EQUILIBRIA IN MARKETS WITH A RIESZ SPACE OF COMMODITIES*

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We present a new proof of the existence of competitive equilibrium for an economy with an infinite dimensional space of commodities.

1. Introduction

In the Arrow–Debreu model of a Walrasian economy, the commodity space is $\mathbb{R}^n$ and the price space is $\mathbb{R}^n_+$, where $n$ is the number of commodities. Agent’s characteristics such as consumption sets, production sets, utility functions, the price simplex, excess demand functions, etc. are introduced in terms of subsets of $\mathbb{R}^n$ or $\mathbb{R}^n_+$, or functions on $\mathbb{R}^n$ or $\mathbb{R}^n_+$. As is well known, the Arrow–Debreu model allows consumption and production to be contingent on time and the state of the world, when there is a finite number of periods and a finite number of states.

In a world of uncertainty where there are an infinite number of states or an intertemporal economy having an infinite number of time periods (e.g., an infinite horizon), the appropriate model for the space of commodities is an infinite-dimensional vector space. In particular, ordered locally convex topological vector spaces are the most frequently used models of infinite-dimensional commodity spaces. In such models, the price space is the cone of positive continuous linear functionals in the dual space. See Brown–Lewis (1981), Kreps (1981) and Mas-Colell (1975).

The paradigmatic example being the dual pair $\langle L_\infty(\mu), L_1(\mu) \rangle$, where $\mu$ is a

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σ-finite measure on the underlying state space; \( L_\infty(\mu) \) is the space of commodities; \( L^+_\infty(\mu) \) is the space of prices. This model was introduced by Bewley (1972), and he proved the existence of a Walrasian equilibrium in both exchange economies and economies with production.

The spaces \( \mathbb{R}^n \) and \( L_\infty(\mu) \) in addition to being ordered vector spaces are also Riesz spaces, i.e., vector spaces which are also lattices. In fact, considered as Banach spaces they belong to the special class of Banach lattices, i.e., to the class of normed Riesz spaces which are complete under their norms.

In the papers cited above, preferences and endowments are taken as the primitive characteristics of consumers and demand functions are derived from utility maximization. If the commodity space is infinite-dimensional, then an agent’s budget set may not be compact. Hence, the existence of demand functions need not be a consequence of utility maximization. Following Bojan (1974), we shall take demand functions and endowments as the basic characteristics of a household.

The main contribution of this paper is a new proof of the existence of equilibrium prices. The idea is as follows: Let \( E(p) \) be an excess demand function on a price simplex \( \Delta \). Define an ordering on \( \Delta \) by saying that \( p > q \) whenever \( p \cdot E(q) > 0 \). A maximal element in this ordering is an equilibrium price vector. The KKM lemma is then used to prove that the ordering has a maximal element. We then apply this argument to the case where the commodity space is an infinite-dimensional Riesz space.

A Riesz dual system is a pair of Riesz spaces \( \langle L, L' \rangle \), where \( L' \) is the topological dual of \( L \) for some locally convex-solid topology, i.e., a locally convex topology for which the lattice operations on \( L \) are uniformly continuous. In our existence proof, the following property of Riesz dual systems is essential: If \( \langle L, L' \rangle \) is a Riesz dual system and \( f \in L \), then \( f \geq 0 \) holds if and only if \( \phi(f) \geq 0 \) for all \( 0 \leq \phi \in L' \).

Fortunately, Riesz dual systems are quite common. For instance, any Banach lattice paired with its norm dual is a Riesz dual system.

In the next section, we give the relevant facts about Riesz spaces. The equilibrium theorems appear in section 3. In the final section, we present a number of examples, which encompass the finite-dimensional cases, say in Debreu (1981), Bewley’s (1972) results for \( L_\infty \) in the pure exchange case, and several new infinite-dimensional markets.

2. Riesz spaces

As we have said in the introduction, this paper utilizes the theory of Riesz spaces (vector lattices). For this reason, this section presents a brief introduction to Riesz spaces. For details concerning the lattice properties of Riesz spaces we refer the reader to Luxemburg–Zaanen (1971), and for the
topological concepts on Riesz spaces to Aliprantis–Burkinshaw (1978) and Schaefer (1974).

A relation $\geq$ on a non-empty set $X$ is said to be an order relation whenever

(a) $x \geq x$ holds for all $x \in X$;
(b) $x \geq y$ and $y \geq x$ imply $x = y$; and
(c) $x \geq y$ and $y \geq z$ imply $x \geq z$.

An ordered set is a non-empty set $X$ together with an order relation $\geq$. The symbol $x \leq y$ is an alternative notation for $y \geq x$.

Now let $A$ be a non-empty set in an ordered set $X$. Then an element $y \in X$ is said to be a least upper bound (or a supremum) for $A$ whenever

1. $y$ is an upper bound for $A$, i.e., $x \leq y$ holds for all $x \in A$; and
2. if $x \leq z$ holds for all $x \in A$, then $y \leq z$.

Clearly, a set $A$ can have at most one least upper bound, and if it does have one, then it is denoted by $\sup A$. The greatest lower bound (or infimum) of a set $A$ is defined similarly, and is denoted by $\inf A$. A lattice is an ordered set $X$ such that $\sup\{x, y\}$ and $\inf\{x, y\}$ exist for each pair $x, y \in X$. As usual we write

$$x \vee y = \sup\{x, y\} \quad \text{and} \quad x \wedge y = \inf\{x, y\}.$$

In this paper, all vector spaces are real vector spaces. The symbol $R$ will stand for the set of real numbers. An ordered vector space is a vector space $L$ together with an order relation $\geq$ which is compatible with the algebraic structure of $L$ in the following manner:

(i) $f \geq g$ in $L$ implies $f + h \geq g + h$ for all $h \in L$; and
(ii) $f \geq g$ in $L$ implies $xf \geq zg$ for all $x \geq 0$.

The set $L^+ = \{f \in L : f \geq 0\}$ is called the positive cone of $L$, and its members are called the positive elements of $L$. An ordered vector space $L$ which is also a lattice is referred to as a Riesz space (or a vector lattice).

Typical examples of Riesz spaces are provided by the function spaces. A function space $L$ is a vector space of real-valued functions defined on a non-empty set $\Omega$ such that for each $f, g \in L$ the two functions $f \vee g$ and $f \wedge g$, defined by

$$f \vee g(\omega) = \max\{f(\omega), g(\omega)\} \quad \text{and} \quad f \wedge g(\omega) = \min\{f(\omega), g(\omega)\},$$
belong to $L$. Clearly, every function space $L$ under the ordering $f \geq g$ whenever $f(\omega) \geq g(\omega)$ for all $\omega \in \Omega$, is a Riesz space. Also, $f \geq 0$ in $L$ means $f(\omega) \geq 0$ for all $\omega \in \Omega$. Here are some examples of function spaces:

1. $\mathbb{R}^\Omega$, all real-valued functions on a set $\Omega$;
2. $C(\Omega)$, all continuous functions on a topological space $\Omega$;
3. $C_b(\Omega)$, all bounded continuous functions on a topological space $\Omega$;
4. $L^\infty(\Omega)$, all bounded real-valued functions on a set $\Omega$;
5. $L_p(1 \leq p < \infty)$, all sequences $(x_1, x_2, \ldots)$ with $\sum_{n=1}^{\infty} |x_n|^p < \infty$;
6. $l^\infty$, all bounded sequences.

