

On the Existence of an Optimal Income Tax Schedule

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1. INTRODUCTION

Since the pioneering work of Mirrlees (1971), many authors have considered the non-linear optimal taxation problem in models with two kinds of goods—labour and consumption—in terms of the variational method or the Pontryagin maximum principle. That is, several necessary conditions have been derived and investigated assuming the existence of an optimal tax schedule. These works, however, have been done without any verification of the existence of an optimal tax schedule. This problem is not a simple maximization problem but a compound maximization problem, i.e. it has individuals' utility maximization and government's social welfare maximization depending upon each other. So, the existence of an optimal tax schedule is not clear either mathematically or economically. On the other hand, Kaneko (1979) proved the existence of an optimal tax schedule for the class of progressive (convex) ones, which is narrower than that employed in the preceding works. Although the progressiveness of tax schedules is an important condition to consider in an optimal taxation problem, it should be checked whether or not this condition can be derived in a more general theory which permits non-progressive tax schedules.¹ Thus we have two general problems—one is the existence of an optimal tax schedule in a class which permits non-progressive (non-convex) tax schedules and the other one is the progressiveness (in some sense) of the optimal tax schedules. Here we consider only the existence problem.

In this paper, we model the optimal taxation problem in the context of general equilibrium analysis. We consider an economy with a finite number of individuals, a finite number of firms and a government, where a finite number of consumption goods and one public good are produced by the firms and the government respectively using labour as input. An optimal tax schedule means one which gives a competitive equilibrium with the maximum value of a social welfare function. Thus the model of this paper is much wider than those of the preceding works, though this paper concerns only the existence of an optimal tax schedule in an economy with a finite number of individuals. The existence problem in an economy with a continuum of individuals is important for the further consideration of the shape of an optimal tax schedule, because tax rates or disposable incomes of rich individuals can be considered naturally in such a model.² The technique given in this paper, however, can not be applied to the existence problem in such a case. This author continues to investigate the continuum case.

This paper is written as follows. In Section 2, we describe the model and define the optimal taxation problem. In Section 3, we provide one important theorem and prove the existence of an optimal tax schedule.

2. TAX SCHEDULE AND COMPETITIVE EQUILIBRIUM

We consider an economy consisting of a finite number of individuals, a finite number of firms and an agent called "government". Let $N = \{1, 2, \dots, n\}$ be the set of all individuals.

We assume that leisure, c -kinds of consumption goods and one public good enter the individuals utility functions $U^i(t, x, Q)$ ($i \in N$), where t denotes leisure time, $x = (x_1, \dots, x_c)$ a level of consumption goods and Q a level of the public good supplied by the government. Every U^i is defined on the consumption set $Y \equiv [0, L] \times E_+^{c+1}$, where $L > 0$ is the initial endowment of leisure time and E_+^{c+1} the non-negative orthant of the $c+1$ -dimensional Euclidean space E^{c+1} . We assume:

Assumption A. For all $i \in N$, $U^i(t, x, Q)$ is monotonically increasing, continuous and quasi-concave function of (t, x, Q) .

The following argument can be directly applicable to the case of more than one public goods. But we discuss the economy with one public good for notational simplicity.

Each individual i owns a labour production function $f^i(h)$. That is, if he works for h -hours, he can provide a quantity $f^i(h)$ of service called "labour". For example, labour may be measured in terms of the unit of manpower/hour. We assume:

Assumption B. For all $i \in N$, $f^i(h)$ is a continuous and concave function of $h \in [0, L]$ with $f^i(0) = 0$ and $f^i(h) > 0$ for some $h > 0$.

We assume that all the individuals are endowed with no consumption goods. The consumption goods are produced by firms. There are m firms in the economy. Each firm j ($j = 1, \dots, m$) has a production set $Z^j \subset E \times E^c$. We also denote $z^j \in Z^j$ by $(z_0^j, z_C^j) = (z_0^j, z_1^j, \dots, z_c^j)$. We make the following assumptions:

Assumption C. $Z^j \subset E_- \times E^c$ for all $j = 1, \dots, m$.

Assumption D. Z^j is a closed convex cone for all $j = 1, \dots, m$.

Assumption E. There are $z^1 = (z_0^1, z_C^1) \in Z^1, \dots, z^m = (z_0^m, z_C^m) \in Z^m$ such that $\sum_{j=1}^m z_C^j > 0$, but there are no $z^1 \in Z^1, \dots, z^m \in Z^m$ such that $\sum_{j=1}^m z^j \geq 0$.

