A Bilateral Monopoly and the Nash Cooperative Solution*

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1. Introduction and the Bilateral Economy

Let us consider a bilateral exchange economy, which consists of a seller $S$ and a buyer $B$. The seller owns $Q > 0$ amount of a commodity, which he wants to sell to the buyer. The buyer owns no amount of the commodity but owns $M > 0$ amount of another commodity, which we call "money." "Money" should be interpreted as the composite commodity of all commodities other than the commodity in question. Of course, the seller may own a positive amount of money, but since we will consider only the case where the seller never buys any amount of the commodity from the buyer, it is convenient to represent the initial level of money by zero. That is, the initial endowments of the seller and buyer are represented as $w^s = (Q, 0)$ and $w^b = (0, M)$. The seller and buyer have von Neumann-Morgenstern utility functions $U(x_1, x_2)$ and $V(x_1, x_2)$ defined on the non-negative orthant $E^2_+$ of the two-dimensional Euclidean space.

Nash [1, 3] provided a solution concept for two-person cooperative games. If the theory is applied to the above bilateral economy, it selects an allocation $((Q - q, p), (q, M - p))$ such that

$$
(U(Q - q, p) - U(w^s))(V(q, M - p) - V(w^b))
= \text{max}(U(x_1, x_2) - U(w^s))(V(y_1, y_2) - V(w^b))
$$

subject to $(x_1, x_2) + (y_1, y_2) = (Q, M)$, $(x_1, x_2), (y_1, y_2) \in E^2_+$

and $U(x_1, x_2) \geq U(w^s), V(y_1, y_2) \geq V(w^b)$. (1.1)

We call an allocation satisfying (1.1) a Nash allocation.

Nash provided two complementary approaches to the solution. One is the axiomatic approach, which was well established in Nash [1]. The axiomatic approach focuses its consideration only on outcomes of a bargaining process instead of the bargaining process itself. Nash [3] provided another approach

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to it in which the solution was characterized as a kind of equilibrium point of a game in extensive form. This approach intends to exhibit a structure of bargaining process yielding the Nash co-operative solution. As the latter approach provides an explicit structure of bargaining process, it looks more realistic than the former. But when this approach is applied to the above bilateral economy, the bargaining process never seems to fit the bilateral economy, i.e., the extensive game given by Nash cannot be naturally interpreted in the context of the bilateral economy. His extensive game should be thought of just as an archetype. The purpose of this paper is to modify the latter approach to incorporate the case of the bilateral economy.

Without loss of generality we can put $U(w') = V(w') = 0$. We assume the following conditions on the utility functions $U$ and $V$:

(A) $U$ and $V$ are monotonically increasing, i.e., if $x \geq y$ and $x \neq y$, then $U(x) > U(y)$ and $V(x) > V(y)$;

(B) $U$ and $V$ are functions of $C^2$, i.e., $U$ and $V$ have all second derivatives which are continuous;

(C) $U$ and $V$ are strictly concave functions;

(D) $U(0, M) V(Q, 0) < 0$, but there is an $m$ such that $U(0, m)$, $V(Q, M - m) > 0$;

(E) for each $m$ with $U(0, m) V(Q, M - m) > 0$, there is a $(q, m')$ such that $U(Q - q, m') V(q, M - m') > U(0, m) V(Q, M - m)$.

Assumptions (A), (C), (D) and (E) impose regularities upon the bilateral economy. Assumption (B) is a mathematical condition. Assumptions (D) and (E) are not familiar to us, but they just claim that the Nash allocation does not lies on the boundary.

Before we provide the last assumption, we need to state the following lemmas, which will be proved in the Appendix.

**Lemma 1.** For each $q \in (0, Q]$ there is an $m \in (0, M]$ such that $U(Q - q, m)$, $V(q, M - m) > 0$, but there is no $m$ such that $U(Q - q, m)$, $V(q, M - m) \leq 0$.

**Lemma 2.** For each $q \in (0, Q]$, there is a unique $m(q) \in (0, M)$ such that

$$U(Q - q, m(q)) V(q, M - m(q)) \geq U(Q - q, m) V(q, M - m)$$

for all $m \in (0, M]$. (1.2)

**Lemma 3.** Let $u(q) = U(Q - q, m(q))$, $v(q) = V(q, M - m(q))$ and $L(q) = \log u(q) + \log v(q)$ for all $q \in (0, Q]$. Then $L(q)$ is a strictly concave function.
It should be noted that \( u(q) > 0 \) and \( v(q) > 0 \) for all \( q \in (0, Q] \) by Lemmas 1 and 2.

**Lemma 4.**

(i) There is a unique \( q_N \in (0, Q) \) such that \( L(q_N) \geq L(q) \) for all \( q \in (0, Q] \).

(ii) \( ((Q - q, m(q_N)), (q_N, M - m(q_N))) \) is a unique Nash allocation.

The last assumption is as follows:

(F) \( \log u(q) \) and \( \log v(q) \) are concave functions of \( q \) on \( (0, Q] \).

Since, as shown in Lemma 3, it follows from (A)–(E) that \( \log u(q) + \log v(q) \) is a strictly concave function, Assumption (F) would be a weak condition. But the author has succeeded in neither proving (F) from (A)–(E) nor constructing any counterexample. The following example satisfies all the assumptions.

