initial distributions and the prevalent prices.

Below we write $f_\tau(p; a)$s for the market excess demand functions with a parameter $a \in C(\theta)$, where $p=(p_1, p_2, p_3)$ is a nominal price vector.

For our specified class of economies $p^*=(1, 1, 1)$ is always a general equilibrium price vector up to multiplication, because by assumption (I) each person's demand for each good becomes $\theta/2$ for any assignment $a \in C(\theta)$ and the total supply of each good is θ. It is, however, noted that the multiple equilibria case might emerge dependent on the assignment value of initial distributions.

We consider the following parametric differential equation which is the analogue of a Walrasian price adjustment process:

$$\dot{p}_\tau = f_\tau(p; a), \qquad \tau=1, 2, 3, \tag{1}$$

where $\dot{p}_\tau$ is the time derivative of p_τ. Let us take an arbitrary initial point $p(0)>0$ satisfying

$$(p_1)^2+(p_2)^2+(p_3)^2=3. \tag{2}$$

By Walras' law the solution $p[t; p(0); a]$ to (1) satisfies (2).

Our interest is to give an estimate of a subset of the parameter space $C(\theta)$, for which the solution to (1) is guaranteed to converge to an equilibrium price $p^*=(1, 1, 1)$, i.e.,

$$S(\theta)=\{a \in C(\theta) \mid \lim_{t\to\infty} p[t; p(0); a]=p^*\}.$$

THEOREM. *For arbitrary utility functions taken from our utility space and any positive value of* θ,

$$S(\theta) \supseteq \bar{S}(\theta) \tag{3}$$

and

$$\frac{\text{measure } S(\theta)}{\text{measure } C(\theta)} \geqslant \frac{\text{measure } \bar{S}(\theta)}{\text{measure } C(\theta)} = \frac{3}{2} - \log 2 \doteqdot 0.806, \tag{4}$$

where $\bar{S}(\theta)=\{a \in C(\theta) \mid 4a_{11}a_{22} \geqslant (\theta - a_{12} - a_{21})^2\}$.

Since the measure S can be this large, the phenomenon of global stability dominates the case of instability. The ratio of (4) indicates the probability of (1) being globally stable when each point of the paremeter space C is assumed to occur with equal frequency. As is analyzed in Hirota [2], the right-hand side value of (4) exactly equals the ratio of measure S to measure C when Scarf's utility function $\min[x_{ij}, x_{kj}]$ is assumed. We see, therefore, that substitutability among goods has the effect of increasing the extent of global stability.

3. Proof and Remark

Let us prove our theorem by contrasting our economy here and a limiting one of ours, i.e., the jth person's utility function is given by $\bar{u}^j(x_{ij}, x_{kj}) = \min[x_{ij}, x_{kj}]$ for differing i, j, k.

The demand functions corresponding to $\bar{u}^j(\cdot)$ are

$$\bar{x}_{\tau j}(p_i, p_k; M_j) = \frac{M_j}{p_i + p_k}, \qquad \tau = i, k. \tag{5}$$

Below we write $\bar{f}_\tau(p; a)$s for the market excess demand functions constructed from (5). In the above, a is an element of the parameter space C. We define

$$g_{\tau j}(p_i, p_k; M_j) = x_{\tau j}(p_i, p_k; M_j) - \bar{x}_{\tau j}(p_i, p_k; M_j), \tag{6}$$

where $x_{\tau j}(\cdot)$ is the demand function of our economy. Since $x_{\tau j}(\cdot)$ and $\bar{x}_{\tau j}(\cdot)$ are defined on the same budget equation, we have

$$p_i g_{ij}(p_i, p_k; M_j) + p_k g_{kj}(p_i, p_k; M_j) \equiv 0. \tag{7}$$

By using (6) our market excess demand functions defined in Section 2 can be decomposed into two parts as follows:

$$\begin{aligned} f_1(p; a) &= \bar{f}_1(p; a) + [g_{12}(\cdot) + g_{13}(\cdot)] \\ f_2(p; a) &= \bar{f}_2(p; a) + [g_{21}(\cdot) + g_{23}(\cdot)] \\ f_3(p; a) &= \bar{f}_3(p; a) + [g_{31}(\cdot) + g_{32}(\cdot)]. \end{aligned} \tag{8}$$