Assumption F. There are $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ for any $z \in \sum_{j=1}^m Z^j$ and k_1, k_2 ($1 \leq k_1, k_2 \leq c$) with $z_{k_1} > 0$ such that $z - \varepsilon_1 e^{k_1} + \varepsilon_2 e^{k_2} \in \sum_{j=1}^m Z^j$.

Here E_- is the set of all non-positive real numbers and $z_C > 0$ means $z_k > 0$ for all $k = 1, \dots, c$ and $z \geq 0$ means $z \geq 0$ but $z \neq 0$, and e^k is the unit vector in $E \times E^c$ in the k -direction (all entries zero except the k -th, 1).

Assumption C means that labour can not be produced by the firms. We would not need to explain Assumption D, but this is a crucial assumption in this paper. Assumption E means that the economy is productive as a whole in the capital-theory sense, and the impossibility of free production. Assumption F means that when the economy produces a positive amount of a consumption good, it can decrease an amount of input or can produce a positive amount of any other good by decreasing the production level of the consumption good.

Each firm j has a profit-share vector $d^j = (d_1^j, \dots, d_n^j)$ with $d_i^j \geq 0$ for all $i \in N$ and $\sum_{i \in N} d_i^j = 1$.

The government produces and supplies the public good using labour and consumption goods as inputs. The government has a production set $Z^0 \subset E \times E^c \times E$. We denote, by $z^0 = (z_0^0, z_C^0, z_{c+1}^0)$, a typical element in Z^0 . z_0^0 and z_C^0 mean amount of labour and the consumption goods used as inputs, respectively, and z_{c+1}^0 an amount of the public good produced. We assume:

Assumption G. $Z^0 \subset E_- \times E_- \times E_+$ and $0 \in Z^0$.

Assumption H. Z^0 is a closed convex set.

Assumption I. There is no $z^0 \in Z^0$ such that $z^0 \geq 0$.

Here E^c is the non-positive orthant of E^c .

Assumption G means that labour cannot be produced and the public good cannot be used as inputs. Further it excludes the possibility that the government produces consumption goods. We do not need to explain Assumption H. Assumption I implies the impossibility of free production.

We are now in a position to define our optimal taxation problem. A tax function T is a function from the interval $[0, M]$ to E which satisfies

$$\begin{aligned} T(y) &\text{ is a continuous and non-decreasing function of } y \text{ with} \\ T(y) &\leq y \text{ for all } y \in [0, M]. \end{aligned} \tag{1}$$

Here M is a positive real number with $M \geq \max_i \max_h f^i(h)$. We assume that the government imposes taxes on the individuals' incomes measured in terms of labour. That is, a tax function T implies that when an individual i works for h -hours and earns income $f^i(h)$, he must pay an income tax $T(f^i(h))$ to the government. Hence we assumed that $T(y) \leq y$ for all $y \in [0, M]$. We denote, by \mathcal{F} , the set of all tax functions. For notational simplicity, we may denote, by $Tf(h)$ or Tf , the composite function $T \circ f$ of T and f in the following.

Kaneko (1979) assumed convexity and (1) on tax functions. The significance of convexity (progressiveness) in optimal taxation theory is clear and we do not need discussion of it here. Mathematically, however, convexity is a kind of uniformness condition and a strong one. Disposing of convexity permits a great deal of variety on tax functions. But we will show in the next section that we can restrict our consideration to a certain narrower class than \mathcal{F} , which plays the same role as convexity in the existence proof of Kaneko (1979) in the domain of progressive tax functions.

We say that $((t^1, x^1, Q), \dots, (t^n, x^n, Q), z^0, z^1, \dots, z^m)$ is an allocation iff

$$(t^i, x^i, Q) \in Y \text{ for all } i \in N, z^j \in Z^j \text{ for all } j = 0, 1, \dots, m \tag{2}$$

and $z_{c+1}^0 = Q$,

$$\sum_{i=1}^n f^i(L - t^i) + \sum_{j=0}^m z_j^i = 0, \tag{3}$$

$$\sum_{i=1}^n x^i - \sum_{j=0}^m z_c^j = 0. \tag{4}$$

It is easily verified that the set of all allocations is a compact set under our assumptions.

The economy works as follows. The government plans to employ a tax function $T \in \mathcal{F}$ and a production schedule $z^0 \in Z^0$. We call (T, z^0) a tax schedule and denote, by \mathcal{S} , the set of all tax schedules. The government announces the tax schedule employed to the individuals. Under the tax schedule, the individuals and the firms behave as price takers, and the prices of consumption goods and labour are determined by the market mechanism (but not the government). Thus we get the following definition.