**Example 1.** Let \( U(x_1, x_2) = x_1^{1/2} + x_2 - 10 \), \( V(x_1, x_2) = \frac{3}{2} \cdot x_1^{1/2} + x_2 - 20 \), \( Q = 100 \) and \( M = 20 \). It is not difficult to verify that Assumptions (A)–(E) are satisfied. Then the program

\[
\max_{0 < m < M} \log(\sqrt{100 - q} + m - 10) + \log(\frac{3}{2} \sqrt{q} - m)
\]

has the solution \( m(q) = (\frac{3}{2} \sqrt{q} - \sqrt{100 - q} + 10)/2 \) for each \( q \in (0, Q] \). Hence \( u(q) \) and \( v(q) \) are represented as

\[
u(q) = v(q) = (\frac{3}{2} \sqrt{q} + \sqrt{100 - q} - 10)/2.
\]

It is clear that this function itself is a concave function with \( u(q) = v(q) > 0 \) for all \( q \in (0, Q] \).

2. **The Bargaining Game** \( G(f_0, g_0) \)

The seller and buyer have a negotiation to abide by the rules of the bargaining game \( G(f_0, g_0) \). We assume that the bargaining game \( G(f_0, g) \) is divided into two stages 1 and 2. At the 1st stage, an amount of the commodity to be traded is decided, and at the 2nd stage, an amount of money to be paid to the seller from the buyer is decided. The rules of the bargaining game are as follows:

**The 1st stage.** The seller \( S \) and buyer \( B \) select amounts of the commodity that they want to trade from \( S \) to \( B \). These decisions are made independently. That is, each must select an amount without the knowledge of the decision of the other. Then they inform each other of the amounts
selected. If the decisions are the same, then the negotiation goes on to the 2nd stage, but otherwise, the negotiation is broken off.

The strategy-spaces of $S$ and $B$ at the 1st stage are $(0, Q]$. It should be noted that 0 does not belong to the strategy-space, since it is assumed that they want to trade the commodity. Let $f_0(q_s - q_b)$ be the function on $(-Q, Q)$ defined by

$$f_0(q_s - q_b) = \begin{cases} 1 & \text{if } q_s - q_b = 0 \\ 0 & \text{otherwise,} \end{cases}$$

(2.1)

where $q_s$ and $q_b$ are amounts of the commodity selected by $S$ and $B$, respectively. The rule can be rewritten by this notation as follows. If $S$ and $B$ select strategies $q_s$ and $q_b$, then the negotiation goes on to the 2nd stage with probability $f_0(q_s - q_b)$ and is broken off with probability $1 - f_0(q_s - q_b)$.

The 2nd stage. Let the negotiation go on to the 2nd stage. This case happens if and only if $q_s = q_b = \bar{q}$. But if $S$ and $B$ violate the rule and if they make the negotiation go on to the 2nd stage and be concluded in a case of $q_s \neq q_b$, then it is assumed that $(q_s + q_b)/2 = \bar{q}$ amount of the commodity is traded. At this stage, $S$ and $B$ select prices of the amount $\bar{q}$ of the commodity. These decisions are also made independently, and they inform each other of the prices. The negotiation is concluded if and only if price $p_s$ set by $S$ coincides with price $p_b$ set by $B$. Similar to the 1st stage, we take the violation of this rule into account and assume that if they make the negotiation be concluded in a case of $p_s \neq p_b$, then the final price is $\bar{p} = (p_s + p_b)/2$. It should be noted that $p_s$, $p_b$, or $\bar{p}$ are the total amount of money to be paid from $B$ to $S$, and that the unit prices are given as $p_s/\bar{q}$, $p_b/\bar{q}$ or $\bar{p}/\bar{q}$, respectively. Let $g_0$ be the function defined by

$$g_0(p_s - p_b) = \begin{cases} 1 & \text{if } p_s - p_b = 0 \\ 0 & \text{otherwise.} \end{cases}$$

(2.2)

The set of prices to be set by $S$ or $B$ is $(0, M]$. Hence the domain of $g_0$ is the open interval $(-M, M]$. 0-price is omitted from the set of prices by the same reason as that of the omission of 0-quantity. The above rule can be also rewritten by $g_0$ as follows: The negotiation is concluded with probability $g_0(p_s - p_b)$ and is broken off with probability $1 - g_0(p_s - p_b)$.

The bargaining game $G(f_0, g_0)$ is formulated as a game in extensive form. The game tree of the game is drawn as Fig. 1. Of course, the order of $S$'s and $B$'s information sets is exchangeable. Even if $q_s \neq q_b$, then the game tree is assumed to have the 2nd stage, though the probability that the negotiation goes on to the 2nd stage is zero. A behavior strategy of $S$ is a pair $(q_s, t_s)$ of a point $q_s$ in $(0, Q]$ and a function $t_s$ from $[0, Q] \cong (0, Q] \times (0, Q]$ to $(0, M]$. A behavior strategy of $B$ is also a pair $(q_b, t_b)$ of a point $q_b$ in $(0, Q]$. 
and a function $t_b$ from $(0, Q)^2$ to $(0, M]$. The set of all behavior strategies of $S$ is the same as that of all behavior strategies of $B$, which is denoted by $BS = (0, Q) \times T$. The payoff functions $K_s$ and $K_b$ of the bargaining game $G(f_0, g_0)$ are real valued functions on $BS^2 = BS \times BS$. If $S$ and $B$ use behavior strategies $b_s = (q_s, t_s)$ and $b_b = (q_b, t_b)$, respectively, the payoff functions are represented as

$$
K_s(b_s, b_b) = U(Q - \bar{q}, \bar{p})f_0(q_s - q_b)g_0(p_s - p_b),
$$

$$
K_b(b_s, b_b) = V(\bar{q}, M - \bar{p})f_0(q_s - q_b)g_0(p_s - p_b),
$$

(2.3)

where $p_s = t_s(q_s, q_b)$, $p_b = t_b(q_s, q_b)$, $\bar{q} = (q_s + q_b)/2$ and $\bar{p} = (p_s + p_b)/2$.