Let $L$ be a Riesz space. Then for each $f \in L$ we put

$$f^+ = f \vee 0, \quad f^- = (-f) \vee 0, \quad |f| = f \vee (-f).$$

The element $f^+$ is called the positive part of $f$, $f^-$ the negative part, and $|f|$ the absolute value of $f$. The following identities hold:

$$f^+ \wedge f^- = 0, \quad f = f^+ - f^-, \quad |f| = f^+ + f^-.$$

In particular, every element of $L$ can be written as a difference of two positive elements. Every set of the form $[-u, u] = \{ f \in L : -u \leq f \leq u \}$ ($u \in L^+$) is called an order interval of $L$. A linear functional $\phi : L \to \mathbb{R}$ is said to be order bounded whenever $\phi$ maps order intervals of $L$ onto bounded subsets of $\mathbb{R}$; i.e., whenever $\phi([-u, u])$ is a bounded subset of $\mathbb{R}$ for each $u \in L^+$. Clearly, the set of all order bounded linear functionals on $L$ is a vector space. This vector space is called the order dual of $L$, and is denoted by $L^\sim$. In $L^\sim$ an ordering $\geq$ is introduced by saying that $\phi \geq \psi$ whenever $\phi(f) \geq \psi(f)$ holds for all $f \in L^+$. Under this ordering $L^\sim$ becomes an ordered vector space. In fact, $L^\sim$ is a Riesz space; see Aliprantis–Burkinshaw (1978, theorem 3.3, p. 20 or 1981, theorem 24.2, p. 189). The lattice operations of $L^\sim$ are given by

$$\phi^+(f) = \sup \{ \phi(g) : 0 \leq g \leq f \},$$

$$\phi^-(f) = \sup \{ -\phi(g) : 0 \leq g \leq f \},$$

$$|\phi| (f) = \sup \{ \phi(g) : -f \leq g \leq f \}.$$

Let $L_+^\sim$ denote the positive cone of $L^\sim$. It should be clear that $L_+^\sim$ consists precisely of all linear functionals $\phi : L \to \mathbb{R}$ for which $\phi(f) \geq 0$ holds for all $f \in L^+$. The members of $L_+^\sim$ are called the positive linear functionals on $L$. Every order bounded linear functional on $L$ can be written as a difference of two positive linear functionals on $L$. A positive linear functional $\phi \in L_+^\sim$ is
said to be strictly positive, in symbols $\phi \gg 0$, whenever $f > 0$ in $L$ (i.e., $f \geq 0$ and $f \neq 0$) implies $\phi(f) > 0$.

An ideal $A$ of a Riesz space $L$ is a vector subspace of $L$ such that $|f| \leq |g|$ and $g \in A$ imply $f \in A$. Every ideal $A$ is a Riesz subspace, i.e., $f, g \in A$ implies that $f \vee g$ and $f \wedge g$ both belong to $A$.

We now turn our attention to topological concepts on Riesz spaces. A seminorm $\| \cdot \|$ on a Riesz space $L$ is said to be a lattice (or a Riesz) seminorm whenever

$$|f| \leq |g| \text{ in } L \text{ implies } \|f\| \leq \|g\|.$$  

A locally convex-solid Riesz space $(L, \tau)$ is a Riesz space $L$ equipped with a Hausdorff linear topology $\tau$ that is generated by a family of lattice seminorms. In a locally convex-solid Riesz space the lattice operations are $\tau$-continuous; for instance,

$$f \xrightarrow{s} f \text{ implies } f \xrightarrow{+} f^+.$$  

Recall that the vector space of all $\tau$-continuous linear functionals on a topological vector space $(L, \tau)$ is known as the topological dual of $(L, \tau)$, and is denoted by $L^\prime$. A locally convex topology $t$ on $L$ is compatible with the dual system $\langle L, L' \rangle$ if $(L, t)$ has topological dual $L$. The Mackey topology $\tau(L, L')$ on $L$ is the finest locally convex topology on $L$ compatible with $\langle L, L' \rangle$. The weak topology $\sigma(L, L')$ is the coarsest locally convex topology on $L$ compatible with $\langle L, L' \rangle$. It is well known that a locally convex topology $t$ on $L$ is compatible with $\langle L, L' \rangle$ if and only if $\sigma(L, L) \subseteq t \subseteq \tau(L, L')$ holds. The weak topology $\sigma(L^\prime, L)$ on $L^\prime$ will be denoted by $w^\prime$.

Regarding topological duals of locally convex-solid Riesz spaces the following results holds. For a proof see Aliprantis–Burkinshaw (1978, theorem 5.7, p. 36).

**Theorem 2.1.** The topological dual $L'$ of a locally convex-solid Riesz space $(L, \tau)$ is an ideal in its order dual $L^\prime$, that is, $|\psi| \leq |\phi|$ and $\phi \in L'$ imply $\psi \in L'$. In particular, $L'$ is a Riesz space in its own right.

The concept of a 'Riesz dual system' will be the basis for our economic model, and is defined as follows:

A Riesz dual system $\langle L, L' \rangle$ is a pair of Riesz spaces, where $L'$ is the topological dual of $L$ for some Hausdorff locally convex-solid topology.

It is well known that a vector $x \in \mathbb{R}^n$ is a positive vector if and only if $p \cdot x \geq 0$ holds for all $p \in \mathbb{R}^n$. As we shall see, this characterization of the
positive elements holds true in any Riesz dual system. The following 'Riesz space' theorem plays an essential role in our existence proofs.

**Theorem 2.2.** Let \( \langle L, L' \rangle \) be a Riesz dual system, and let \( f \in L \). Then \( f \geq 0 \) holds if and only if we have \( \phi(f) \geq 0 \) for all \( 0 \leq \phi \in L' \).

**Proof.** Assume that \( \phi(f) \geq 0 \) holds for all \( \phi \in L' \). Fix \( 0 \leq \psi \in L \). Combining the above Theorem 2.1 with Theorem 3.5 in Aliprantis–Burkinshaw (1978, p. 22) we see that

\[
\psi(f) = \sup \{-\phi(f) : \phi \in L' \text{ and } 0 \leq \phi \leq \psi\} \geq 0.
\]

On the other hand, \( \psi(f) \geq 0 \) holds trivially, and so, \( \psi(f) = 0 \) holds for all \( 0 \leq \psi \in L \). Since \( L \) is a Riesz space, this implies \( \psi(f) = 0 \) for all \( \psi \in L' \), from which it follows that \( f = f^* - f^- \geq 0 \), as desired. ■

An important density property of the Riesz dual systems is described next.