Definition 1. $\gamma = (p, (t^1, x^1, Q), \dots, (t^n, x^n, Q), z^1, z^2, \dots, z^m)$ is said to be a competitive equilibrium under a tax schedule $\tau = (T, z^0)$ iff

$$p = (p_0, p_c) = (p_0, p_1, \dots, p_c) \in E_+^{1+c} \tag{5}$$

with $p_0 > 0$ and $((t^1, x^1, Q), \dots, (t^n, x^n, Q), z^1, \dots, z^m)$ is an allocation,

$$p_c x^i \leq p_0(1 - T)[f^i(L - t^i) + \sum_{j=1}^m d_j^i p z^j / p_0] \text{ for all } i \in N, \tag{6}$$

for all $i \in N, U^i(t^i, x^i, Q) \geq U^i(t, x, Q)$ for all (t, x) such that

$$p_c x \leq p_0(1 - T)[f^i(L - t) + \sum_{j=1}^m d_j^i p z^j / p_0], \tag{7}$$

for all $j = 1, \dots, m$,

$$pz^j \geq pz \quad \text{for all } z \in Z^j. \tag{8}$$

We call $\tau = (T, z^0)$ a feasible tax schedule iff there exists a competitive equilibrium under τ which satisfies

$$-p(z_0^0, z_0^0) \leq p_0 \sum_{i=1}^n T[f^i(L - t^i) + \sum_{j=1}^m d^j pz^j / p_0]. \tag{9}$$

We denote, by \mathcal{S}_f , the set of all feasible tax schedules and by $C(\tau)$ the set of all competitive equilibria under τ with (9).

Condition (5) guarantees a positive price of labour and the coincidence of total demands and total supplies of consumption goods and labour. Condition (6) is the individuals' budget constraint and (7) is the individuals' utility maximization under the budget constraint. Condition (8) is the firms' profit maximization. Condition (9) means that the government's expenditure for the production of the public good does not exceed revenue.

Let $(T^0, 0)$ be the trivial tax schedule, i.e. $T^0(y) = 0$ for all $y \in [0, M]$ and $z^0 = 0$. That is, the government does nothing. If there exists a competitive equilibrium under $(T^0, 0)$, then (2.9) is automatically satisfied by it. The definition of competitive equilibrium under the trivial tax schedule $(T^0, 0)$ is reduced to the standard one, so the existence of a competitive equilibrium in this case can be proved, slightly modifying the standard existence proof, e.g. Nikaido (1970). Thus we get the following theorem.

Theorem 1. *There exists a feasible tax schedule τ , i.e. $\mathcal{S}_f \neq \emptyset$.*

Under our assumptions, we can easily prove:

Lemma 1. *$p > 0$ for any competitive equilibrium $(p, (t^1, x^1, Q), \dots, (t^n, x^n, Q), z^1, \dots, z^m) \in C(\tau)$ and any $\tau \in \mathcal{S}_f$.*

Since Z^j is a convex cone for all $j = 1, \dots, m$, firm j 's profit pz^j is always zero in equilibrium for all $j = 1, \dots, m$. In the following, we will use this fact and Lemma 1 without any remark.

The government employs a social welfare function such that

$$\sum_{i=1}^n G^i[U^i(\cdot)], \tag{10}$$

where G^i is a monotonically increasing and continuous function on $[U^i(L, 0, 0), +\infty)$. For example, if $G^i[U^i(\cdot)] = \log[U^i(\cdot) - U^i(0, 0, 0)]$ for all $i \in N$, then this welfare function is the Nash social welfare function of Kaneko and Nakamura (1979). When the government employs a feasible tax schedule $\tau = (T, z^0)$ and so a competitive equilibrium $\gamma = (p, (t^1, x^1, Q), \dots, (t^n, x^n, Q), z^1, \dots, z^m)$ in $C(\tau)$ results in the economy, the value of the social welfare function is represented as

$$W(\tau, \gamma) = \sum_{i=1}^n G^i[U^i(t^i, x^i, Q)].$$

The government tries to maximize the social welfare function over the feasible tax schedules.

Definition 2. $\hat{\tau} \in \mathcal{S}_f$ is said to be an optimal tax schedule iff for some $\hat{\gamma} \in C(\hat{\tau})$,

$$W(\hat{\tau}, \hat{\gamma}) = \sup_{\substack{\tau \in \mathcal{S}_f \\ \gamma \in C(\tau)}} W(\tau, \gamma). \tag{11}$$

The purpose of this paper is to prove the existence of an optimal tax schedule. Although the meaning of an optimal tax schedule is clear, the definition deserves a special critical comment and another note:

Remark 1. Let us remember the role of the government in this economy. The government has tax functions and production schedules as the controllable variables. When the government employs and announces a tax schedule (T, z^0) , the environment of the economy is determined. Under the environment the participants, i.e. the individuals and firms adopt plans independently to maximize their utilities and profits. Then a competitive equilibrium is determined by the market mechanism but not the government. If there are multiple-equilibria, i.e. $C(T, z^0)$ consists of more than one equilibrium, then we cannot give a unique answer to the question which equilibrium occurs. We can answer only that $C(T, z^0)$ corresponds to a tax schedule (T, z^0) . It is natural to assume that the government cannot select one equilibrium from $C(T, z^0)$ but that it can manipulate only T and z^0 . This situation is like a “game against nature”. See Luce and Raiffa (1957, Chapter 13). In this case, the government plays against the market mechanism. There are many solution concepts for a “game against nature” (see Luce and Raiffa (1957, Chapter 13)). Our definition “max max” (an optimal tax schedule satisfies $W(\hat{\tau}, \hat{\gamma}) = \max_{\tau \in \mathcal{S}_f} \max_{\gamma \in C(\tau)} W(\tau, \gamma)$) is the most optimistic criterion. The opposite criterion, the most pessimistic one is “max min”, i.e. we define an optimal tax schedule $\hat{\tau}$ by

$$W(\hat{\tau}, \hat{\gamma}) = \max_{\tau \in \mathcal{S}_f} \min_{\gamma \in C(\tau)} W(\tau, \gamma). \tag{12}$$

This definition means that an optimal tax schedule $\hat{\tau}$ maximizes the guaranteed level of social welfare $\min_{\gamma \in C(\tau)} W(\tau, \gamma)$. Therefore this definition would be safer than the former. But this author has not succeeded in proving the existence of an optimal tax schedule using the “max min” criterion. If the correspondence of equilibria to τ was always lower-semi-continuous, the existence would be proved without much difficulty, but unfortunately, lower-semi-continuity does not necessarily hold (see Hildenbrand (1974, Section 2.3)).

Remark 2. We have chosen labour as numeraire, and so we have assumed that the government imposes taxes upon the individuals’ incomes measured in terms of labour. But it is also possible to choose any other good as numeraire. Or, when the government does not measure the individuals’ incomes by any particular good, the argument of this paper remains true if we define the unit of incomes appropriately. For example, we always normalize the price vector p such that $\sum_{k=0}^c p_k = 1$ and assume that incomes are measured in terms of the value defined by p , i.e. when individual i works for h -hours, his gross income and disposable income are $p_0 f^i(L-h)$ and $p_0 f^i(L-h) - T[p_0 f^i(L-h)]$. But since the individuals can get incomes only by selling their labour in our economy, there is no substantial difference among these alternative settings.

3. THE RESTRICTION THEOREM AND THE EXISTENCE THEOREM

Before we state and show the existence of an optimal tax schedule, we need to state the theorem which makes it possible to restrict our consideration within a narrower class, \mathcal{S}_0 , than \mathcal{S} . Here \mathcal{S}_0 designates the set of all tax functions $T \in \mathcal{S}$ having the following property:

$$\frac{T(y_1) - T(y_2)}{y_1 - y_2} \leq 1 \quad \text{for all } y_1, y_2 \in [0, M] \quad \text{with } y_1 \neq y_2. \tag{13}$$

A tax function $T \in \mathcal{S}_0$ has the property that its marginal tax rate is not greater than 100% everywhere.

Let (T_1, z^0) and (T_2, z^0) be feasible tax schedules. Then (T_1, z^0) is said to be equivalent to (T_2, z^0) iff $C(T_1, z^0) = C(T_2, z^0)$. The equivalence of (T_1, z^0) and (T_2, z^0) means that even if the government employs either, the same equilibria can be expected to result. Of course, this relation is an “equivalence relation” on \mathcal{S}_f .

Theorem 2. (The Restriction Theorem). *For any feasible tax schedule (T, z^0) , there exists a tax function $T_0 \in \mathcal{F}_0$ such that (T, z^0) is equivalent to (T_0, z^0) .*

Theorem 2 says that for every feasible tax schedule, we can curve the tax function so that the new tax function has the same equilibria but has marginal tax rates not greater than 100%. Hence this theorem ensures that we can restrict our consideration to \mathcal{F}_0 instead of \mathcal{F} .

This theorem can be explained intuitively as follows. Suppose T is not in \mathcal{F}_0 . Then there are points y_1 and y_2 in $[0, M]$ such that $(T(y_1) - T(y_2))/(y_1 - y_2) > 1$ and $y_1 > y_2$. This is equivalent to $y_1 - T(y_1) < y_2 - T(y_2)$. Any individual i does not choose labour time h_1 with $y_1 = f^i(h_1)$, because he can get more net income $y_2 - T(y_2)$ than $y_1 - T(y_1)$, working for the less time h_2 with $y_2 = f^i(h_2)$. So, we can flatten out T without any effect on individuals' behaviour so that a new tax function belongs to \mathcal{F}_0 . A precise proof will be given in the Appendix.