The 2nd stage of the bargaining game $G(f_0, g_0)$ is subgames of $G(f_0, g_0)$. For definition of subgame, see Selten [4, 5]. When $q_s$ and $q_b$ are selected at the 1st stage, the subgame of the 2nd stage is denoted by $SG(q_s, q_b, g_0)$. $SG(q_s, q_b, g_0)$ is characterized as a game as follows. The set of strategies of $S$ or $B$ is $(0, M]$, which is the set of values of behavior strategies $t_s(q_s, q_b)$ or $t_b(q_s, q_b)$, i.e., $(0, M] = \{t(q_s, q_b) | t \in T \}$. The payoff functions $k_s$ and $k_b$ of $SG(q_s, q_b, g_0)$ are given as

$$
k_s(p_s, p_b) = U(Q - \tilde{q}, \tilde{p})g_0(p_s - p_b),
$$

$$
k_b(p_s, p_b) = V(\tilde{q}, M - \tilde{p})g_0(p_s - p_b),
$$

(2.4)

where $\tilde{q} = (q_s + q_b)/2$ and $\tilde{p} = (p_s + p_b)/2$. We call $k_s$ and $k_b$ the induced payoff functions of $K_s$ and $K_b$ on $SG(q_s, q_b, g_0)$. We call $p_s = t_s(q_s, q_b)$ and
\[ \rho_b = t_b(q_s, q_b) \] the induced strategies of \( b_s = (q_s, t_s) \) and \( b_b = (q_b, t_b) \) on \( S\Gamma_\rho(q_s, q_b, g_0) \).

When strategies \( t_s \) and \( t_b \) in \( T \) at the 2nd stage are fixed, \( \Gamma_\rho(f_0, g_0) \) is reduced to a game with one stage. We call the game a reduced game, which is denoted by \( \Gamma_\rho(f_0, g_0, t_s, t_b) \). It is characterized as follows. The sets of all strategies are \( (0, Q) \), and \( S\Gamma_\rho \) and \( B\Gamma_\rho \)'s payoff functions \( r_s \) and \( r_b \) of \( \Gamma_\rho(f_0, g_0, t_s, t_b) \) are represented as

\[
\begin{align*}
\rho_s(q_s, q_b) &= K_s((q_s, t_s), (q_b, t_b)), \\
\rho_b(q_s, q_b) &= K_b((q_s, t_s), (q_b, t_b)).
\end{align*}
\tag{2.5}
\]

Let \( (X_1, X_2, k_1, k_2) \) be a two-person game, where \( X_1 \) and \( X_2 \) are the sets of strategies, and where \( k_1 \) and \( k_2 \) are the payoff functions. Then we call a pair \((x^+_1, x^+_2) \in X_1 \times X_2\) of strategies an equilibrium point if

\[
\begin{align*}
k_1(x^+_1, x^+_2) &\geq k_1(x_1, x^+_2) \quad \text{for all} \quad x_1 \in X_1, \\
k_2(x^+_1, x^+_2) &\geq k_2(x^+_1, x_2) \quad \text{for all} \quad x_2 \in X_2.
\end{align*}
\tag{2.6}
\]

We call a pair \((b^+_1, b^+_2) = ((q^+_s, t^+_s), (q^+_b, t^+_b))\) of behavior strategies of \( \Gamma_\rho f_0, g_0 \) an subgame perfect equilibrium point if \( b^+_1 \) satisfies the following (2.7) and (2.8):

\[
\begin{align*}
\text{For each } (q_s, q_b) \in (0, Q)^2, \text{ the pair } (\rho_s, \rho_b) \text{ of the induced strategies of } b^+_1 \text{ and } b^+_2 \text{ on } S\Gamma_\rho(q_s, q_b, g_0) \text{ is an equilibrium point of } S\Gamma_\rho(q_s, q_b, g_0). \tag{2.7}
\end{align*}
\]

\[
\begin{align*}
(q^+_s, q^+_b) \text{ is an equilibrium point of the reduced game } \\
R\Gamma_\rho(f_0, g_0, t^+_s, t^+_b). \tag{2.8}
\end{align*}
\]

This solution concept is defined by Selten, and the basic idea was explained in Selten [4, 5].

In the bargaining game \( \Gamma_\rho(f_0, g_0) \) there are a lot of subgame perfect equilibrium points. For example, if a pair \((b^+_1, b^+_2) = ((q^+_s, t^+_s), (q^+_b, t^+_b))\) satisfies

\[
\begin{align*}
q^+_s &= q^+_b \quad \text{and} \quad t^+_s(q_s, q_b) = t^+_b(q_s, q_b) \\
&\quad \text{for all } (q_s, q_b) \in (0, Q)^2, \tag{2.9}
\end{align*}
\]

\[
\begin{align*}
U(Q - (q_s + q_b)/2, t^+_s(q_s, q_b)) &\geq 0 \quad \text{and} \quad V(q_s + q_b)/2, M - t^+_b(q_s, q_b)) \geq 0 \\
&\quad \text{for all } (q_s, q_b) \in (0, Q)^2, \tag{2.10}
\end{align*}
\]

then it is a subgame perfect equilibrium point. Hence we would like to select an equilibrium point from the set of these equilibrium points by comparing
relative stabilities of these points. In the next section, we shall define a solution concept for this purpose.

