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EXAMPLES OF EXCESS DEMAND FUNCTIONS ON INFINITE-DIMENSIONAL COMMODITY SPACES

C. D. Aliprantis[1], D. J. Brown[2] and O. Burkinshaw[1]

1. INTRODUCTION

The Walrasian equilibrium problem with a finite dimensional commodity space has been studied rather extensively in the past. The existence of equilibrium prices in economies with a finite dimensional commodity space has been demonstrated very satisfactorily; see [8,9]. However, a number of economic situations lead naturally to infinite dimensional commodity spaces. In such a case, the mathematical tools employed in the finite dimensional case do not yield similar equilibrium results. Due to the nature of infinite dimensional spaces, questions about compactness of budget sets, continuity of utility and excess demand functions, utility maximization problems, etc. are very subtle. For this very reason, there are no satisfactory results guaranteeing the existence of equilibrium prices in economies with infinite dimensional commodity spaces. However, in spite of these difficulties, considerable progress has been made on the equilibrium problem with infinite dimensional commodity spaces.

In 1972 T. F. Bewley [4] demonstrated the existence of Walrasian equilibrium in both pure exchange economies and economies with production whenever the commodity space is $L_\infty(\mu)$ and the space of prices is $L_1^+(\mu)$ for a σ-finite measure μ. Bewley's work was a breakthrough for the subject. Subsequently, A. Mas-Colell [17], D. J. Brown and L. M. Lewis [6], and D. M. Kreps [15] obtained equilibrium results by using an ordered locally convex topological space as the commodity space and the cone of positive continuous linear functionals in the dual as its price space. This type of model was also used in [10],[11],[12],[19],[21], and [22].

In 1982 C. D. Aliprantis and D. J. Brown [1] introduced Riesz spaces as commodity spaces and were able to obtain equilibrium prices using the lattice structure of Riesz spaces. The Aliprantis-Brown economic model was built upon the concept of a Riesz dual system and by taking the excess demand function as a primitive notion. (The latter idea was

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first suggested by P. Bojan in [5].) In this approach the usual "Pareto optimality" questions are, of course, vacuous. On the other hand, the Aliprantis-Brown work showed that Banach lattices constitute a natural class of commodity spaces for infinite dimensional economies. Recently, A. Mas-Colell [18], A. M. Khan [14], N. C. Yannelis and W. R. Zame [23], G. Chichilnisky and G. Heal [7], L. Jones [13], and J. M. Ostroy [19] have also used Banach lattices as commodity spaces.

A Banach lattice version of the Aliprantis-Brown model is as follows. The economy is associated with a dual system $\langle L,L'\rangle$, where L is a Banach lattice and L' is its norm dual. The price simplex Δ is a non-empty convex w*-compact subset of L'_+ (= the cone of all positive linear functionals on L) such that:

a) The convex set S of all strictly positive prices of Δ (a price $p \in L'_+$ is said to be *strictly positive* whenever $x > 0$ implies $p(x) > 0$) is w*-dense in Δ; and
b) The cone generated by S (i.e., the set $\bigcup\{\lambda S\colon \lambda \geqslant 0\}$) is w*-dense in L'_+.

Now an excess demand function E for Δ is a mapping $p \longmapsto E_p$, from a convex subset D (= domain of E) of Δ into L, satisfying the following conditions.

1. D is w*-dense in Δ.
2. There exists a locally convex topology τ on L for which every functional of D is τ-continuous and $E\colon (D,w^*) \longrightarrow (L,\tau)$ is continuous.
3. If a net $\{p_\alpha\} \subseteq D$ satisfies $p_\alpha \xrightarrow{w^*} p \in \Delta \sim D$, then $\limsup q(E_{p_\alpha}) > 0$ holds for some $q \in D$.
4. E satisfies Walras' law, i.e., $p(E_p) = 0$ holds for all $p \in D$.

The triplet $(\langle L,L'\rangle,\Delta,E)$ is now the concept of an economy. A price $p \in D$ is said to be *a free disposal equilibrium price* (resp., *an equilibrium price*) whenever $E_p \leqslant 0$ (resp., $E_p = 0$) holds. The following result was established in [1].

THEOREM 1.1. (Aliprantis-Brown) *For an economy $(\langle L,L'\rangle,\Delta,E)$ the following statements hold:*

1. *The set of free disposal equilibrium prices is non-empty and w*-compact.*
2. *If $D = S$, then the economy has a non-empty w*-compact set of equilibrium prices.*

It may happen (as it was pointed out in [1]) that the zero price is a free disposal equilibrium price. However, in [1] it was also shown that for a special class of Banach lattices (the AM-spaces with units) with $x_i > 0$ for each i is a quasi-interior point), while ϕ does not have any quasi-interior points. On the other hand, c is an AM-space with unit $\vec{e}$ the constant sequence one (i.e., $\vec{e} = (1,1,\ldots)$), but its norm is not order continuous and fails to be σ-Dedekind complete.