We are now in a position to state the main result of this paper.

Theorem 3. (The Existence Theorem). *There exists an optimal tax schedule.*

Let $\mathcal{S}_0 = \{(T, z^0) \in \mathcal{S}_f: T \in \mathcal{F}_0\}$. Then it follows from Theorem 2 that

$$\sup_{\substack{\tau \in \mathcal{S}_0 \\ \gamma \in C(\tau)}} W(\tau, \gamma) = \sup_{\tau \in \mathcal{S}_f} W(\tau, \gamma). \tag{14}$$

Since U^i and G^i are continuous functions for all $i \in N$ and the set of all allocations is a compact set, we have $\sup W(\tau, \gamma) < +\infty$. Hence there is a sequence $\{(\tau^s, \gamma^s)\} = \{(T^s, z^{0s}), (p^s, (t^{1s}, x^{1s}, Q^s), \dots, (t^{ns}, x^{ns}, Q^s), z^{1s}, \dots, z^{ms})\}$ such that $\gamma^s \in C(\tau^s)$ and $\tau^s \in \mathcal{S}_0$ for all $s \geq 1$ and $\lim_{s \rightarrow \infty} W(\tau^s, \gamma^s) = \sup W(\tau, \gamma)$. The outline of the following proof is that we can choose a convergence subsequence $\{(\tau^{s'}, \gamma^{s'})\}$ from $\{(\tau^s, \gamma^s)\}$ and then the limit point (τ^*, γ^*) really attains (14).

Lemma 2. $\inf_s T^s(0) > -\infty$.

Proof. Suppose $\inf_s T^s(0) = -\infty$. Since each T^s belongs to \mathcal{F}_0 , $T^s(y) \leq T^s(0) + y$ for all $y \in [0, M]$ and all s . Hence there is an s_0 such that $T^{s_0}(y) \leq T^{s_0}(0) + M < 0$ for all $y \in [0, M]$ and all $s \geq s_0$. This means that for large s , the government's revenue is negative. This contradicts the supposition that $\gamma^s \in C(\tau^s)$ for all s . \parallel

Let $K = \inf_s T^s(0)$. We define $C[0, M]$ by

$$C[0, M] = \{t: t \text{ is a continuous non-decreasing function with } (t(y_1) - t(y_2))/(y_1 - y_2) \leq 1 \text{ for all } y_1, y_2 (y_1 \neq y_2) \text{ and } K \leq t(0) \leq 0\}. \tag{15}$$

Lemma 3. $C[0, M]$ is a compact set with respect to the topology of uniform convergence.

Proof. By Ascoli's theorem (Simmons (1963, Section 25, Theorem C)), it is sufficient to show that $C[0, M]$ is closed, bounded and equicontinuous.

It is easily verified that $C[0, M]$ is closed and bounded. By (15) it holds that for all $t \in C[0, M]$,

$$|y_1 - y_2| \leq \delta \text{ implies } |t(y_1) - t(y_2)| \leq \delta.$$

This means that $C[0, M]$ is equicontinuous. \parallel

We can normalize $\{p^s\}$ without loss of generality such that each p^s belongs to $P \equiv \{p \in E_+^{1+c} : \sum_{k=0}^c p_k = 1\}$. Let A be the set of all allocations. Then the sequence $\{(\tau^s, \gamma^s)\}$ is in $C[0, M] \times P \times A$. Clearly $C[0, M] \times P \times A$ is a compact set, so there is a subsequence $\{(\tau^{s^*}, \gamma^{s^*})\}$ of $\{(\tau^s, \gamma^s)\}$ which converges to $(\tau^*, \gamma^*) = ((T^*, z^{0*}), (p^*, (t^{1*}, x^{1*}, Q^*), \dots, (t^{n*}, x^{n*}, Q^*), z^{1*}, \dots, z^{m*}))$ in the sense of the product topology.

We assume for notational simplicity that $\{(\tau^s, \gamma^s)\}$ itself converges to (τ^*, γ^*) . Then since U^i and G^i are continuous functions for all $i \in N$, we have

$$\sup W(\tau, \gamma) = \lim_{s \rightarrow \infty} \sum_{i \in N} G^i[U^i(t^{is}, x^{is}, Q^s)] = \sum_{i \in N} G^i[U^i(t^{i*}, x^{i*}, Q^*)].$$

Further it holds that $T^* \in C[0, M] \subset \mathcal{F}_0$. Hence it is sufficient to show that γ^* is a competitive equilibrium under τ^* with (9).