**Theorem 2.3.** If \( \langle L, L' \rangle \) is a Riesz dual system, then \( L_+ \) is w*-dense in \( L_+ \).

**Proof.** Let \( L_+ \) be the w*-closure of \( L_+ \) in \( L_+ \). Clearly, \( L_+ \subseteq L_+ \). Assume, by way of contradiction, that there exist some \( f \in L_+ \). Then by the Hahn–Banach theorem there exist \( c \in \mathbb{R} \) and a w*-continuous linear functional on \( L \) (i.e., some \( f \in L \)) satisfying \( \psi(f) < c \) and \( \psi(f) \geq c \) for all \( \psi \in L_+ \). Since \( \psi \in L_+ \), holds for all \( \psi \in L_+ \), and all \( \psi \geq 0 \), it follows that \( c \geq 0 \) and \( \psi(f) \geq 0 \) for all \( \psi \in L_+ \). By Theorem 2.2, we have \( f \geq 0 \). But then \( 0 \leq \psi(f) \leq c \leq 0 \) holds, which is impossible. Therefore, \( L_+ = L_+ \) holds. ■

A special class of locally convex-solid Riesz spaces is the class of Banach lattices. A Riesz space that under a lattice norm is a complete metric space is referred to as a **Banach lattice**. Some important examples of Banach lattices are:

(a) Euclidean spaces \( \mathbb{R}^n \);
(b) \( C(\Omega) \) spaces, \( \Omega \) compact, with the sup norm \( \|f\|_\infty = \sup \{|f(\omega)| : \omega \in \Omega\} \);
(c) \( L_p(\mu) \) spaces \( (1 \leq p < \infty) \) under the norm \( \|f\|_p = \left( \int |f|^p d\mu \right)^{1/p} \);
(d) \( L_\infty(\mu) \) spaces under the essential sup norm;
(e) \( C_0(\Omega) \) spaces under the sup norm.

The Banach lattices have a number of remarkable properties. For instance, a given Riesz space admits at most one lattice norm (up to an equivalence, of course) that makes it a Banach lattice. (The sup norm, for example, is the only lattice norm that makes \( C[0,1] \) a Banach lattice.) On the other hand, every positive linear functional on a Banach lattice is norm continuous; see Aliprantis–Burkinshaw (1981, theorem 24.10, p. 192). Thus, if \( L \) is a Banach lattice, then its norm dual \( L^* \) coincides with its order dual \( L^- \) (i.e., \( L^* = L^- \)).
holds), and moreover $L^*$ is a Banach lattice. In particular, for any Banach lattice $L$ the dual system $\langle L, L^* \rangle$ is in fact a Riesz dual system. However, not every Riesz dual system comes from a Banach lattice. For example, the important Riesz dual systems $\langle L_0(\mu), L_1(\mu) \rangle$ and $\langle C_0(\Omega), M_1 \rangle$ (both discussed in section 4) are not generated by Banach lattices.

Now let $L$ be a Banach lattice. Then there exists a natural embedding $f \mapsto \hat{f}$ of $L$ into its double norm dual $L^{**}$ defined by

$$\hat{f}(\phi) = \phi(f) \text{ for all } f \in L \text{ and } \phi \in L^*.$$  

The embedding is linear, norm, and lattice preserving. Thus, $L$ is a Banach sublattice of the Banach lattice $L^{**}$. For more about Banach lattices the reader can consult Schaefer (1974).

3. Equilibrium theorems

As we have said before, our economic model will be based upon the concept of a Riesz dual system. In this section we discuss this model and derive the basic equilibrium theorems.

Let $\langle L, L' \rangle$ be a Riesz dual system. A price simplex for $\langle L, L' \rangle$ is a non-empty convex $w^*$-compact subset $A$ of $L_+^*$ (the members of which are called prices) satisfying the following two properties:

(a) the convex set $S$ of all strictly positive prices, i.e., the convex set

$$S = \{ p \in L_+^* \cap A : p \not\geq 0 \},$$

is $w^*$-dense in $A$; and

(b) the cone generated by $S$ (i.e., the set $\bigcup_{\lambda \geq 0} \lambda S$) is $w^*$-dense in $L_+^*$.

Now let $A$ be a price simplex for a Riesz dual system $\langle L, L' \rangle$. An excess demand function $E$ for $A$ is a mapping $p \mapsto E_p$, from a convex subset $D (= \text{dom } E)$ of $L_+^*$ into $L$, satisfying the following four properties:

1. Density Condition: $D$ is a $w^*$-dense subset of $A$.
2. Continuity Condition: There exists a locally convex topology $t$ on $L$ for which every functional in $D$ is $t$-continuous and $E:(D, w^*) \to (L, t)$ is continuous.
3. Boundary Condition: If a net $\{ p_n \} \subseteq D$ satisfies $p_n \overset{w^*}{\to} q \in A \sim D$, then $\lim p(E_{p_n}) = 0$ holds for some $p \in D$.
4. Walras’ Law: $p(E_p) = 0$ holds for all $p \in D$.

We now come to the definition of an economy.

**Definition 3.1.** An economy $\mathcal{E}$ is a triplet $(\langle L, L' \rangle, \Delta, E)$ where $\Delta$ is a price simplex for the Riesz dual system $\langle L, L' \rangle$ and $E$ is an excess demand function for $\Delta$.
According to our definition, the \( w^* \)-compact convex set,

\[
\Delta = \{ p \in l_2^+ : p \geq 0 \text{ and } \|p\|_2 \leq 1 \},
\]

is a price simplex for the Riesz dual system \( \langle l_2, l_2^* \rangle \). Now if \( u, v \in l_2^+ \) are two fixed elements, then the function \( E: (\Delta, w^*) \to l_2^* \), defined by \( E_p = p(u)v - p(v)u \), is continuous, and from this it follows that \( \langle l_2^*, l_2 \rangle, \Delta, E \rangle \) is an economy in our sense. However, since \( 0 \in \Delta \), and \( 0 \) is an equilibrium price for this excess demand function, this example is of little economic significance. In section 4, we shall consider ‘flat’ simplices which exclude \( 0 \).

We continue by recalling a few definitions about binary relations. Let \( > \) be a binary relation on a convex set \( D \) of a topological vector space \( (L, \tau) \). As usual, for each \( p \in D \) we write

\[
(-\infty, p) = \{ q \in D : p > q \} \quad \text{and} \quad (p, \infty) = \{ q \in D : q > p \}.
\]

The binary relation \( > \) is said to be:

1. **irreflexive**, whenever \( p \not> p \) holds for all \( p \in D \);
2. **convex valued**, whenever \( (p, \infty) \) is convex for each \( p \in D \); and
3. **\( \tau \)-upper semicontinuous**, whenever \( (-\infty, p) \) is \( \tau \)-open in \( D \) for each \( p \in D \).