As Banach spaces c and c_0 are linearly homeomorphic. For instance, if $T: c \longrightarrow c_0$ is defined by

$$T(x_1, x_2, \ldots) = (x_\infty, x_1 - x_\infty, x_2 - x_\infty, \ldots), \quad x_\infty = \lim x_n ,$$

then T is an onto linear homeomorphism. Indeed, note first that T is linear, one-to-one, and onto, and then use the inequalities

$$|x_i - x_\infty| \leqslant |x_i| + |x_\infty| \leqslant 2\|\vec{x}\|_\infty , \text{ and}$$
$$|x_i| \leqslant |x_i - x_\infty| + |x_\infty| \leqslant 2\|T\vec{x}\|_\infty$$

to see that

$$\tfrac{1}{2}\|\vec{x}\|_\infty \leqslant \|T\vec{x}\|_\infty \leqslant 2\|\vec{x}\|_\infty$$

holds for all $\vec{x} \in c$. (The norm bounds are exact:

$$\|T(1,-1,-1,-1,\ldots)\|_\infty = \|(-1,2,0,0,\ldots)\|_\infty = 2, \text{ and}$$
$$\|T(1,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\ldots)\|_\infty = \|(\tfrac{1}{2},\tfrac{1}{2},0,0,0,\ldots)\|_\infty = \tfrac{1}{2} .)$$

As Banach lattices c and c_0 cannot be lattice isomorphic. (Every lattice isomorphism between two Banach lattices preserves the order continuity of the norms.)

The Banach spaces c and c_0 both have Schauder bases (and hence they are both separable Banach spaces). Indeed, $\vec{x} = \sum_{n=1}^{\infty} x_n \vec{e}_n$ holds for each $\vec{x} = (x_1, x_2, \ldots) \in c_0$, and $\vec{x} = x_\infty \vec{e} + \sum_{n=1}^{\infty} (x_n - x_\infty)\vec{e}_n$ holds for each $\vec{x} \in c$. (As usual, $\vec{e}_n = (0,\ldots,0,\underset{n}{1},0,0,\ldots)$ and $\vec{e} = (1,1,1,\ldots)$.)

The norm duals of ϕ and c_0 coincide (as Banach lattices) with ℓ_1. The duality is given by

$$\langle \vec{x}, \vec{p} \rangle = \sum_{n=1}^{\infty} x_n p_n$$

for each $\vec{x} = (x_1, x_2, \ldots) \in c$ and each $\vec{p} = (p_1, p_2, \ldots) \in \ell_1$. (Following the common practice in economics, if $\langle X, X' \rangle$ is a dual system under the duality $\langle x, x' \rangle$, then we shall also denote $\langle x, x' \rangle$ by $x' \cdot x$. For instance, in this notation, $\langle \vec{x}, \vec{p} \rangle$ will be written as $\vec{p} \cdot \vec{x}$.)

The situation with the dual of c is a little bit different. The details are included in the next theorem. (A usual, whenever the sequence $\vec{x} = (x_1, x_2, \ldots)$ belongs to c, then we shall write $x_\infty = \lim x_n$.)

THEOREM 2.1. *The norm dual of* c *as a Banach lattice coincides with* $\ell_1 \oplus \mathbb{R}$ *(whose norm is given by* $\|\vec{p} \oplus r\| = \|\vec{p}\|_1 + |r| = \sum_{n=1}^{\infty} |p_n| + |r|$*).*

this unpleasant situation does not occur. In the next sections we shall present some more commodity spaces that illustrate the above theorem. These examples will have some natural economic significance and the excess demand functions will have equilibria at strictly positive prices. On the other hand, we shall present some functions which are not excess demand functions (according to [1]) but nevertheless they exhibit properties similar to those of excess demand functions, and have equilibrium prices.

For the theory of Riesz spaces (vector lattices) we refer the reader to [2] and [16]. For the theory of Banach lattices see [20] and the forthcoming book by C. D. Aliprantis and O. Burkinshaw [3].

2. THE NORMED RIESZ SPACES ϕ, c_0, AND c

The normed spaces ϕ, c_0, and c appear quite often in mathematical economics. In this section, we shall exhibit their basic topological and lattice properties. Although the norm structure of these spaces is well known, some of their lattice properties are not even available in the mathematical literature.

As usual, ℓ_∞ denotes the vector space of all bounded real-valued sequences. The elements of ℓ_∞ will be distinguished by the use of arrows. For instance, we shall write

$$\vec{x} = (x_1, x_2, \ldots),\quad \vec{y} = (y_1, y_2, \ldots),\quad \vec{a} = (a_1, a_2, \ldots),\ \text{etc.}$$

We consider ℓ_∞ as a Banach lattice under the sup norm

$$\|\vec{x}\|_\infty = \sup\{|x_i|:\ i = 1, 2, \ldots\}.$$

The Riesz subspaces ϕ, c_0, and c of ℓ_∞ are defined as follows.