Let us take a Liapunov function

$$v(p) = [q-1]^2 \tag{9}$$

(where $q=\prod_{i=1}^{3} p_i$), the time derivative of which is given by $\dot{v}=2[q-1]\,\dot{q}$. Since the solution to (1) with an initial point $p(0)$ satisfying (2) always satisfies (2), $(q-1)$ is negative for all $p \neq p^*$. On the other hand, simple calculation yields

$$\begin{aligned}\dot{q} &= p_2 p_3 f_1(p; a) + p_1 p_3 f_2(p; a) + p_1 p_2 f_3(p; a) && \text{by (1)},\\ &= \phi(p; a) + \psi(p; a) && \text{by (8)},\end{aligned}$$

where

$$\phi(p; a) = p_2 p_3 \tilde{f}_1(p; a) + p_1 p_3 \tilde{f}_2(p; a) + p_1 p_2 \tilde{f}_3(p; a) \tag{10}$$

and

$$\begin{aligned}\psi(p; a) = {} & p_1[p_2 g_{31}(\cdot) + p_3 g_{21}(\cdot)]\\ & + p_2[p_1 g_{32}(\cdot) + p_3 g_{12}(\cdot)]\\ & + p_3[p_2 g_{13}(\cdot) + p_1 g_{23}(\cdot)].\end{aligned} \tag{11}$$

Next we find a subset of C such that $\dot{q}$ is positive for all $p \neq p^*$; such a subset leads to the global stability of (1); (10) actually becomes a quadratic form

$$\begin{aligned}\phi(p; a) &= (\tfrac{1}{2}) \cdot p[A + A' - \theta \cdot I^c]\, p'\\ &= (\tfrac{1}{2})[p_1 - p_3,\ p_2 - p_3][\tilde{A} + \tilde{A}' - \theta \cdot \tilde{I}^c]\begin{bmatrix} p_1 - p_3 \\ p_2 - p_3 \end{bmatrix},\end{aligned} \tag{12}$$

where $\tilde{A} = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]$, A' and $\tilde{A}'$ denote the transposed matrices of A and $\tilde{A}$, and I^c and $\tilde{I}^c$ denote matrices such that all the diagonal elements are nil and all off-diagonal elements are unity.

Equation (12) tells us that $\phi(p; a)$ is nonnegative for all p if and only if the right-hand side of (12) is positive semi definite, i.e.,

$$4a_{11}a_{22} \geqslant (\theta - a_{12} - a_{21})^2. \tag{13}$$ [1]

Next we show the positivity of (11) for all $p \neq p^*$. First consider the first parenthesis of (11). (7) leads to

$$p_2 g_{31}(\cdot) + p_3 g_{21}(\cdot) = [(p_2 + p_3)/p_3](p_3 - p_2) \cdot g_{21}(\cdot) \tag{14}$$

[1] (13) with inequality is equivalent to boundary condition (#) $\tilde{f}_i(p, a) > 0$ if $p_i = 0$, $p_j > 0$, and $p_k > 0$ for differing i, j, k (13) with equality is delicate for Scarf's type example the system $\dot{p}_\tau = \tilde{f}_\tau(p, a)$ $(\tau = 1, 2, 3)$ is subject to a limit cycle when $a_{11} = a_{22} = (\theta - a_{21} - a_{12})$, and is globally stable when $a_{11} + a_{22} > 0$ ad $4a_{11}a_{22} = (\theta - a_{21} - a_{12})^2$ See Propositions 2 and 3 of Hirota [2]

Let us show (14) is positive unless $p_2 = p_3$. Suppose $x_{21}(\cdot) \geqslant x_{31}(\cdot)$ for $p_2 > p_3$. Then from the condition of the first person's utility maximization subject to his budget equation, we have

$$(\partial u^1/\partial x_{21})/(\partial u^1/\partial x_{31}) \geqslant p_2/p_3 > 1.$$

(At the corner maximum case the marginal rate of substitution may be larger than the price ratio.) On the other hand, the diminishing marginal rate of substitution and assumption (I) in Section 2 together say that if $x_{21}(\cdot) \geqslant x_{31}(\cdot)$, then $(\partial u^1/\partial x_{21})/(\partial u^1/\partial x_{31})$ must be less than or equal to unity. Therefore $0 \leqslant x_{21}(\cdot) < x_{31}(\cdot)$ has to hold for $p_2 > p_3$. Figure 1 illustrates this. It is also clear that $x_{21}(\cdot) < \bar{x}_{21}(\cdot) = \bar{x}_{31}(\cdot) < x_{31}(\cdot)$ holds for $p_2 > p_3$, so that $g_{21}(\cdot) = x_{21}(\cdot) - \bar{x}_{21}(\cdot) < 0$. Similarly we have $g_{21}(\cdot) > 0$ for $p_2 < p_3$. Thus, (14) is positive unless $p_2 = p_3$.