An element \( p \in D \) is said to be a **maximal element** for \( > \) whenever \( q \not> p \) holds for all \( q \in D \).

For the rest of the discussion in this section \( \mathcal{E} = (\langle L, L^* \rangle, \Delta, E) \) will be a fixed economy. The economy \( \mathcal{E} \) defines a binary relation \( > \) on \( D \), which will be referred to as the **revealed preference relation**, as follows: If \( p, q \in D \), then we say that \( p \) dominates \( q \) (in symbols, \( p > q \)) whenever \( p(E_q) > 0 \) holds. [The definition of \( > \) was suggested by Nikaidō's (1968, p. 297) discussion of economies with gross substitutability.]

The properties of the revealed preference relation are included in the next theorem.

**Theorem 3.2.** The revealed preference relation for an arbitrary economy is irreflexive, convex valued, and \( w^* \)-upper semicontinuous.

**Proof.** Since \( p(E_p) = 0 \) holds for all \( p \in D \), we see that \( p \not> p \) for each \( p \in D \), and so, \( > \) is irreflexive.

To see that \( > \) is convex valued, let \( q_1, q_2 \in (p, \infty) \) and \( 0 < \alpha < 1 \). Thus, \( q_1(E_p) > 0 \) and \( q_2(E_p) > 0 \) both hold, and so, if \( q = \alpha q_1 + (1 - \alpha)q_2 \), then \( q(E_p) = \alpha q_1(E_p) + (1 - \alpha)q_2(E_p) > 0 \). That is, \( q > p \) holds, proving that \( (p, \infty) \) is a convex set.

Finally, let us establish that \( > \) is \( w^* \)-upper semicontinuous. To this end,
let \( q \in (-\infty, p) \). This means that \( p(E_q) > 0 \). Since \( p \) is a \( t \)-continuous linear functional on \( L \), it follows that the function \( f(t) = p(E_t) \) from \( (D, w^*) \) into \( \mathbb{R} \) is continuous. Consequently, there exists a \( w^* \)-neighborhood \( V \) of \( q \) such that \( r \in D \cap V \) implies \( p(E_r) > 0 \) (i.e., \( p > r \)). Therefore, \( q \) is an interior point of \( (-\infty, p) \), and hence, \( (-\infty, p) \) is an open subset of \( (D, w^*) \). ■

Recall that an element \( p \in D \) is said to be a free disposal equilibrium price whenever \( E_p \leq 0 \) holds.

The theory of Riesz spaces allows us to characterize the free disposal equilibrium prices as follows.

**Theorem 3.3.** For an element \( p \in D \) the following statements are equivalent:

1. \( p \) is a free disposal equilibrium price;
2. \( p \) is a maximal element for the revealed preference relation \( \succ \); and
3. \( q(E_p) \leq 0 \) holds for all \( q \in D \).

**Proof.**

(1)\( \Rightarrow \) (2) Let \( E_p \leq 0 \). Then \( q(E_p) \leq 0 \) holds for all \( q \in D \), and so, \( q \succ p \) for all \( q \in D \). That is, \( p \) is a maximal element for \( \succ \).

(2)\( \Rightarrow \) (3) Obvious.

(3)\( \Rightarrow \) (1) Since \( D \) is \( w^* \)-dense in \( A \), statement (3) implies that \( q(E_p) \leq 0 \) holds for all \( q \in A \). From this it follows easily that \( q(E_p) \leq 0 \) holds for all \( q \in L^*_+ \). But then Theorem 2.2 shows that \( E_p \leq 0 \). Thus, \( p \) is a free disposal equilibrium price, and the proof of the theorem is finished. ■

It is well known that proofs of equilibrium existence theorems require some fixed point argument, and our approach is no exception. We shall use the following generalization (due to K. Fan) of the classical Knaster–Kuratowski–Mazurkiewicz theorem; for a proof see Fan (1961, lemma 1, p. 305).

**Lemma 3.4.** Let \( X \) be an arbitrary set in a topological vector space \( Y \). To each \( x \in X \), let a closed set \( F(x) \) in \( Y \) be given such that the following two conditions are satisfied:

(a) The convex hull of any finite subset \( \{x_1, \ldots, x_n\} \) of \( X \) is contained in \( \bigcup_{i=1}^n F(x_i) \); and

(b) \( F(x) \) is compact for at least one \( x \in X \).

Then \( \bigcap_{x \in X} F(x) \neq \emptyset \).

The following theorem will be of fundamental importance. As usual, in a vector space, \( \text{co} A \) denotes the convex hull of the set \( A \).
Theorem 3.5. Let $L$ be a Hausdorff topological vector space, and let $D$ be a non-empty, convex, and $\tau$-compact subset of $L$. If $\succ$ is an irreflexive, convex valued, and upper semicontinuous binary relation on $D$, then the set of all maximal elements of $\succ$ is non-empty and $\tau$-compact.

Proof. Let $F(p) = D \sim (-\infty, p)$. Since $p \notin (-\infty, p)$, it follows that $F(p)$ is non-empty for all $p \in D$. Also, by the upper semicontinuity of $\succ$, we see that each $F(p)$ is $\tau$-closed, and hence, $\tau$-compact. Now it is a routine matter to verify that $\bigcap_{p \in D} F(p)$ is precisely the set of all maximal elements of $\succ$. Thus it remains to be shown that $\bigcap_{p \in D} F(p)$ is non-empty. To establish this, it is enough to prove that condition (a) of Lemma 3.4 is satisfied.

To this end, let $p_1, \ldots, p_n \in D$. Then we claim that $\cap_{i=1}^n F(p_i)$ holds. Indeed, if $q = \sum_{i=1}^n x_i p_i$ is a convex combination and $q \notin \bigcup_{i=1}^n F(p_i)$, then $p_i \succ q$ holds for all $i = 1, \ldots, n$, and so (since $\succ$ is convex valued) we must have $q = \sum_{i=1}^n x_i p_i \succ q$, contrary to the irreflexivity of $\succ$. Thus, condition (a) of Lemma 3.4 is satisfied, and so, $\bigcap_{p \in D} F(p) \neq \emptyset$, as desired. ■

We now come to one of the main results of this paper.

Theorem 3.6. Every economy $\mathcal{E}$ has a non-empty $w^*$-compact set of free disposal equilibrium prices.