$\phi = \{\vec{x} \in \ell_\infty:\ x_i = 0$ for all but finitely many indices $i\}$,
$c_0 = \{\vec{x} \in \ell_\infty:\ \lim x_n = 0\}$, and
$c = \{\vec{x} \in \ell_\infty:\ \lim x_n$ exists in $\mathbb{R}\}$.

Clearly, the following Riesz subspace inclusions hold:

$$\phi \subseteq c_0 \subseteq c \subseteq \ell_\infty .$$

With the sup norm ϕ, c_0, and c are AM-spaces.

The Riesz space ϕ is Dedekind complete but not norm complete, and its norm is order continuous. (Recall that a normed Riesz space L has *order continuous norm* whenever $x_\alpha \downarrow 0$ in L implies $\|x_\alpha\| \downarrow 0$.) The normed Riesz space c_0 is a Dedekind complete AM-space with order continuous norm but without a unit. The norm completion of ϕ is precisely c_0. The AM-space c_0 has plenty of quasi-interior points (every $\vec{x} \in c_0$

with $x_i > 0$ for each i is a quasi-interior point), while ϕ does not have any quasi-interior points. On the other hand, c is an AM-space with unit $\vec{e}$ the constant sequence one (i.e., $\vec{e} = (1,1,\ldots)$), but its norm is not order continuous and fails to be σ-Dedekind complete.

As Banach spaces c and c_0 are linearly homeomorphic. For instance, if $T: c \longrightarrow c_0$ is defined by

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THEOREM 2.1. *The norm dual of c as a Banach lattice coincides with $\ell_1 \oplus \mathbb{R}$ (whose norm is given by $\|\vec{p} \oplus r\| = \|\vec{p}\|_1 + |r| = \sum_{n=1}^{\infty}|p_n| + |r|$).*

Moreover, the duality of the system $\langle c, \ell_1 \oplus \mathbb{R}\rangle$ *satisfies*

$$\langle \vec{x}, \vec{p}\oplus r\rangle = (\vec{p}\oplus r)\cdot\vec{x} = rx_\infty + \sum_{i=1}^{\infty} p_i x_i$$

for all $\vec{x} = (x_1, x_2, \ldots) \in c$ *and all* $\vec{p}\oplus r \in \ell_1 \oplus \mathbb{R}$.

PROOF. Consider the mapping $\vec{p}\oplus r \longmapsto f_{\vec{p}\oplus r}$ (from $\ell_1 \oplus \mathbb{R}$ into the norm dual c' of c) defined by

$$f_{\vec{p}\oplus r}(\vec{x}) = rx_\infty + \sum_{n=1}^{\infty} p_n x_n$$

for each $\vec{x} = (x_1, x_2, \ldots) \in c$. Clearly, $\vec{p}\oplus r \longmapsto f_{\vec{p}\oplus r}$ is linear and one-to-one, and moreover, we claim that it is onto. To see this, let $0 \leqslant f \in c'$. Then for each $\vec{x} \in c$ we have

$$f(\vec{x}) = f(x_\infty\vec{e} + \sum_{n=1}^{\infty}(x_n - x_\infty)\vec{e}_n) = x_\infty f(\vec{e}) + \sum_{n=1}^{\infty}(x_n - x_\infty) f(\vec{e}_n)$$

$$= [f(\vec{e}) - \sum_{n=1}^{\infty} f(\vec{e}_n)]\,x_\infty + \sum_{n=1}^{\infty} f(\vec{e}_n)x_n = f_{\vec{p}\oplus r}(\vec{x}),$$

where $r = f(\vec{e}) - \sum_{n=1}^{\infty} f(\vec{e}_n)$ and $\vec{p} = (f(\vec{e}_1), f(\vec{e}_2), \ldots) \in \ell_1$. (To see that $\vec{p}$ belongs to ℓ_1 argue as follows: Since $\vec{0} \leqslant \sum_{n=1}^{k} \vec{e}_n \leqslant \vec{e}$ holds, the positivity of f implies $\sum_{n=1}^{k} f(\vec{e}_n) \leqslant f(\vec{e})$ for each k, and so the series $\sum_{n=1}^{\infty} f(\vec{e}_n)$ converges.)

Next note that $f_{\vec{p}\oplus r} \geqslant 0$ holds if and only if $\vec{p} \geqslant \vec{0}$ and $r \geqslant 0$ both hold. This implies [3, Theorem 7.3] that $\vec{p}\oplus r \longmapsto f_{\vec{p}\oplus r}$ is an onto lattice isomorphism. Finally, we see that

$$\|f_{\vec{p}\oplus r}\| = \|\,|f_{\vec{p}\oplus r}|\,\| = \|f_{|\vec{p}|\oplus|r|}\|$$

$$= f_{|\vec{p}|\oplus|r|}(\vec{e}) = \sum_{n=1}^{\infty} |p_n| + |r| = \|\vec{p}\oplus r\|,$$

and so $\vec{p}\oplus r \longmapsto f_{\vec{p}\oplus r}$ is a lattice isometry from $\ell_1 \oplus \mathbb{R}$ onto c'. Therefore, $\ell_1 \oplus \mathbb{R}$ is the norm dual of c.■