3. The Perfect Equilibrium Point of $G(f_0, g_0)$

We shall define a solution concept being based on the idea of the smoothing procedure introduced by Nash [3]. In our case, the basic idea is that the bargaining game $G(f_0, g_0)$ is thought of as a limit case of bargaining games in which the probabilities that $S$ and $B$ violate the rules of the 1st and 2nd stages, i.e., $f_0$ and $g_0$, take positive values. Selten [5] reconsidered the concept of his subgame perfect equilibrium point in extensive games and defined a solution concept called “perfect equilibrium point” due to the similar procedure. The approach of this paper has the same thought with Selten’s.

We shall approximate $f_0$ and $g_0$ by classes of smooth functions. Let $h_0$ be the function on $(-K, K)$ such that

$$h_0(w) = \begin{cases} 1 & \text{if } w = 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $K$ is a positive constant. Of course, when $K = Q$ or $K = M$, $h_0 = f_0$ or $h_0 = g_0$, respectively. We call $\{h_n\}$ an approximate sequence of $h_0$ if

for all $n \geq 1$, $h_n(w)$ is a continuously differentiable function on $(-K, K)$ with $0 < h_n(w) \leq 1$ and $h_n(0) = 1$, (3.1)

log $h_n(w)$ is a strictly concave function for all $n \geq 1$, (3.2)

$\{h_n'/h_n\}$ converges pointwise to $+\infty$ on $(-K, 0)$ and to $-\infty$ on $(0, K)$. (3.3)

Here $h'_n = dh_n/dw$. It is not difficult to verify that these conditions imply

$$\lim_{n \to \infty} h_n(w) = h_0(w) \quad \text{for each } w \in (-K, K),$$

log $h_n(w)$, i.e., $h_n(w)$ itself is increasing on $(-K, 0)$ and decreasing on $(0, K)$ for each $n \geq 1$, (3.5)

$\{h_n'/h_n\}$ converges uniformly to $+\infty$ on $(-K, -\varepsilon)$ and to $-\infty$ on $(\varepsilon, K)$ for any $\varepsilon > 0$. (3.6)

For example, it is the case if $h_0(w) = e^{-nw}$.

Let $\{f_n\}$ and $\{g_n\}$ be approximate sequences of $f_0$ and $g_0$, respectively. In
the following, we shall consider the games $SG(q_a, q_b, g_a)$ and $RG(f_n, g_0, t_i, t_0)$ in which $f_0$ and $g_0$ are substituted by $f_n$ and $g_n$. The definitions of games $SG(q_a, q_b, g_a)$ and $RG(f_n, g_0, t_i, t_0)$ are clear and unmistakable. We shall think of $SG(q_a, q_b, g_a)$ and $RG(f_n, g_0, t_i, t_0)$ as limit cases of $SG(q_a, q_b, g_0)$ and $RG(f_n, g_0, t_i, t_0)$.

A function $f_n$ assigns the probability that the negotiation goes on to the 2nd stage. When $q_a$ is not equal to $q_b$, $f_n(q_a - q_b)$ is the probability that $S$ and $B$ violate the rule of the 1st stage and make the negotiation go on to the 2nd stage. Similarly, $g_n$ assigns the probability that the negotiation is concluded, assuming that it goes on to the 2nd stage. That is, when $p_s \neq p_b$, $g_n(p_s - p_b)$ is the probability that $S$ and $B$ violate the rule of the 2nd stage and make the negotiation be concluded. Condition (3.1) means that the probability of the violation is always positive, but (3.3) that it becomes quickly smaller as the disparity between $q_a$ and $q_b$ ($p_s$ and $p_b$) becomes larger. Equation (3.2) is a regularity condition.

We call a pair $(b_n^*, b_n^*) = ((q_n^*, t_n^*), (q_n^*, t_n^*))$ of behavior strategies a \textit{perfect equilibrium point} if it satisfies (3.7), (3.8) and (3.9):

$$(b_n^*, b_n^*) \text{ is a subgame perfect equilibrium point of } G(f_n, g_0). \quad (3.7)$$

Let $(g_n)$ be an arbitrary approximate sequence of $g_0$ and let $(q_n, q_b)$ be arbitrary in $(0, Q]^2$. Let $\{E_n\}$ be the sequence such that each $E_n$ is the set of equilibrium points of $SG(q_n, q_b, g_n)$. Then the sequence $\{E_n\}$ converges to $\{(p_s, p_b)\}$ in the sense of the Hausdorff metric for subsets. Here $(p_s, p_b)$ is the induced strategy pair of $(b_n^*, b_n^*)$ on $SG(q, q_b, g_0)$. \quad (3.8)

Let $\{f_n\}$ be an arbitrary approximate sequence of $f_0$. Let $\{Q_n\}$ be the sequence such that each $Q_n$ is the set of equilibrium points of $RG(f_n, g_0, t_i^*, t_0^*)$. Then the sequence $\{Q_n\}$ converges to $\{(q_n^*, q_n^*)\}$ in the sense of the Hausdorff metric for subsets. \quad (3.9)
BILATERAL MONOPOLY

In fact, since $E_n$ and $Q_n$ are singleton sets for sufficiently large $n$ as will be shown in the subsequent section, it is not necessary to use the Hausdorff metric for subsets instead of the usual Euclidean metric in (3.8) and (3.9).