Proof. We first show that the set of free disposal equilibrium prices is non-empty. Let $\mathcal{A}$ denote the collection of all finite subsets of $D$. For each $\alpha \in \mathcal{A}$, let $D_\alpha$ denote the convex hull of $\alpha$. Clearly, each $D_\alpha$ is $w^*$-compact, and $\bigcup_{\alpha \in \mathcal{A}} D_\alpha = D$ holds. Also, the collection $\{D_\alpha\}$ is directed upwards by inclusion. Now let $\alpha \in \mathcal{A}$ be fixed. By Theorem 3.2 the revealed preference relation $\succ$ restricted to $D_\alpha$ is irreflexive, convex valued, and $w^*$-upper semicontinuous. Hence, by Theorem 3.5, there exists some $p_\alpha \in D_\alpha$ that is maximal for $\succ$ on $D_\alpha$ (i.e., $p(E_{p_\alpha}) \leq 0$ holds for each $p \in D_\alpha$). Next consider the net $\{p_\alpha : \alpha \in \mathcal{A}\}$, where $\mathcal{A}$ is directed by the inclusion $\supseteq$. Since $\mathcal{A}$ is $w^*$-compact, we can assume (by passing to a subnet if necessary) that $p_\alpha \to q$ holds in $\mathcal{A}$.

We claim that $q \in D$. Indeed, if $q \notin D$, then by our boundary condition there exists some $p \in D$ with $\lim p(E_{p_\alpha}) > 0$. On the other hand, $p$ must lie in some $D_\alpha$; and since $\{D_\alpha\}$ is directed upwards, there exists some $\beta \in \mathcal{A}$ so that $p \in D_\beta$ holds for all $\alpha \supseteq \beta$. But then for each $\alpha \supseteq \beta$ we have $p(E_{p_\alpha}) \leq 0$, and so, $\lim p(E_{p_\alpha}) \leq 0$ holds, which is a contradiction. Thus, $q \in D$.

We now establish that $q$ is a free disposal equilibrium price. To this end, let $p \in D$. Since the function $f(r) = p(E_r)$, from $(D, w^*)$ into $R$, is continuous, it

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1The argument in this proposition is due to Sonnenschein (1971), who used it to prove Theorem 3.5 for a finite-dimensional topological vector space. We thank Kim Border for calling our attention to Sonnenschein's theorem.
follows that \( p(E_\beta) = \text{w*}-\text{lim}_n p(E_{\beta_n}) \). As above, we see that there exists some \( \beta \in \mathcal{A} \) satisfying \( p(E_{\beta_n}) \leq 0 \) for all \( \alpha \leq \beta \), and so, \( p(E_{\beta}) \leq 0 \) holds for all \( \beta \in D \). By Theorem 3.3, \( q \) is a free disposal equilibrium price.

Finally, let us show that the set of all free disposal equilibrium prices is \( \text{w*}-\text{compact} \). It is enough to show that this set is \( \text{w*}-\text{closed} \) in \( \mathcal{A} \). So, let \( \{p_\alpha\} \) be a net of free disposal equilibrium prices satisfying \( p_\alpha \to q \in \mathcal{A} \). If \( q \in \mathcal{A} \sim D \), then our boundary condition implies the existence of some \( \beta \in D \) with \( \lim p(E_{\beta_n}) > 0 \). On the other hand, since each \( p_\alpha \) is a maximal element, \( p(E_{\beta_n}) \leq 0 \) holds for each \( \alpha \), which contradicts \( \lim p(E_{\beta_n}) > 0 \). Thus, \( q \in D \). From the continuity of the function \( f(r) = p(E_r) \) we get \( p(E_{\beta}) \leq 0 \) for all \( \beta \in D \), and so, by Theorem 3.3, \( q \) is a free disposal equilibrium price. The proof of the theorem is now complete.

A strictly positive price \( \beta \in D \) is said to be an equilibrium price for the economy \( \mathcal{E} \) whenever \( E_\beta = 0 \) holds. The following fundamental theorem guarantees the existence of equilibrium prices.

**Theorem 3.7.** Every economy \( \mathcal{E} \) with \( D = S \) has a non-empty \( \text{w*}-\text{compact} \) set of equilibrium prices.

**Proof.** If \( D = S \) holds, then we claim that \( \beta \in S \) satisfies \( E_\beta \leq 0 \) if and only if \( E_\beta = 0 \). Indeed, if \( E_\beta \leq 0 \) holds, then either \( E_\beta < 0 \) or \( E_\beta = 0 \). If \( E_\beta < 0 \) holds, then (since \( \beta \in S \)) we have \( p(E_\beta) < 0 \), which contradicts Walras' law. Consequently, \( E_\beta = 0 \).

Thus, an element \( \beta \in D \) is a free disposal equilibrium price if and only if \( \beta \) is an equilibrium price. The conclusion now follows from Theorem 3.6.

Finally, we note in passing that if the excess demand function \( E \) satisfies Samuelson's weak axiom of revealed preference, i.e., \( p, q \in D \) and \( p(E_q) \leq 0 \) implies \( q(E_p) > 0 \), then the economy \( \mathcal{E} \) has precisely one free disposal equilibrium price. Indeed, note first that Samuelson's axiom is equivalent to the statement: if \( p, q \in D \) satisfy \( p \not\succ q \), then \( q \succ p \) holds. Thus, if \( p, q \in D \) are two distinct free disposal equilibrium prices, then by the maximality of \( p \) and \( q \) we have \( p \not\succ q \) and \( q \not\succ p \), which contradicts Samuelson's axiom.

4. Examples

First, we present examples of price simplices. Most of our examples are AM-spaces with unit. A Banach lattice \( L \) is said to be an AM-space (abstract M-space) whenever for each \( f, g \in L^+ \) we have

\[
\|f \vee g\| = \max \{\|f\|, \|g\|\}.
\]
(The letter M stands for maximum.) The element $e > 0$ in an AM-space $L$ is said to be the unit of $L$ whenever the order interval $[-e, e]$ coincides with the closed unit ball of $L$, i.e., whenever \([-e, e] = \{ f \in L : \| f \| \leq 1 \}\) holds. Here are some examples of AM-spaces with units:

1. The Riesz space $C_b(\Omega)$ of all continuous bounded real-valued functions on a topological space $\Omega$, with the sup norm $\| f \|_\infty = \sup \{|f(\omega)| : \omega \in \Omega\}$. The unit $e$ is the constant function one on $\Omega$.
2. The Riesz space $L_\infty(\mu)$ of all essentially bounded measurable functions on a measure space $(X, \Sigma, \mu)$, with the essential sup norm $\| f \|_\infty = \text{ess sup} |f|$.
   The constant function one is the unit.
3. The Riesz space $l_\infty$ of all bounded real sequences with the sup norm.
   Again, the constant sequence one is the unit.

The Banach lattice $c_0$ of all null sequences with the sup norm is an AM-space without a unit.

Let $L$ be an AM-space with unit $e$. Then it is easy to check that for every positive linear functional $p$ on $L$ we have $\| p \| = p(e)$. This implies that the set

$$\Delta = \{ p \in L_*^\infty : \| p \| = 1 \}$$

is convex and $w^*$-closed, and hence, by Alaoglu's theorem $w^*$-compact. As we shall see, $\Delta$ will be our primary price simplex, which (in view of $0 \notin \Delta$) is of economic importance.