3. THE BANACH LATTICE c AS A COMMODITY SPACE

As we mentioned before, c is an AM-space having unit the constant sequence one (i.e., $\vec{e} = (1,1,1,\ldots)$). Our next objective is to consider c as a commodity space with prices in its dual $\ell_1 \oplus \mathbb{R}$. We start by considering the following convex sets of $c' = \ell_1 \oplus \mathbb{R}$.

$$\Delta = \{\vec{p}\oplus r \in (\ell_1\oplus\mathbb{R})^+ : \|\vec{p}\oplus r\| = (\vec{p}\oplus r)\cdot\vec{e} = r + \sum_{i=1}^{\infty} p_i = 1\},$$

$$\mathscr{D} = \{\vec{p}\oplus r \in \Delta : r \neq 1\} = \Delta \sim \{\vec{0}\oplus 1\}$$

$$S = \{\vec{p}\oplus r \in \Delta : p_i > 0 \text{ for each } i = 1,2,\dots\}, \text{ and}$$

$$D = \{\vec{p}\oplus 0 : \vec{p} \geqslant \vec{0} \text{ and } \|\vec{p}\|_1 = \sum_{i=1}^{\infty} p_i = 1\}.$$

Note that S consists precisely of all strictly positive prices of Δ. Clearly, we have the following proper inclusions

$$D \subseteq S \subseteq \mathscr{D} \subseteq \Delta .$$

By Alaoglu's theorem Δ is w*-compact (i.e., $\sigma(\ell_1\oplus\mathbb{R},c)$-compact), and we claim that D is w*-dense in Δ. (In the sequel, if $\{\vec{p}_n\}$ is a sequence in some ℓ_s, then we shall write $\vec{p}_n$ in component form as $\vec{p}_n = (p_1^n, p_2^n, \dots)$.)

LEMMA 3.1. *Let $\vec{p}\oplus r \in \Delta$, and let $\{\vec{p}_n\oplus r_n\} \subseteq \Delta$ be a sequence. Then*

$$\vec{p}_n\oplus r_n \xrightarrow{w^*} \vec{p}\oplus r$$

holds if and only if $\vec{p}_n \xrightarrow{\sigma(\ell_1, c_0)} \vec{p}$.

PROOF. If $\vec{p}_n\oplus r_n \xrightarrow{w^*} \vec{p}\oplus r$ holds, then it is easy to see that $\vec{p}_n \xrightarrow{\sigma(\ell_1, c_0)} \vec{p}$.

Conversely, assume that $\vec{p}_n \xrightarrow{\sigma(\ell_1, c_0)} \vec{p}$ holds. If $\vec{x} = (x_1, x_2, \dots)$ belongs to c, then by taking into account that $\vec{p}_n\cdot\vec{e} + r_n = \|\vec{p}_n\|_1 + r_n = 1$, we see that

$$(\vec{p}_n\oplus r_n)\cdot\vec{x} = r_n x_\infty + \vec{p}_n\cdot\vec{x} = (r_n + \vec{p}_n\cdot\vec{e})x_\infty + \vec{p}_n\cdot(\vec{x} - x_\infty\vec{e})$$

$$= x_\infty + \vec{p}_n\cdot(\vec{x} - x_\infty\vec{e}) \longrightarrow x_\infty + \vec{p}\cdot(\vec{x} - x_\infty\vec{e}) = (1 - \vec{p}\cdot\vec{e})x_\infty + \vec{p}\cdot\vec{x}$$

$$= r x_\infty + \vec{p}\cdot\vec{x} = (\vec{p}\oplus r)\cdot\vec{x} .$$

That is, $\vec{p}_n\oplus r_n \xrightarrow{w^*} \vec{p}\oplus r$ holds.■

If c is the commodity space for an economy, then (according to [1]) D can serve as the domain of an excess demand function provided that D is w*-dense in Δ. This is guaranteed by the next theorem.

THEOREM 3.2. *The convex set D is w*-dense in Δ.*

PROOF. Let $\vec{p}\oplus r \in \Delta$. For each n define $\vec{p}_n = (p_1^n, p_2^n, \dots, p_n^n, \dots)$ as follows. Consider first $1 \leqslant i \leqslant n$. If $p_i > 0$ holds, then choose p_i^n to satisfy $0 < p_i^n < p_i$ and $p_i - p_i^n < \frac{1}{n}2^{-i}$. If $p_i = 0$, then let $p_i^n = \frac{1}{n}2^{-i}$. Now for each $i > n$ pick $p_i^n > 0$ so that

$$\sum_{i=n+1}^{\infty} p_i^n = 1 - \sum_{i=1}^{n} p_i^n .$$

Clearly, $\vec{p}_n \oplus 0 \in D$, and we claim that $\vec{p}_n \oplus 0 \xrightarrow{w^*} \vec{p} \oplus r$ holds. To establish this, it suffices (by Lemma 3.1) to show that