This definition is based on the intuitive idea that a reasonable equilibrium point should be stable against arbitrarily small imperfections of rationality. This paper makes probabilities of violations of or deviations from the rules of game represent imperfections of rationality. Approximate sequences $\{f_n\}$ and $\{g_n\}$ represent arbitrarily small probabilities of violations of or deviations from the rules of game. The above intuitive idea is rewritten in terms of approximate sequences as follows: For any approximate sequences a reasonable equilibrium point in the limit of the sets of all equilibrium points of corresponding games. That is, when a small violation of the rules of game occurs, a reasonable equilibrium point changes a little. This is a continuity property, which is really the requirement of the definition of perfect equilibrium point. It is noted that we impose this requirement upon every subgame $SG(q_s, q_b, g_0)$ and the reduced game $RG(f_0, g_0, t^*_s, t^*_b)$. The reason for this is the same with that for the definition of subgame perfect equilibrium point by Selten. The basic assumption for this is that when a player makes a decision on an information set in a game tree, he uses a rationality to make it supposing that when he will make a decision on a succeeding information set, he will use the same rationality again. This would be a natural assumption. This is also the reason why only $f_0$ is substituted by $f_n$ but not $g_0$ in $RG(f_0, g_0, t^*_s, t^*_b)$ of (3.9). In sum, we require perfect equilibrium point to have the stability against small imperfections of rationality on every information set.

Our main result of this paper is the following theorem:

**Main Theorem.** There exists a unique perfect equilibrium point of the bargaining game $G(f_0, g_0)$. The perfect equilibrium point $(b^*_s, b^*_b) = ((q^*_s, t^*_s), (q^*_b, t^*_b))$ is represented as

$$q^*_s = q_s^0 = q_b^0 \quad \text{and} \quad t^*_s(q_s, q_b) = t^*_b(q_s, q_b) = \frac{m(q_s + q_b)}{2}$$

for all $(q_s, q_b) \in [0, Q]^2$, (3.10)

where $q_s$ and $m(q)$ are given in Lemmas 4 and 2, respectively.

This theorem means that the perfect equilibrium point yields the Nash allocation $((Q - q_s, m(q_s)), (q_b, M - m(q_b)))$, and that even if $(q_s, q_b) \neq (q_s^0, q_b^0)$ are played but if the negotiation goes on to the 2nd stage, the perfect equilibrium point also yields the Nash cooperative solution $((Q - (q_s + q_b)/2, m((q_s + q_b)/2)), ((q_s + q_b)/2, M - m((q_s + q_b)/2)))$ in the subgame of the 2nd stage.
Example 2. Let us consider the bilateral economy given in Example 1. Then the perfect equilibrium point \((b^s, b^h) = ((q^s, t^s), (q^h, t^h))\) is given as

\[ q^s = q^h = q_N = 900/13, \]
\[ t^s(q_N, q_h) = t^h(q_N, q_h) = m((q_N + q_h)/2) \]
\[ = (\frac{1}{2} \sqrt{(q_N + q_h)/2} - \sqrt{100 - (q_N + q_h)/2 + 10})/2 \]

for all \((q_N, q_h) \in (0, Q)^2\).

The Nash allocation is \(((Q - q_N, m(q_N)), (q_N)), (q_N, M - m(q_N)) = ((400/13, (25 + 10 \sqrt{13})/2 \sqrt{13}), (900/13, (30 \sqrt{13} - 25)/2 \sqrt{13})\) \(\simeq ((30.8, 8.5), (69.2, 11.5))\).

We have shown that the Nash cooperative solution is derived as the perfect equilibrium point in the bargaining game \(G(f_0, g_0)\). The bargaining game \(G(f_0, g_0)\), however, is one of several models which we can consider. For example, we can formulate other models of the bilateral monopoly, in which a quantity of the 1st commodity and a price of it are decided simultaneously, or in which a price of it is decided before a quantity is done, etc. In such models our basic approach of smoothing procedure may also choose a narrower class of equilibrium points from those of an original game. It should be, however, noted that the Nash allocation is not necessarily derived. The reason for the variety of models of the bilateral monopoly is that it has two decision-variables, i.e., it has the two kinds of commodities. For example, when an amount of the 1st commodity to be traded is fixed by an exogeneous decision, or when the commodity has a very large and indivisible unit, e.g., a house, a tanker, a petrochemical plant, etc., the game of the 2nd stage would be relevant to such an economy. It may be, however, more natural to assume instead of (2.2) in the 2nd stage:

\[ g_0(p_s - p_h) = 1 \quad \text{if} \quad p_s - p_h \leq 0 \]
\[ = 0 \quad \text{otherwise}. \]

In this case, our theorem still hold. Thus it is thought that our basic approach may be applied to wide fields.

4. Proof of Main Theorem

Initially, apart from the bargaining game \(G(f_0, g_0)\), we prepare three auxiliary theorems which will be used in the proof of the Main Theorem.
Let us consider a sequence of games \( \{J_n\} = \{(X_1, X_2, k_1^n, k_2^n)\} \) such that for all \( n \geq 1 \),

\[
X_1 = X_2 = (0, K),
\]

\[
k_1(x_1, x_2) = u_1((x_1 + x_2)/2) h_n(x_1 - x_2)
\]

\[
k_2(x_1, x_2) = u_2((x_1 + x_2)/2) h_n(x_1 - x_2)
\]

for all \( (x_1, x_2) \in (0, K)^2 \),

where \( u_1(z) \) and \( u_2(z) \) are real-valued functions on \( (0, K) \) and \( \{h_n\} \) is an approximate sequence of \( h_0 \).