Lemma 4. $p_0^* > 0$.

Proof. Let $p_0^* = 0$. By Assumption E, there are $z^1 \in Z^1, \dots, z^m \in Z^m$ such that $\sum_{j=1}^m z^j > 0$. Then for some $s_0, p^s \circ \sum_{j=1}^m z^j > 0$ for all $s \geq s_0$. That is, there is a j such that $p^s z^j > 0$. Then $p^s z^{is} \geq p^s z^j > 0$ by (8). This is impossible. \parallel

Suppose:

$$f^i(L - t^{i*}) - T^* f^i(L - t^{i*}) = 0 \quad \text{for all } i \in N. \tag{16}$$

Then it is easily verified that if $p_k^* > 0$, then $\lim_{s \rightarrow \infty} x_k^{is} = 0$ for all $i \in N$ because of Lemma 4 and (16). Let $p_C^* = 0$. Then since $p_C^s \circ \sum_{j=1}^m z^j$ converges to 0 by the compactness of the set of allocations, $p^s \circ \sum_{j=1}^m z^j = p_0^s \circ \sum_{j=1}^m z^j + P_C \sum_{j=1}^m z^j = 0$ for all s implies $\lim_{s \rightarrow \infty} \sum_{j=1}^m z_0^s = 0$, because of Lemma 4. This implies $\sum_{j=1}^m z^j = 0$ for all $j = 1, \dots, m$ by Assumption E, and so $x^{i*} = 0$ for all $i \in N$. Finally let us consider the case where $p_{k_1}^* = 0$ and $p_{k_2}^* > 0$ for some k_1, k_2 ($1 \leq k_1, k_2 \leq c$). In this case, if $\sum_{j=1}^m z_{k_1}^{j*} > 0$, then there are $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ by Assumption F such that $\sum_{j=1}^m z^{j*} - \varepsilon_1 e^{k_1} + \varepsilon_2 e^{k_2}$ is in $\sum_{j=1}^m Z^j$. Hence $p^s (\sum_{j=1}^m z^{j*} - \varepsilon_1 e^{k_1} + \varepsilon_2 e^{k_2}) \rightarrow p_{k_2}^* \varepsilon_2 > 0$, which is a contradiction to the fact that $\max_{z \in Z^j} p^s z = 0$ for all $j = 1, \dots, m$ and all s . Thus we have shown that $\sum_{j=1}^m z_{k_1}^{j*} = 0$ for all j . Hence it follows that $x_{k_1}^{i*} = 0$ for all $i \in N$. Thus, it is always true that $x^{i*} = 0$ for all $i \in N$.

Since U^i is continuous for all $i \in N$, it follows from the above argument that $\lim_{s \rightarrow \infty} U^i(t^{is}, x^{is}, Q^s) = U^i(t^{i*}, 0, Q^*)$ for all $i \in N$. If $t^{i*} < L$, then for a sufficiently small $\varepsilon > 0$, there is an s_0 such that $U^i(L, 0, Q^*) - \varepsilon > U^i(t^{is}, x^{is}, Q^s)$ for all $s \geq s_0$. Further there is an s_1 such that $U^i(L, 0, Q^s) > U^i(L, 0, Q^*) - \varepsilon$ for all $s \geq s_1$. These imply $U^i(L, 0, Q^s) > U^i(t^{is}, x^{is}, Q^s)$ for all $s \geq \max(s_0, s_1)$, which is a contradiction because $(L, 0)$ always satisfies his budget. Hence we have $t^{i*} = L$ for all $i \in N$. This also implies $Q^* = 0$. Hence we have

$$\sup W(\tau, \gamma) = \sum_{i \in N} G^i[U^i(L, 0, 0)].$$

Let $\tau^0 = (T^0, 0)$ be the trivial tax schedule and $\gamma^0 = (p^0, (t^{10}, x^{10}, 0), \dots, (t^{n0}, x^{n0}, 0), z^{10}, \dots, z^{m0}) \in C(\tau^0)$. The existence of such a γ^0 has been stated in the preceding paragraph of Theorem 1. It is clear that $U^i(t^{i0}, x^{i0}, 0) \geq U^i(L, 0, 0)$ for all $i \in N$. Hence we have

$$W(\tau^0, \gamma^0) = \sup W(\tau, \gamma). \tag{17}$$

In the following we consider the case where

$$f^i(L - t^{i*}) - T^* f^i(L - t^{i*}) > 0 \quad \text{for some } i \in N. \tag{18}$$

Lemma 5. $p^* > 0$.