$$\vec{p}_n \xrightarrow{\sigma(\ell_1, c_0)} \vec{p} .$$

To see this, let $\vec{x} = (x_1, x_2, \ldots) \in c_0$ be fixed. Then

$$|(\vec{p}_n - \vec{p}) \cdot \vec{x}| \leqslant \sum_{i=1}^{n} |(p_i^n - p_i)x_i| + \sum_{i=n}^{\infty} |(p_i^n - p_i)x_i|$$

$$\leqslant \frac{1}{n} \sup\{|x_i|: i = 1,2,\ldots\} + 2\sup\{|x_i|: i \geqslant n\} \longrightarrow 0 ,$$

and so $\vec{p}_n \oplus 0 \xrightarrow{w^*} \vec{p} \oplus r$ holds.∎

Next, we present some examples of functions that have properties very close to those of the excess demand functions of [1] but they are not excess demand functions. These examples illustrate the many things that can go wrong in the infinite dimensional case.

EXAMPLE 3.3. Consider the function $\zeta : \Delta \longrightarrow c$ defined by

$$\zeta(\vec{p} \oplus r) = \vec{p} - (\vec{p} \cdot \vec{p})\vec{e} = (p_1 - \vec{p} \cdot \vec{p}, p_2 - \vec{p} \cdot \vec{p}, \ldots) ,$$

where, of course, $\vec{p} \cdot \vec{p} = \sum_{i=1}^{\infty} (p_i)^2$. The function ζ has the following properties.

a) ζ *satisfies Walras' law, i.e.,* $(\vec{p} \oplus r) \cdot \zeta(\vec{p} \oplus r) = 0$ *holds for all* $\vec{p} \oplus r \in \Delta$.

Indeed, if $\vec{p} \oplus r \in \Delta$, then

$$(\vec{p} \oplus r) \cdot \zeta(\vec{p} \oplus r) = -r\vec{p} \cdot \vec{p} + \sum_{i=1}^{\infty} p_i (p_i - \vec{p} \cdot \vec{p})$$

$$= (-r + 1 - \sum_{i=1}^{\infty} p_i) \vec{p} \cdot \vec{p} = 0(\vec{p} \cdot \vec{p}) = 0 .$$

b) $\zeta : (\Delta, w^*) \longrightarrow (c, w)$ *is not continuous.*

From Lemma 3.1, it follows that $\vec{e}_n \oplus 0 \xrightarrow{w^*} \vec{0} \oplus 1$ in Δ. (Recall that $\vec{e}_n$ denotes the sequence whose nth component is one and the rest are zero.) Now if $\vec{0} < \vec{x} \in \ell_1$, then $\vec{x} \oplus 0 \in c'$, $\zeta(\vec{e}_n \oplus 0) = \vec{e}_n - \vec{e}$, and

$$(\vec{x} \oplus 0) \cdot \zeta(\vec{e}_n \oplus 0) = (\vec{x} \oplus 0) \cdot (\vec{e}_n - \vec{e})$$

$$= \vec{x} \cdot (\vec{e}_n - \vec{e}) \longrightarrow -\vec{x} \cdot \vec{e} \neq 0 = (\vec{x} \oplus 0) \cdot \zeta(\vec{0} \oplus 1) .$$

c) *For each* n *we have* $\zeta(\vec{e}_n \oplus 0) = \vec{e}_n - \vec{e} \leqslant \vec{0}$.

In other words, every $\vec{e}_n \oplus 0$ is a free disposal equilibrium price for ζ .

d) *The price* $\vec{0}\oplus 1$ *is the only equilibrium price of* ζ *(i.e.,* $\zeta(\vec{0}\oplus 1)=\vec{0}$ *holds).*

If $\zeta(\vec{p}\oplus r)=\vec{0}$ holds, then $\vec{p}\cdot\vec{p}=p_n \longrightarrow 0$ implies $\vec{p}\cdot\vec{p}=0$. Thus, $\vec{p}=\vec{0}$, and so $\vec{p}\oplus r=\vec{0}\oplus 1$.

Since the domain of ζ is Δ, the function ζ also satisfies the boundary condition of [1]. However, since $\zeta:(\Delta,w^*)\longrightarrow(c,w)$ fails to be continuous, ζ is not an excess demand function according to [1]. ■

EXAMPLE 3.4. Consider the function $Z:\mathscr{D}\longrightarrow c$ defined by

$$Z(\vec{p}\oplus r)=\vec{p}/\vec{p}\cdot\vec{p}-\vec{e}=\zeta(\vec{p}\oplus r)/\vec{p}\cdot\vec{p},$$

where ζ is the function of Example 3.3. Then Z has the following properties.