**Theorem 4.1.** Let $\langle L, L' \rangle$ be a Riesz dual system with $L$ an AM-space with unit. If $L'_+\,$ has a strictly positive functional, then

$$\Delta = \{ p \in L_*^\infty : \| p \| = 1 \}$$

is a price simplex for $\langle L, L' \rangle$.

**Proof.** Let $e > 0$ be the unit of $L$. It is enough to show that the convex set

$$S = \{ p \in L'_+ : p \geq 0 \}$$

is $w^*$-dense in $\Delta$. To this end, fix some $p \in S$, and let $\bar{S}$ denote the $w^*$-closure of $S$ in $L_*^\infty$. Clearly, $\bar{S} \subseteq \Delta$.

Let $q \in L'_+ \cap \Delta$. Then $zp + (1 - z)q \in S$ for all $0 < z < 1$. Since $q = w^* - \lim_{z \to 0^+} [zp + (1 - z)q]$, we see that $q \in \bar{S}$. Therefore $L'_+ \cap \Delta \subseteq \bar{S}$.

Now let $q \in \Delta$. Since (by Theorem 2.3) $L'_+$ is $w^*$-dense in $L_*^\infty$, there exists a net $\{ p_\alpha \} \subseteq L'_+$ with $p_\alpha \to w^* q$. We can assume $p_\alpha \neq 0$ for all $\alpha$. From $\lim p_\alpha(e) = q(e) = 1$, $\{ p_\alpha / p_\alpha(e) \} \subseteq L'_+ \cap \Delta \subseteq \bar{S}$, and $q = w^* - \lim [p_\alpha / p_\alpha(e)]$, it follows that $q \in S$. Hence, $S = \Delta$ holds, and the proof of the theorem is finished. ■
The specific examples of price simplices we have in mind are all special cases of the preceding theorem. The finite-dimensional case comes first.

**Example 4.2.** The convex compact set

\[ A = \left\{ (p_1, \ldots, p_n) \in \mathbb{R}^n : p_i \geq 0 \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad \sum_{i=1}^{n} p_i = 1 \right\} \]

is a price simplex for the Riesz dual system \( \langle \mathbb{R}^n, \mathbb{R}^n \rangle \).

To see this, consider \( \mathbb{R}^n \) as an AM-space with unit. [The norm is \( \|x\|_\infty = \max \{ |x_i| : i = 1, \ldots, n \} \) and the unit \( e = (1, \ldots, 1) \).] The rest follows from Theorem 4.1 by observing that if \( p = (p_1, \ldots, p_n) \geq 0 \), then in the norm dual of \( (\mathbb{R}^n, \| \cdot \|_\infty) \) we have \( \|p\| = p(e) = \sum_{i=1}^{n} p_i \). Also, note that

\[ S = \left\{ (p_1, \ldots, p_n) \in \mathbb{R}^n : p_i > 0 \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad \sum_{i=1}^{n} p_i = 1 \right\} \]

The next case is that of \( C(\Omega) \) with \( \Omega \) a compact metric space.

**Example 4.3.** If \( L = C(\Omega) \) for some compact metric space, then the convex set

\[ A = \{ p \in L^*_+ : \|p\| = 1 \} \]

is a price simplex for the Riesz dual system \( \langle L, L^* \rangle \), where \( L^* \) is the norm dual of \( C(\Omega) \) equipped with the sup norm.

According to Theorem 4.1 it is enough to show that there exists a strictly positive linear functional on \( L \). To this end, start by observing that since \( \Omega \) is a compact metric space, \( C(\Omega) \) is a separable Banach lattice; see, for instance, Schaefer (1974, proposition 7.5, p. 105). This implies that \( B^*_+ = \{ p \in L^*_+ : \|p\| \leq 1 \} \) with the w*-topology is a compact metric space [Dunford–Schwartz (1958, theorem 1, p. 426)], and in particular, that \( (B^*_+, \text{w}^*) \) is separable. Fix a countable w*-dense subset \( \{ p_1, p_2, \ldots \} \) of \( B^*_+ \), and let \( p = \sum_{n=1}^{\infty} 2^{-n} p_n \in L^*_+ \). Then \( p \) is strictly positive on \( L \).

For if \( p(f) = 0 \) holds for some \( f \geq 0 \), then \( p_n(f) = 0 \) likewise holds for all \( n \), and so, by the w*-denseness of \( \{ p_1, p_2, \ldots \} \) in \( B^*_+ \) we get \( p(f) = 0 \) for all \( p \in B^*_+ \), from which it follows that \( f = 0 \). This implies that \( p \) is strictly positive on \( L \).

In our next example, the commodity space is \( L_\sigma(\mu) \) for a \( \sigma \)-finite measure space \( (X, \Sigma, \mu) \). It is a fact that \( L_1(\mu) \) is an ideal in its second norm dual — see Theorem 9.2 on p. 61 in Aliprantis–Burkinshaw (1978) — and from this
and Theorem 6.6 on p. 40 it follows that \(\langle L_\infty(\mu), L_1(\mu) \rangle\) is a Riesz dual system. Another important feature of the Riesz dual system \(\langle L_\infty(\mu), L_1(\mu) \rangle\) is the following: The Mackey topology \(\tau(L_\infty(\mu), L_1(\mu))\) is a locally convex-solid topology [Aliprantis–Burkinshaw (1978, p. 163)]. This property will be used later.

Example 4.4. If \((X, \Sigma, \mu)\) is a \(\sigma\)-finite measure space, then the convex set

\[ A = \{ p \in L^*_\infty(\mu); p \geq 0 \text{ and } \|p\| = 1 \} \]

is a price simplex for the Riesz dual system \(\langle L_\infty(\mu), L_1(\mu) \rangle\).

According to Theorem 4.1 we have to show that some \(g \in L_1(\mu)\) is strictly positive on \(L_\infty(\mu)\). To this end, pick a disjoint sequence \(\{A_n\}\) of \(\Sigma\) satisfying \(\bigcup_{n=1}^\infty A_n = X\) and \(\mu(A_n) < \infty\) for each \(n\). Choose \(0 < \lambda_n \leq 1\) with \(\lambda_n \mu(A_n) < 2^{-n}\), and then let \(g = \sum_{n=1}^\infty \lambda_n \chi_{A_n} \in L_1(\mu)\). Now note that linear functional

\[ p(f) = \int fg \, d\mu, \quad f \in L_\infty(\mu), \]

is strictly positive on \(L_\infty(\mu)\). □

An important special case of the previous example is the Riesz dual system \(\langle L_\infty, L_1 \rangle\).