Proof. Let $p_k^* = 0$ for some k ($1 \leq k \leq c$). Then it is easily verified that $x_k^{is} \rightarrow \infty$ ($s \rightarrow \infty$) for all i with $f^i(L - t^{i*}) - T^* f^i(L - t^{i*}) > 0$. This contradicts the impossibility of free production. \parallel

It is easily verified that γ^* satisfies (5), (6) and (8) under τ^* . We show that γ^* satisfies (7). Suppose that there is a (t^0, x^0) such that $U^i(t^0, x^0, Q^*) > U^i(t^{i*}, x^{i*}, Q^*)$ and $p_c^* x^0 \leq p_0^* (1 - T^*) f^i(L - t^0)$. Now let us show $(1 - T^*) f^i(L - t^0) > 0$. On the contrary, let $(1 - T^*) f^i(L - t^0) = 0$. Then $x^0 = 0$ because $p^* > 0$. Hence $U^i(L, 0, Q^*) \geq U^i(t^0, 0, Q^*) > U^i(t^{i*}, x^{i*}, Q^*)$. Since $\{(t^{is}, x^{is}, Q^s)\}$ converges to (t^{i*}, x^{i*}, Q^*) , there is an s_0 such that $U^i(L, 0, Q^s) > U^i(t^{is}, x^{is}, Q^s)$ for all $s \geq s_0$. But $(L, 0)$ satisfies i 's budget constraint under (T^s, z^s) . This is a contradiction. Hence $(1 - T^*) f^i(L - t^0) > 0$. Then there is another (\tilde{t}, \tilde{x}) in a neighbourhood of (t^0, x^0) by the continuity of U^i and Lemma 5 such that $U^i(\tilde{t}, \tilde{x}, Q^*) > U^i(t^{i*}, x^{i*}, Q^*)$ and $p_c^* \tilde{x} < p_0^* (1 - T^*) f^i(L - \tilde{t})$. Since $\{T^s\}$ converges uniformly to T^* and $\{p^s\}$ converges to p^* , there is an s_1 such that $p_c^s \tilde{x} < p_0^s (1 - T^s) f^i(L - \tilde{t})$ for all $s \geq s_1$. Since (t^{is}, x^{is}, Q^s) converges to (t^{i*}, x^{i*}, Q^*) , there is an s_2 for some $\varepsilon > 0$ such that $U^i(\tilde{t}, \tilde{x}, Q^*) - \varepsilon > U^i(t^{is}, x^{is}, Q^s)$ for all $s \geq s_2$. Since $\{Q^s\}$ converges to Q^* , there is an s_3 such that $U^i(\tilde{t}, \tilde{x}, Q^s) \geq U^i(\tilde{t}, \tilde{x}, Q^*) - \varepsilon$ for all $s \geq s_3$. Hence it holds that $U^i(\tilde{t}, \tilde{x}, Q^s) > U^i(t^{is}, x^{is}, Q^s)$ and $p_c^s \tilde{x} < p_0^s (1 - T^s) f^i(L - \tilde{t})$ for all $s \geq \max(s_1, s_2, s_3)$. This contradicts the fact that γ^s is a competitive equilibrium under τ^s for all $s \geq 1$. Thus we have proved (7).

Finally we show that γ^* satisfies (9) under τ^* . Since $\{\gamma^s\}$ and $\{z^{0s}\}$ converges to γ^* and z^{0*} , and since $\{T^s\}$ converges uniformly to T^* , $\{T^s f^i(L - t^{is})\}$ converges to $T^* f^i(L - t^{i*})$ for all $i \in N$. Hence the condition

$$-p^s(z_0^{0s}, z_C^{0s}) \leq p_0^s \circ \sum_{i=1}^n T^s f^i(L - t^{is}) \quad \text{for all } s \geq 1$$

implies $-p^*(z_0^{0*}, z_C^{0*}) \leq p_0^* \circ \sum_{i=1}^n T^* f^i(L - t^{i*})$.

APPENDIX

Proof of Theorem 2. Let (T, z^0) be any feasible tax schedule. Let $c(y)$ be the net income function corresponding to T , i.e. $c(y) = y - T(y)$ for all $y \in [0, M]$. Using this function $c(y)$, we define $c_0(y)$ and $T_0(y)$ by

$$c_0(y) = \max_{0=y_1 \leq y} c(y_1) \quad \text{for all } y \in [0, M], \quad (\text{A.1})$$

$$T_0(y) = y - c_0(y) \quad \text{for all } y \in [0, M]. \quad (\text{A.2})$$

Lemma 6. (i) c_0 is a non-decreasing continuous function with property (13). (ii) $T_0(y)$ is a nondecreasing continuous function with property (13) and $T_0(y) \leq y$ for all $y \in [0, M]$, i.e. $T_0 \in \mathcal{F}_0$.