1. Z *satisfies Walras' law.*

2. Z *does not satisfy the boundary condition of* [1].

To see this, note that $\vec{e}_n\oplus 0 \xrightarrow{w^*} \vec{0}\oplus 1$ holds, and so if $\vec{p}\oplus r$ belongs to $\mathscr{D}$, then

$$(\vec{p}\oplus r)\cdot Z(\vec{e}_n\oplus 0)=r(-1)-\sum_{i\neq n}p_i=-r-\sum_{i=1}^{\infty}p_i+p_n=-1+p_n\longrightarrow -1<0.$$

3. $Z:(\mathscr{D},w^*)\longrightarrow(c,w)$ *is not continuous.*

To see this, let $\vec{p}_n=\tfrac{1}{2}(\vec{e}_1+\vec{e}_n)$, $n=1,2,\dots$. Then, $\{\vec{p}_n\oplus 0\}\subseteq\mathscr{D}$ and $\vec{p}_n\oplus 0\xrightarrow{w^*}\tfrac{1}{2}\vec{e}_1\oplus\tfrac{1}{2}$ holds in $\mathscr{D}$. On the other hand, note that

$$Z(\vec{p}_n\oplus 0)=2\vec{p}_n-\vec{e}=\vec{e}_1+\vec{e}_n-\vec{e}\xrightarrow{w}\vec{e}_1-\vec{e}\neq 2\vec{e}_1-\vec{e}=Z(\tfrac{1}{2}\vec{e}_1\oplus\tfrac{1}{2}).$$

4. Z *has a non-empty set of free disposal equilibrium prices which is not* w^**-compact.*

Indeed, note that $Z(\vec{e}_n\oplus 0)=\vec{e}_n-\vec{e}\leqslant\vec{0}$ holds for each n, $\vec{e}_n\oplus 0\xrightarrow{w^*}\vec{0}\oplus 1$, end $\vec{0}\oplus 1\notin\mathscr{D}$.

5. Z *does not have any equilibrium prices.*

If $Z(\vec{p}\oplus r)=\vec{0}$, then $\vec{p}\cdot\vec{p}=p_n\longrightarrow 0$, and so $\sum_{i=1}^{\infty}(p_i)^2=0$. This implies $\vec{p}\oplus r=\vec{0}\oplus 1\notin\mathscr{D}$. ■

EXAMPLE 3.5. Consider the function Z of the preceding example and define $W:\mathscr{D}\longrightarrow c$ by

$$W(\vec{p}\oplus r)=-Z(\vec{p}\oplus r)=\vec{e}-\vec{p}/\vec{p}\cdot\vec{p}.$$

Then W has the following properties.

a) W *satisfies Walras' law.*

b) W *does not satisfy the boundary condition of* [1].

To see this, let $\vec{p}_n=(2^{-n-1},2^{-n-2},\dots)$, and let $r_n=1-2^{-n}$.

Then $\{\vec{p}_n \oplus r_n\} \subseteq \mathscr{D}$, and by Lemma 3.1, it is easy to see that $\vec{p}_n \oplus r_n \xrightarrow{w^*} \vec{0} \oplus 1$ holds. Also, note that

$$\vec{p}_n \cdot \vec{p}_n = \sum_{i=1}^{\infty} (p_i^n)^2 = \sum_{i=1}^{\infty} (2^{-n-1})^2 = \frac{4^{-n}}{3}.$$

Now if $\vec{q} \oplus r \in \mathscr{D}$, then

$$\begin{aligned}(\vec{q} \oplus r) \cdot W(\vec{p}_n \oplus r_n) &= r \cdot 1 + \sum_{i=1}^{\infty} q_i (1 - p_i^n / \vec{p}_n \cdot \vec{p}_n) \\ &= r + \sum_{i=1}^{\infty} q_i - 3 \cdot 4^n \sum_{i=1}^{\infty} q_i p_i^n \\ &= 1 - 3 \cdot 4^n \sum_{i=1}^{\infty} q_i 2^{-n-1} \\ &= 1 - 3 \cdot 2^n \sum_{i=1}^{\infty} q_i 2^{-i} \longrightarrow -\infty .\end{aligned}$$

c) $W: (\mathscr{D}, w^*) \longrightarrow (c, w)$ *is not continuous.*

d) W *does not have any free disposal equilibrium prices.*

If $W(\vec{p} \oplus r) \leqslant \vec{0}$ holds, then $0 \leqslant \vec{p} \cdot \vec{p} \leqslant p_n \longrightarrow 0$, and so $\vec{p} \cdot \vec{p} = 0$. This implies $\vec{p} \oplus r = \vec{0} \oplus 1 \notin \mathscr{D}$, which is impossible.∎

By introducing "weights" we can make the functions continuous.

EXAMPLE 3.6. Fix a strictly positive sequence $\vec{w} = (w_1, w_2, \ldots) \in c_0$ (i.e., $w_i > 0$ for each i), and then define $\zeta_1: \mathscr{D} \longrightarrow c$ by

$$\zeta_1(\vec{p} \oplus r) = (p_1 w_1, p_2 w_2, \ldots) / \sum_{i=1}^{\infty} (p_i)^2 w_i - \vec{e}.$$

Then ζ_1 has the following properties.

a) ζ_1 *satisfies Walras' law.*

b) $\zeta_1: (\mathscr{D}, w^*) \longrightarrow (c, w)$ *is continuous.*

To see this, assume $\vec{p}_n \oplus r_n \xrightarrow{w^*} \vec{p} \oplus r$ in $\mathscr{D}$. (Since w^* is metrizable on $\mathscr{D}$, we need only consider sequences.) Fix $\vec{x} \oplus t \in \ell_1 \oplus \mathbb{R} = c'$.