Now let us consider a completely regular Hausdorff topological space \(\Omega\). Sentilles (1972) defined a notion of a topology \(\beta\) on \(C_0(\Omega)\) which extends the notion of the strict topology introduced by Buck (1958) for locally compact \(\Omega\). The strict topology \(\beta\) has the following properties (a), (b) and (c) as was shown by Sentilles on pp. 328, 327 and 332, respectively:

(a) \(\beta\) is the finest locally convex topology on \(C_0(\Omega)\) for which Dini's property holds [i.e., for any net \(\{f_\alpha\} \subseteq C_0(\Omega)\) such that \(f_\alpha(\omega) \to 0\) for each \(\omega \in \Omega\) and \(f_\alpha(\omega) \geq f_\gamma(\omega)\) for \(\alpha \geq \gamma\) and all \(\omega \in \Omega\) imply \(f_\alpha \leq 0\)];

(b) \(\beta\) is locally solid; and

(c) when \(\Omega\) is either \(\sigma\)-compact or complete separable metric, then the topological dual of \(\langle C_0(\Omega), \beta \rangle\) is precisely the vector space \(M_1\) of all linear functionals on \(C_0(\Omega)\) that are representable as an integral with respect to a unique compact-regular Borel measure on \(\Omega\).

Thus, when \(\Omega\) is a complete separable metric space, then \(\langle C_0(\Omega), M_1 \rangle\) is a Riesz dual system.
Example 4.5. If $\Omega$ is a complete separable metric space, then the convex set

$$\Delta = \{ p \in C_0^*(\Omega); p \geq 0 \text{ and } \|p\| = 1 \}$$

is a price simplex for the Riesz dual system $\langle C_0(\Omega), M_i^* \rangle$.

The proof goes as follows: By Theorem 14 on p. 192 and 18 on p. 194 in Varadarajan (1961), $(M_i^*, w^*)$ is complete, separable, and metrizable. If $\{p_1, p_2, \ldots \}$ is a countable $w^*$-dense subset of $M_i^* \sim \{0\}$, then

$$p = \sum_{n=1}^{\infty} 2^{-n} (p_n/\|p_n\|) \in M_i^+$$

is strictly positive on $C_0(\Omega)$ (see the corresponding proof in Example 4.3), and our conclusion follows. ■

An interesting special case of the preceding example is when $\Omega = C[0,1]$ with the topology generated by the sup norm.

Finally, we close this section with a few examples of economies.

Example 4.6. Consider the Riesz dual system $\langle R^n, R^n \rangle$ of Example 4.2 with the price simplex

$$\Delta = \left\{ (p_1, \ldots, p_n) \in R^n : \sum_{i=1}^{n} p_i = 1 \right\}.$$

In his section 3 Debreu (1981) derives the excess demand correspondence for an economy having a finite number of profit-maximizing firms and a finite number of utility-maximizing consumers. If, in addition to his assumptions, we suppose that production sets are strictly convex and that utility functions are strictly quasi-concave, then the excess demand correspondence is a function, which we denote by $E$. Debreu shows that the domain of $E$ is $\Delta$; $E$ is continuous; and that $E$ satisfies Walras' law. Hence, $(\langle R^n, R^n \rangle, \Delta, E)$ is an economy in our sense, and so, by Theorem 3.6 there exists some $p \in \Delta$ with $E_p \leq 0$ and $p(E_p) = 0$. This proves Debreu's Theorem 6 for the special case where the excess demand correspondence is a function. ■

Example 4.7. Again consider the Riesz dual system $\langle R^n, R^n \rangle$ with $\Delta$ and $S$ as in Example 4.2.

Consider $R^n$ as the commodity space of an exchange economy having a finite number of consumers whose consumption sets are $R^n_+$, and we assume
that agents are utility maximizers having quasi-concave, continuous, monotone utility functions. Again Debreu (1981) shows that for this economy the excess demand function $E$ has as its domain $S$; that $E$ is continuous; and that it satisfies Walras’ law. Moreover, $E$ is bounded from below (i.e., there exists some $x \in \mathbb{R}^n$ satisfying $x \leq E_p$ for all $p \in S$), and $\|E_p\| \to \infty$ whenever $\{p_n\} \subseteq S$ satisfies $p_n \to q \in \Delta \sim S$. It is easy to see that the latter two properties of $E$ imply our boundary condition: $\{p_n\} \subseteq S$ and $p_n \to q \in \Delta \sim S$ imply $\lim p(E_{p_n}) > 0$ for some $p \in S$. Hence, by Theorem 3.7 there exists some $p \in S$ with $E_p = 0$, and this is the conclusion of Theorem 8.3 of Dierker (1974).

In the next example, we present a new proof of Bewley’s existence theorem for a finite exchange economy, with commodity space $L_x(\mu)$. Since the excess demand function need not be continuous on the price simplex, we cannot invoke Theorem 3.6 directly. Rather we use the argument in the proof of Theorem 3.6, by observing that the excess demand function restricted to the finite-dimensional subsimplices is continuous. Since a finite-dimensional simplex is norm compact, the inner product $p \cdot x$ is jointly continuous, so that Debreu’s argument demonstrating the continuity of demand functions for finite-dimensional commodity spaces, is applicable. This allows us to construct a net of prices which converges to an equilibrium price, by considering the maximal elements of the revealed preference relation restricted to the finite-dimensional subsimplices. As observed by Florenzano (1982), our argument can be generalized to commodity spaces which have a predual.

Example 4.8. Let $\langle L_x(\mu), L_1(\mu) \rangle$ be the Riesz dual system of Example 4.4, with the price simplex $\Delta = \{p \in L_x^+(\mu); p \geq 0 \text{ and } \|p\| = 1\}$. Note that $S = \{p \in L^1(\mu); p > 0 \text{ on } L_x(\mu) \text{ and } \|p\|_1 = 1\}$. For simplicity we shall write $L_x$ and $L_1$ instead of $L_x(\mu)$ and $L_1(\mu)$.

Consider $L_x$ with the Mackey topology $\tau(L_x, L_1)$ as the commodity space of an exchange economy having a finite number of consumers. Each consumer has $L_x^+$ with the relative $\tau(L_x, L_1)$-topology as his consumption set. Endowments are in $L_x^+ \sim \{0\}$. Agents maximize strictly quasi-concave, monotone, and $\tau(L_x, L_1)$-continuous preferences.