Proof. It is clear that $c_0(y)$ is a non-decreasing continuous function. Since $T(y)$ is non-decreasing, we have, for $y_1 > y_2$,

$$\begin{aligned} c(y_1) - c(y_2) &= y_1 - T(y_1) - (y_2 - T(y_2)) \\ &= y_1 - y_2 - (T(y_1) - T(y_2)) \leq y_1 - y_2, \end{aligned}$$

which is (13). It follows from this and (A.1) that $c_0(y)$ also satisfies (13). It is clear that $T_0(y)$ is a continuous function. Since $c_0(y)$ is nondecreasing and satisfies (13), we have, for $y_1 > y_2$,

$$\begin{aligned} T_0(y_1) - T_0(y_2) &= y_1 - c_0(y_1) - (y_2 - c_0(y_2)) \\ &= y_1 - y_2 - (c_0(y_1) - c_0(y_2)) \leq y_1 - y_2, \end{aligned}$$

and

$$T_0(y_1) - T_0(y_2) = y_1 - y_2 - (c_0(y_1) - c_0(y_2)) \geq 0.$$

Since $T_0(0) = T(0)$, we have, by the above inequality, $T_0(y) \leq y$ for all $y \in [0, M]$. \parallel

We define the set $I(T, T_0)$ for T and T_0 by

$$I(T, T_0) = \{y \in [0, M]: T(y) = T_0(y)\}. \quad (\text{A.3})$$

Lemma 7. (i). If $\gamma = (p, (t^1, x^1, Q), \dots, (t^n, x^n, Q), z^1, \dots, z^n)$ is a competitive equilibrium under (T, z^0) , then $f^i(L - t^i) \in I(T, T_0)$ for all $i \in N$. (ii). If γ is a competitive equilibrium under (T_0, z^0) , then $f^i(L - t^i) \in I(T, T_0)$ for all $i \in N$.

Proof. We prove only (i), but can prove (ii) analogously. Suppose $f^i(L - t^i) \notin I(T, T_0)$. Then $T(y) > T_0(y)$, i.e. $c(y) < c_0(y)$, where $y = f^i(L - t^i)$. Hence there is a $y_1 < y$ such that $c_0(y) = c(y_1)$. If he works for $L - t_1$ hours such that $f^i(L - t_1) = y_1$ and $t_1 > t^i$, then the disposable income $c(y_1)$ is greater than $c(y)$. This means that the individual can increase his utility. This is a contradiction to the supposition that γ is a competitive equilibrium under (T, z^0) . \parallel

Lemma 8. $\gamma \in C(T, z^0)$ if and only if $\gamma \in C(T_0, z^0)$.

Proof. We prove that if $\gamma \in C(T, z^0)$, then $\gamma \in C(T_0, z^0)$. It is sufficient to show that γ satisfies (6), (7) and (9) under (T_0, z^0) . It follows from Lemma 7(i) that γ satisfies (6) and (9). Let (7) be not satisfied. Then there is a (t, x) such that $f^i(L - t) \notin I(T, T_0)$, $U^i(t, x, Q) > U^i(t^i, x^i, Q)$ and $p_C x \leq p_0(1 - T_0)f^i(L - t)$. Since $f^i(L - t) = y \notin I(T, T_0)$, $T(y) > T_0(y)$, i.e. $c(y) < c_0(y)$, which implies that there is a $y_1 < y$ such that $c_0(y) = c(y_1)$. Let t_1 be the real number such that $y_1 = f^i(L - t_1)$ and $t_1 > t$. Then $f^i(L - t_1) - T f^i(L - t_1) = c(y_1) = c_0(y) = f^i(L - t) - T_0 f^i(L - t)$. In sum, we have

$$f^i(L - t_1) \in I(T, T_0), \quad U^i(t_1, x, Q) > U^i(t, x, Q) > U^i(t^i, x^i, Q)$$

and

$$p_C x \leq p_0(1 - T_0)f^i(L - t) = p_0(1 - T)f^i(L - t_1).$$

This is a contradiction to the supposition $\gamma \in C(T, z^0)$.

We can analogously prove the converse part of this lemma using Lemma 7(ii). \parallel

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NOTES

1. A result of Seade (1977) appears, to a certain extent, to contradict the progressiveness of an optimal tax schedule. Ordover and Phelps (1979) also achieved a similar conclusion in a different context. But Kaneko (1979) proved that the optimal marginal and average tax rates tend to 100% as income level becomes arbitrarily high under certain assumptions. We can interpret this result as progressiveness in a different sense broader than convexity. Anyway, the author does not think that the optimal taxation problem has been studied fully nor that we can generally answer the problem of progressiveness.

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