By Lemma 3.1, it follows that $\vec{p}_n \xrightarrow{\sigma(\ell_1, c_0)} \vec{p}$, and so, in view of $x_n w_n \longrightarrow 0$, we see that

$$\sum_{i=1}^{\infty} x_i p_i^n w_i \xrightarrow[n \to \infty]{} \sum_{i=1}^{\infty} x_i p_i w_i .$$

On the other hand, from the inequalities

$$\begin{aligned}\sum_{i=1}^{\infty} (p_i^n)^2 w_i - \sum_{i=1}^{\infty} (p_i)^2 w_i &\leqslant \sum_{i=1}^{k} |(p_i^n)^2 - (p_i)^2| w_i + \sum_{i=k}^{\infty} |(p_i^n)^2 - (p_i)^2| w_i \\ &\leqslant \sum_{i=1}^{k} |(p_i^n)^2 - (p_i)^2| w_i + 2\sup\{w_i : i \geqslant k\},\end{aligned}$$

it follows that

$$\sum_{i=1}^{\infty}(p_i^n)^2 w_i \xrightarrow[n\to\infty]{} \sum_{i=1}^{\infty}(p_i)^2 w_i \neq 0 .$$

Now note that

$$(\vec{x}\oplus t)\cdot\zeta_1(\vec{p}_n\oplus r_n) = -t\cdot 1 + \sum_{i=1}^{\infty} x_i p_i^n w_i / \sum_{i=1}^{\infty}(p_i^n)^2 w_i \xrightarrow[n\to\infty]{}$$

$$-t\cdot 1 + \sum_{i=1}^{\infty} x_i p_i w_i / \sum_{i=1}^{\infty}(p_i)^2 w_i = (\vec{x}\oplus t)\cdot\zeta_1(\vec{p}\oplus r) .$$

c) *The set of all free disposal equilibrium prices of* ζ_1 *is non-empty but is not w*-compact.*

Note that $\zeta_1(\vec{e}_n\oplus 0) \leqslant \vec{e}_n - \vec{e} \leqslant \vec{0}$ holds for all n, and that $\vec{e}_n\oplus 0 \xrightarrow{w^*} \vec{0}\oplus 1 \notin \mathscr{D}$.

d) ζ_1 *does not have any equilibrium prices.*

As before, the only possible equilibrium price is $\vec{0}\oplus 1$ which does not belong to $\mathscr{D}$.

e) ζ_1 *does not satisfy the boundary condition of* [*1*].

Indeed, note that $\vec{e}_n\oplus 0 \xrightarrow{w^*} \vec{0}\oplus 1$, and that for each $\vec{p}\oplus r\in\mathscr{D}$

$$(\vec{p}\oplus r)\cdot\zeta_1(\vec{e}_n\oplus 0) = (\vec{p}\oplus r)\cdot(\vec{e}_n-\vec{e}) = r(-1) - \sum_{i\neq n} p_i = -1 + p_n \longrightarrow -1 < 0$$

holds.■

Finally, if we consider the function $z_2 : \mathscr{D} \longrightarrow c$ defined by

$$z_2(\vec{p}\oplus r) = -\zeta_1(\vec{p}\oplus r) ,$$

then we have:

1) z_2 satisfies Walras' law.
2) $z_2 : (\mathscr{D}, w^*) \longrightarrow (c,w)$ is continuous.
3) z_2 does not satisfy the boundary condition of [1].

   To see this, use the sequence $\{\vec{p}_n\}$ of part (b) of Example 3.5.

4) z_2 does not have any free disposal equilibrium prices.

4. ECONOMIES WITH COMMODITY SPACE ℓ_p

Consider the Riesz dual systems

$$\langle \ell_r, \ell_s \rangle,\ 1 \leqslant r < \infty ,\ \frac{1}{r} + \frac{1}{s} = 1 .$$

Let $\Delta = \{\vec{p}\in(\ell_s)^+ : \|\vec{p}\|_s \leqslant 1\}$, and note that (by Alaoglu's theorem) Δ is w*-compact. Since ℓ_r (for $1 \leqslant r < \infty$) is separable, it follows that

w^* is metrizable on Δ; see, for example, [3, Theorem 10.7]. Also, note that the set of strictly positive prices

$$S = \{\vec{p} = (p_1, p_2, \ldots) \in \Delta : \ p_i > 0 \ \text{ for each } \ i\}$$

is non-empty. Now fix $\vec{0} \leqslant \vec{u} = (u_1, u_2, \ldots) \in \ell_r$ and $\vec{0} \leqslant \vec{v} = (v_1, v_2, \ldots)$ in ℓ_1.