**Auxiliary Theorem A.** Assume:

(A1) \( u_1 \) and \( u_2 \) are differentiable functions of \( z \).

(A2) \( \log u_1 \) and \( \log u_2 \) are concave functions on \( \{z \in (0, K) : u_1(z) > 0\} \) and \( \{z \in (0, K) : u_2(z) > 0\} \), respectively.

(A3) There is a unique \( z^* \in (0, K) \) such that \( \log u_1(z^*) + \log u_2(z^*) \geq \log u_1(z) + \log u_2(z) \) for all \( z \in \{z \in (0, K) : u_1(z), u_2(z) > 0\} \).

(A4) There is an integer \( N \) such that for all \( n \geq N \), there exists a unique equilibrium point \( (x_1^n, x_2^n) \) of \( J_n \) with \( x_1^n, x_2^n \in (0, K) \), \( u_1((x_1^n + x_2^n)/2) > 0 \) and \( u_2((x_1^n + x_2^n)/2) > 0 \).

Then it holds that \( (x_1^n + x_2^n)/2 = z^* \) for all \( n \geq N \) and \( \lim_{n \to \infty} x_1^n = \lim_{n \to \infty} x_2^n = z^* \).

**Proof.** It follows from (A1), (A2) and (A4) that for each \( n \geq N \), \( (x_1^n, x_2^n) \) is an equilibrium point of \( J_n \) if and only if \( (\partial/\partial x_1) \log k_1^n = (\partial/\partial x_2) \log k_2^n = 0 \), that is,

\[
\frac{u_1(z^*)}{u_1(z^*)} = \frac{u_2(z^*)}{u_2(z^*)} = \frac{h_n}{h_n}.
\]

(4.2)

It also follows from (A3) and (A4) that \( z \) coincides with \( z^* \) if and only if \( (d/dz)(\log u_1(z) + \log u_2(z))|_{z=z^*} = 0 \), that is,

\[
\frac{u_1'(z^*)}{u_1(z^*)} = \frac{u_2'(z^*)}{u_2(z^*)}.
\]

(4.3)

Since there is a unique \( z^* \) satisfying (4.3) by (A3), we have \( z^* = (x_1^n + x_2^n)/2 \) for each \( n \geq N \). If \( \{x_1^n - x_2^n\} \) does not converge to \( 0 \), then there are an \( \epsilon > 0 \) and a subsequence \( \{x_1^{n_r} - x_2^{n_r}\} \) such that \( |x_1^{n_r} - x_2^{n_r}| > \epsilon \) for all \( n_r \). Hence \( \{h_n(x_1^{n_r} - x_2^{n_r})/h_n(x_1^{n_r} - x_2^{n_r})\} \) converges uniformly to \( +\infty \) or \( -\infty \) by (3.6). This contradicts (4.2). Hence \( \{x_1^n - x_2^n\} \) must converge to \( 0 \). Hence we have \( \lim_{n \to \infty} x_1^n = \lim_{n \to \infty} x_2^n = z^* \).

Q.E.D.
Auxiliary Theorem B. Assume:

(B₁) \( u₁ \) and \( u₂ \) are differentiable and concave functions of \( x \).

(B₂) \( u₁ \) and \( u₂ \) are increasing and decreasing on \((0, K)\), respectively.

(B₃) There is a \( \tilde{z} \in (0, K) \) such that \( u₁(\tilde{z}) > 0 \) and \( u₂(\tilde{z}) < 0 \).

(B₄) \( \inf u₁(x) < 0 \) and \( \inf u₂(K) < 0 \).

Then Assumptions \((A₁), (A₂), (A₃)\) and \((A₄)\) of Auxiliary Theorem A hold.

Proof. It is easy to verify that \((B₁)-(B₄)\) imply \((A₁), (A₂)\) and \((A₃)\). We will show \((A₄)\). To do so, we need to prove two claims.

Claim 1. If \( u₁((x₁ + x₂)/2) \leq 0 \) or \( u₂((x₁ + x₂)/2) \leq 0 \), then there is an integer \( N₁ \) such that \((x₁, x₂)\) can not be an equilibrium point of \( Jₙ \) for all \( n \geq N₁ \).

It follows from \((B₂)\) and \((B₃)\) that \( u₁((x₁ + x₂)/2) \leq 0 \) implies \( u₂((x₁ + x₂)/2) > 0 \) and \( u₂((x₁ + x₂)/2) \leq 0 \) implies \( u₁((x₁ + x₂)/2) > 0 \). Let us consider the first case, i.e., \( u₁((x₁ + x₂)/2) \leq 0 \) and \( u₂((x₁ + x₂)/2) > 0 \). Let \( \epsilon \) be a sufficiently small positive number. If \( |x₁ - x₂| \geq \epsilon \), then there is an integer \( N₁ \) by (3.6) such that for all \( n \geq N₁ \),

\[
\frac{u₁((x₁ + x₂)/2)}{u₁((x₁ + x₂)/2)} > 2 \frac{h₂(x₁ - x₂)}{h₁(x₁ - x₂)} \quad \text{if} \quad x₁ > x₂,
\]

\[
\frac{u₁((x₁ + x₂)/2)}{u₂((x₁ + x₂)/2)} < 2 \frac{h₂(x₁ - x₂)}{h₁(x₁ - x₂)} \quad \text{if} \quad x₁ < x₂.
\]