The social endowment $\omega$ is a vector of $L_x^+$ uniformly bounded away from zero. By Alaoglu’s theorem $[0, \omega]$ is $\sigma(L_x, L_1)$-compact. Consequently, each agent’s attainable consumption set is $\sigma(L_x, L_1)$-compact. We choose a $\sigma(L_x, L_1)$-compact convex subset of $L_x^+$ containing all attainable consumption sets, say the interval $[0, 2\omega]$. Consider a new exchange economy where each agent has $[0, 2\omega]$ as his consumption set and retains his original endowment and preferences. For this economy the ‘excess demand function $E$ is well defined, with domain $D = \{p \in L_1^+; \|p\|_1 = 1\}$. Note that $E$ need not be continuous on $D$. 
As in the proof of Theorem 3.6, let \( \{D_z : z \in A\} \) be the family of all finite-dimensional simplices contained in \( D \). It is easy to show that 
\( E: (D, w^*) \rightarrow (L, \tau(L, L_1)) \) is continuous, using arguments as, say, in Debreu (1981). Then the revealed preference relation \( \succ \) restricted to each \( D_z \) is irreflexive, convex valued, and \( w^* \)-upper semicontinuous (see the proof of Theorem 3.2). Thus, by Theorem 3.5, \( \succ \) has a maximal element on each \( D_z \), say, \( p_z \). Now consider the nets \( \{p_z\} \subseteq A \) and \( \{E_{p_z}\} \subseteq [-\omega, \omega] \). Since \( A \) is \( \sigma(L^*_\omega, L_\omega) \)-compact and \( [-\omega, \omega] \) is \( \sigma(L^*_\infty, L_1) \)-compact, by passing to two subnets (if necessary) we can assume that
\[
p_z \xrightarrow{\sigma(L^*_\omega, L_\omega)} p \in A \quad \text{and} \quad E_{p_z} \xrightarrow{\sigma(L^*_\infty, L_1)} Z_p \in [-\omega, \omega].
\]
Then \( p \in S \), and is an equilibrium price vector. To see this simply use the arguments in the proofs of Theorems 1 and 2 in Bewley (1972).

The next example presents a method of constructing excess demand functions.

**Example 4.9.** Let \( \langle L, L^* \rangle \) be a Riesz dual system with \( L \) an AM-space with unit \( e \), and let \( A = \{p \in L^*_\infty : p(e) = 1\} \) be a price simplex for \( \langle L, L^* \rangle \).

Fix a continuous function \( F: (A, w^*) \rightarrow (L, \|\cdot\|) \). (For instance if \( \{e_n\} \) and \( \{u_n\} \) are two sequences of \( L \) with \( \{e_n\} \) norm bounded and \( \sum \|u_n\| < \infty \), then put
\[
F_p = \sum_{a=1}^\infty p(e_a)u_a \quad \text{for each} \quad p \in A.
\]
Now it is a routine matter to verify that the function \( E: (A, w^*) \rightarrow (L, \|\cdot\|) \), defined by \( E_p = F_p - p(F_p)e \), is an excess demand function (with domain \( A \)).

Finally, for a specific example. Let \( L = C(Q) \) for some compact metric space. Fix a countable norm dense subset \( \{f_n\} \) of \( C(Q) \) consisting of non-zero functions and define \( e_n = u_n = 2^{-n}\|f_n\|_\infty \). Then it is easy to see that \( p(e)/p(F_p)F_p - e \) is an excess demand function (with domain \( L^*_\infty \sim \{0\} \)) which is homogeneous of degree zero.

Our final example uses the method of Example 4.9 to construct an excess demand function \( E \) on a simplex \( A \) where the domain \( D \) of \( E \) is a proper subset of \( A \).

**Example 4.10.** Consider the Riesz dual system \( \langle L, L_1 \rangle \) with the price simplex \( A \sim \{p \in L^*_1 : p \geq 0 \text{ and } \|p\| = 1\} \). Let \( D = \{p \in L^*_1 : \|p\| = 1\} \), \( u_1 = (1, 1, \ldots) = e \), \( u_2 = (0, 1, 1, \ldots) \), \( u_3 = (0, 0, 1, 1, \ldots) \), \( \ldots \), \( e_1 = (1, 0, 0, \ldots) \), \( e_2 = (0, 1, 0, 0, \ldots) \), \ldots

Let \( \xi: [0, 1] \rightarrow [0, \infty) \) be a strictly increasing continuous function; for instance, let \( \xi(x) = x^p \) (\( 0 < p < \infty \)), \( \xi(x) = x \), or \( \xi(x) = 1 - e^{-x} \). Also, let \( \{\lambda_n\} \) be a sequence of real numbers with \( \lambda_n > 0 \) for all \( n \) and \( \sum_{n=1}^\infty \lambda_n < \infty \).
Now define \( F: \Delta \to l_\infty^* \) by

\[
F_p = \sum_{n=1}^{\infty} \frac{\lambda_n}{n} \xi[p(u_n)]e_n = (\lambda_1 \xi[p(u_1)], \lambda_2 \xi[p(u_2)], \ldots).
\]

Then it is easy to verify that the functions \( p \mapsto F_p \) from \((\Delta, w^*)\) into \((l_\infty, \| \cdot \|)\)
and \( p \mapsto p(F_p) \) from \((\Delta, w^*)\) into \(R\) are both continuous. Also, if \( p = (p_1, p_2, \ldots) \in D \), then from the inequality

\[
\lambda_n \xi[p(u_n)] = \lambda_n \xi\left(\sum_{i=n}^{\infty} p_i \right) \geq \lambda_n \xi(p_n),
\]

it follows that

\[
p(F_p) \geq \sum_{n=1}^{\infty} \lambda_n \xi(p_n) p_n > 0.
\]

Thus, the function \( E: (D, w^*) \to (l_\infty, \| \cdot \|) \), defined by

\[
E_p = F_p/p(F_p) - e,
\]

is continuous and satisfies Walras' law. In addition, we claim that it satisfies

our boundary condition, i.e., if a net \( \{ p_s \} \subseteq D \) satisfies \( p_s \rightharpoonup q \in \Delta \sim D \), then

\[
\lim p(F_{p_s}) > 0 \text{ holds for some } p \in D.
\]

To see this, let \( \{ p_s \} \subseteq D \) satisfy \( p_s \rightharpoonup q \in \Delta \sim D \). Write \( q = q_\infty + q_\omega \), with \( q_\infty \in l_1^\sim \)
and \( q_\omega \) purely finitely additive.

**Case I:** \( q_\omega = 0 \) (i.e., \( q = q_\infty \)).

Clearly, \( q(F_{p_s}) = 0 \) holds, and so, the element \( e_1 \in D \) satisfies

\[
\lim e_1(F_{p_s}) = \lim [\lambda_1 \xi(1)/p_2(F_{p_s}) - 1] = \infty.
\]

**Case II:** \( q_\omega > 0 \).

Since \( \|q_\omega\| < \|q_\infty\| + \|q_\omega\| = \|q\| = 1 \), there exists some \( p \in D \) satisfying \( q_\omega < p \).

Also, note that \( q_\omega(u_n) = q_\omega(e) = \|q_\omega\| > 0 \) holds for each \( n \). Therefore,

\[
F_q = \sum_{n=1}^{\infty} \lambda_n \xi[q(u_n)]e_n \geq \sum_{n=1}^{\infty} \lambda_n \xi[q_\omega(e)]e_n > 0.
\]

In particular, \( q(F_q) > 0 \) holds, and so, from \( q(F_q) = q(F_q) \) we see that
0 < q(F_q) < p(F_q). This implies
\[ \lim p(E_p) = p(F_q)/q(F_q) - 1 > 0, \]
and we are finished. ■

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