Next, define the function $F: \Delta \longrightarrow \ell_r$ by

$$F(\vec{p}) = (p_1 v_1, p_2 v_2, \ldots).$$

We claim that the functions $\vec{p} \longmapsto F(\vec{p})$ from (Δ, w^*) into (ℓ_r, w) and $\vec{p} \longmapsto \vec{p} \cdot F(\vec{p}) = \sum_{i=1}^{\infty} (p_i)^2 v_i$ from (Δ, w^*) into $\mathbb{R}$ are both continuous.

For the continuity of $F: (\Delta, w^*) \longrightarrow (\ell_r, w)$, let $\vec{p}_n \xrightarrow{w^*} \vec{p}$ in Δ, and let $\vec{x} = (x_1, x_2, \ldots) \in \ell_s$. Put $\vec{y} = (x_1 v_1, x_2 v_2, \ldots)$, and note that $\vec{y} \in \ell_r$. Hence,

$$\vec{x} \cdot F(\vec{p}_n) = \sum_{i=1}^{\infty} x_i p_i^n v_i = \vec{y} \cdot \vec{p}_n \longrightarrow \vec{y} \cdot \vec{p} = \vec{x} \cdot F(\vec{p}),$$

and the continuity of F follows.

For the continuity of $\vec{p} \longmapsto \vec{p} \cdot F(\vec{p})$, let $\vec{p}_n \xrightarrow{w^*} \vec{p}$ in Δ. Note that

$$\vec{p}_n \cdot F(\vec{p}_n) - \vec{p} \cdot F(\vec{p}) = (\vec{p}_n - \vec{p}) \cdot F(\vec{p}_n) + \vec{p} \cdot [F(\vec{p}_n) - F(\vec{p})]$$

and that by the previous case $\vec{p} \cdot [F(\vec{p}_n) - F(\vec{p})] \longrightarrow 0$. On the other hand, the inequality

$$|(\vec{p}_n - \vec{p}) \cdot F(\vec{p}_n)| = \Big|\sum_{i=1}^{\infty} (p_i^n - p_i) p_i^n v_i\Big| \leqslant \Big|\sum_{i=1}^{k} (p_i^n - p_i) p_i^n v_i\Big| + 2\sum_{i=k}^{\infty} v_i \,,$$

shows that $(\vec{p}_n - \vec{p}) \cdot F(\vec{p}_n) \longrightarrow 0$. Therefore, $\vec{p}_n \cdot F(\vec{p}_n) \longrightarrow \vec{p} \cdot F(\vec{p})$.

Now define $E: \Delta \longrightarrow \ell_r$ by

$$E(\vec{p}) = (\vec{p} \cdot \vec{u}) F(\vec{p}) - [\vec{p} \cdot F(\vec{p})] \vec{u} \,.$$

Then, according to the preceding discussion we have:

a) $E: (\Delta, w^*) \longrightarrow (\ell_r, w)$ is continuous;

b) $\vec{p} \cdot E(\vec{p}) = 0$ holds for all $\vec{p} \in \Delta$; and

c) since $D = \Delta$, the boundary condition of [1] is satisfied.

Hence, $(\langle \ell_r, \ell_s \rangle, \Delta, E)$ is an economy in the sense of [1]. Clearly, $\vec{p} = \vec{0}$ is an equilibrium price. It is interesting to note that the excess demand function E may have equilibrium prices lying in S. Some specific examples follow.

1. Consider $\langle \ell_1, \ell_\infty \rangle$. Then $\Delta = [\vec{0}, \vec{e}]$. Take $\vec{u} = \vec{v} = (1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots)$.

   Note that each price $c\vec{e} = (c, c, c, \ldots)$ $(0 \leqslant c \leqslant 1)$ is an equilibrium price for

$$E(\vec{p}) = (\vec{p}\cdot\vec{u})F(\vec{p}) - [\vec{p}\cdot F(\vec{p})]\vec{u} .$$

2. Let $1 \leqslant r < \infty$, and let $\vec{u} = (1, 2^{-3}, 3^{-3}, 4^{-3}, \ldots)$ and $\vec{v} = (1, 2^{-2}, 3^{-2}, \ldots)$. Then $\vec{p} = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$ is an equilibrium price for $E(\vec{p})$.

3. Let $1 \leqslant r < \infty$, and let $\vec{u} = (2^{-2}, 2^{-4}, 2^{-6}, \ldots)$ and $\vec{v} = (2^{-1}, 2^{-2}, 2^{-3}, \ldots)$. Then $\vec{p} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$ is an equilibrium price for $E(\vec{p})$. Note also that $\sum_{i=1}^{\infty} p_i = 1$.

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C. D. Aliprantis & O. Burkinshaw
DEPARTMENT OF MATHEMATICAL SCIENCES
INDIANA UNIVERSITY - PURDUE UNIVERSITY AT INDIANAPOLIS
INDIANAPOLIS, IN 46223

AND

D. J. Brown
COWLES FOUNDATION
YALE UNIVERSITY
NEW HAVEN, CT 06520