That is,

\[
\frac{\partial}{\partial x₂} \log K₂(x₁, x₂) > 0 \quad \text{if} \quad x₁ > x₂,
\]

\[
\frac{\partial}{\partial x₂} \log K₂(x₁, x₂) < 0 \quad \text{if} \quad x₁ < x₂.
\]

This means that \((x₁, x₂)\) can not be an equilibrium point of \( Jₙ \) for all \( n \geq N₁ \). Suppose \( |x₁ - x₂| < \epsilon \). If \( x₁ = K \) or \( x₂ = K \), then \( u₂((x₁ + x₂)/2) < 0 \) by \((B₁)\), because \( \epsilon \) is sufficiently small, which is a contradiction. Hence we can assume \( x₁, x₂ \in (0, K) \). If \( u₁((x₁ + x₂)/2) = 0 \), then it holds by \((B₂)\) that for \( y₁ > x₁ \),

\[
K₂(y₁, x₂) = u₁((y₁ + x₂)/2) h₁(y₁ - x₂) > 0 = K₂(x₁, x₂),
\]

which means that \((x₁, x₂)\) can not be an equilibrium point of \( Jₙ \) for all \( n \geq 1 \).
Let \( u_1(x_1, x_2) < 0 \). If \( x_1 \leq x_2 \), then by (3.5), \( h_n'(x_1 - x_2)/h_n(x_1 - x_2) \geq 0 \) for all \( n \geq 1 \). Since \( u_2((x_1 + x_2)/2)/u_2((x_1 + x_2)/2) < 0 \) by \((B_2)\), we have
\[
\frac{u_2((x_1 + x_2)/2)}{u_2((x_1 + x_2)/2)} - 2 \frac{h_n'(x_1 - x_2)}{h_n(x_1 - x_2)} < 0 \quad \text{for all} \quad n \geq 1,
\]
i.e.,
\[
\frac{\partial}{\partial x_2} \log k_n^*(x_1, x_2) < 0 \quad \text{for all} \quad n \geq 1.
\]
This means that \((x_1, x_2)\) cannot be an equilibrium point of \( J_n \) for all \( n \geq 1 \). If \( x_1 > x_2 \), then \( \log h_n(y_1 - x_2) < \log h_n(y_1 - x_1) \), i.e., \( h_n(y_1 - x_1) < h_n(y_1 - x_2) \) for all \( y_1 > x_1 \) by (3.5). Further since \( u_1((y_1 + x_2)/2) > u_1((x_1 + x_2)/2) \) for all \( y_1 > x_1 \) by \((B_2)\), we have
\[
k_n^*(y_1, x_2) = u_1((y_1 + x_2)/2)h_n(y_1 - x_2) > u_1((x_1 + x_2)/2)h_n(x_1 - x_2)
\]
\[
= k_n^*(x_1, x_2) \quad \text{for all} \quad y_1 > x_1.
\]
This means that \((x_1, x_2)\) can not be an equilibrium point of \( J_n \) for all \( n \geq 1 \).

In the case where \( u_1((x_1 + x_2)/2) > 0 \) and \( u_2((x_1 + x_2)/2) \leq 0 \), we can analogously prove the claim.

**Claim 2.** If \( x_1 = K \) or \( x_2 = K \), then there is an integer \( N_2 \) such that \((x_1, x_2)\) cannot be an equilibrium point of \( J_n \) for all \( n \geq N_2 \).

We need to consider the case where \( u_2((x_1 + x_2)/2) > 0 \). Let \( x_1 = K \). Since \( u_2(K) \leq 0 \) by \((B_2)\), there is an \( \bar{x}_2 \) such that \( u_2((x_1 + \bar{x}_2)/2) = 0 \) and \( x_2 < \bar{x}_2 < K \). Since \( \{h_n'/h_n\} \) converges uniformly to \(-\infty\) on \((K - \bar{x}_2, K)\), there is an integer \( N_2 \) such that for all \( n \geq N_2 \),
\[
\frac{u_2((x_1 + x_2)/2)}{u_2((x_1 + x_2)/2)} > 2 \cdot \frac{h_n'(x_1 - x_2)}{h_n(x_1 - x_2)},
\]
i.e.,
\[
\frac{\partial}{\partial x_2} \log k_n^*(x_1, x_2) > 0 \quad \text{for all} \quad n \geq N_2.
\]
Since \( x_2 < K \), it follows from this that \((x_1, x_2)\) cannot be an equilibrium point of \( J_n \) for all \( n \geq N_2 \). In the case where \( x_2 = K \), we can analogously prove the claim.

It follows from these claims that for each \( n \geq N_3 \equiv \max(N_1, N_2) \), a necessary and sufficient condition for \((x_1, x_2)\) to be an equilibrium point of \( J_n \) is
\[
\frac{\partial}{\partial x_1} \log k_n^*(x_1, x_2) = \frac{\partial}{\partial x_2} \log k_n^*(x_1, x_2) = 0.
\]
i.e.,
\[ \frac{u'_1((x_1 + x_2)/2)}{u_1((x_1 + x_2)/2)} = \frac{u'_2((x_1 + x_2)/2)}{u_2((x_1 + x_2)/2)} = -2 \cdot \frac{h'_n(x_1 - x_2)}{h_n(x_1 - x_2)}. \] (4.4)

By (4.4) we have \((x_1 + x_2)/2 = z^*\), where \(z^*\) is the number given in \((A_3)\).