The positivity of the second and third parentheses of (11) can be obtained similarly unless $p_1 = p_3$ and $p_1 = p_2$, so that we may summarize these as $\psi(p; a)$ is positive for all $p \neq p^*$.

Let $\bar{S}(\theta)$ be the set defined by

$$\bar{S}(\theta) = \{a \in C(\theta) \mid (13) \text{ is met}\}.$$

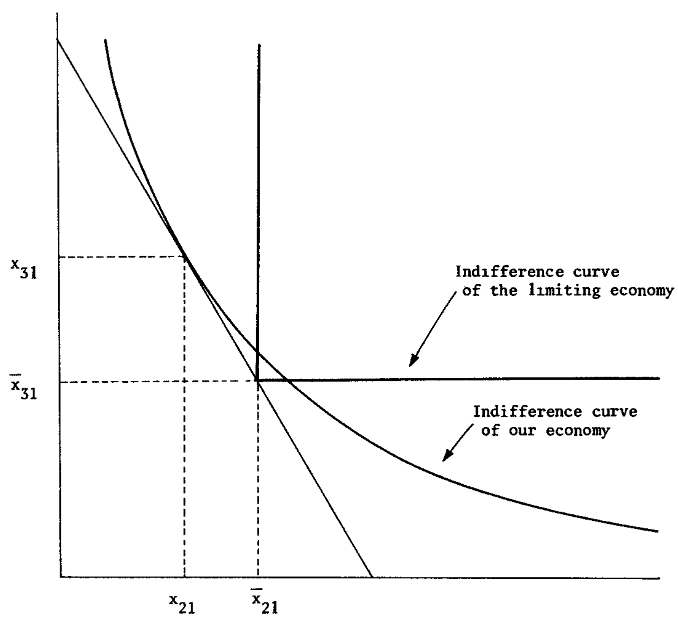

FIG 1. The first person's demands in our economy and the limiting economy when $p_2 > p_3$

Hence we know that for any $\tilde{a} \in \bar{S}(\theta)$, the solution $p[t; p(0); \tilde{a}]$ to (1) converges to $p^* = (1, 1, 1)$. So our set $S(\theta)$ defined in Section 2, contains $\bar{S}(\theta)$.

When $\theta = 1$, the measure of $C1)$ and $\bar{S}(1)$, as shown in Hirota [2], equals

$$\text{measure } C(1) = \int_{a \in C(1)} da = \tfrac{1}{8}$$

$$\text{measure } \bar{S}(1) = \int_{a \in \bar{S}(1)} da = \tfrac{1}{8}(\tfrac{3}{2} - \log 2).$$

Since there are linear maps $\theta \cdot I$: $C(1) \to C(\theta)$ and $\theta \cdot I$: $\bar{S}(1) \to \bar{S}(\theta)$, where I is the identity matrix, we have

$$\text{measure } C(\theta) = \theta^4 \text{ measure } C(1), \qquad \text{measure } \bar{S}(\theta) = \theta^4 \text{ measure } \bar{S}(1).$$

This completes the proof.

We have clarified the extent to which global stability holds for a class of economies and have found it to be quite large. Our class of economies, however, is very restrictive and our work must be viewed as a positive indication that further and more general results are worth pursuing.

From a series of analyses studied by Sonnenschein, Mantel, Debreu, and others we know that a classical assumption on preferences cannot characterize any class of market excess demand functions aside from continuity, homogeneity, and Walras' law. But we also imagine that for given preferences there will be such distributions of initial endowments or incomes that yield a well-behaved market excess demand function very suitable for some problem (stability, uniqueness, comparative statics, and others). Our work here may suggest a possible direction of such a research.

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