Since for any \(\varepsilon > 0\) there is an \(N_{\varepsilon}\) by (3.6) such that for all \(n \geq N_{\varepsilon}\),
\[ -2 \cdot \frac{h'_n(\varepsilon)}{h_n(\varepsilon)} > \frac{u'_1(z^*)}{u_1(z^*)} > -2 \cdot \frac{h'_n(-\varepsilon)}{h_n(-\varepsilon)}. \]

Since \(h'_n(w)/h_n(w)\) is a continuous and decreasing function of \(w\) on \((-K, K)\)
by (3.1) and (3.2), there is a unique \(w_n\) for each \(n \geq N_{\varepsilon}\) such that
\[ -2 \cdot \frac{h'_n(w_n)}{h_n(w_n)} = \frac{u'_1(z^*)}{u_1(z^*)}. \]

Hence \((x^*_1, x^*_2) = (z^* + w_n/2, z^* - w_n/2)\) satisfies (4.4) for each \(n \geq \max(N_{\varepsilon}, N_{\varepsilon})\), i.e., this \((x^*_1, x^*_2)\) is a unique equilibrium point of \(J_n\) for each \(n \geq N = \max(N_{\varepsilon}, N_{\varepsilon})\).

**Q.E.D.**

**Auxiliary Theorem C.** Assume that \(u_1(z) > 0\) and \(u_2(z) > 0\) for all \(z \in (0, K)\) and that \((A_1)\), \((A_2)\) and \((A_3)\) are true. Then \((A_4)\) is true.

**Proof.** Suppose that if \(x_1 = K\) or \(x_2 = K\), then \((x_1, x_2)\) cannot be an equilibrium point of \(J_n\) for all \(n \geq N\). Then it can be proved in the same way with the proof of Auxiliary Theorem B that \((A_4)\) is true. Hence we will show the above supposition.

Since \(z^* \in (0, K)\), we have \(u'_1(K)/u_1(K) + u'_2(K)/u_2(K) < 0\). Then it must hold that \(u'_1(K)/u_1(K) < 0\) or \(u'_2(K)/u_2(K) < 0\). Let \(\delta\) be a positive real number such that \(u'_1(K-\delta)/u_1(K-\delta) + u'_2(K-\delta)/u_2(K-\delta) < 0\).

Suppose \(x_1 = K\) and \(x_2 < K - \delta\). Since \(\{h'_n/h_n\}\) converges uniformly to \(-\infty\) on \((\delta, K)\), there is \(N\) such that
\[ \frac{u'_1((x_1 + x_2)/2)}{u_1((x_1 + x_2)/2)} < -2 \cdot \frac{h'_n(x_1 - x_2)}{h_n(x_1 - x_2)}, \]
i.e., \(\partial/\partial x_i\) log \(k^*_n(x_1, x_2) < 0\). This means that \((x_1, x_2)\) cannot be an equilibrium point of \(J_n\) for all \(n \geq N\). In the case where \(x_1 < K - \delta\) and \(x_2 = K\), we can prove similarly that \((x_1, x_2)\) cannot be an equilibrium point of \(J_n\) for all \(n > N\).

Suppose that \(x_1, x_2 \in [K - \delta, K]\). Let us consider the case where \(u'_1((x_1 + x_2)/2)/u_1((x_1 + x_2)/2) \leq u'_2((x_1 + x_2)/2)/u_2((x_1 + x_2)/2) \leq 0\). Of course, it holds that \(u'_1((x_1 + x_2)/2)/u_1((x_1 + x_2)/2) \leq 0\). If \(u'_1((x_1 + x_2)/2)/u_1((x_1 + x_2)/2) < -2h'_n(x_1 - x_2)/h_n(x_1 - x_2)\), then \(\partial/\partial x_i\) log \(k^*_n(x_1, x_2) < 0\), which implies that \((x_1, x_2)\) cannot be an equilibrium point of \(J_n\).
If \( u'_1((x_1 + x_2)/2)/u_1((x_1 + x_2)/2) \geq -2h'_n(x_1 - x_2)/h_n(x_1 - x_2) \), then \( u'_1((x_1 + x_2)/2)/u_1((x_1 + x_2)/2) < 2h'_n(x_1 - x_2)/h_n(x_1 - x_2) \), i.e., \((\partial/\partial x_1) \log k_2^r(x_1, x_2) < 0\). Hence \((x_1, x_2)\) cannot be an equilibrium point of \(J_n\) for all \(n \geq 1\). In the case where \(u'_1((x_1 + x_2)/2)/u_1((x_1 + x_2)/2) \leq u'_2((x_1 + x_2)/2)/u_2((x_1 + x_2)/2) \leq 0\), we can prove similarly that \((x_1, x_2)\) cannot be an equilibrium point of \(J_n\) for all \(n \geq 1\).

Let us consider the case where \(u'_1((x_1 + x_2)/2)/u_1((x_1 + x_2)/2) < 0\) and \(u'_2((x_1 + x_2)/2)/u_2((x_1 + x_2)/2) > 0\). If \(u'_1((x_1 + x_2)/2)/u_1((x_1 + x_2)/2) < -2h'_n(x_1 - x_2)/h_n(x_1 - x_2)\), then \((\partial/\partial x_1) \log k_2^r(x_1, x_2) < 0\), which implies that \((x_1, x_2)\) cannot be an equilibrium point of \(J_n\). If \(u'_1((x_1 + x_2)/2)/u_2((x_1 + x_2)/2) \geq -2h'_n(x_1 - x_2)/h_n(x_1 - x_2)\), then \(u'_2((x_1 + x_2)/2)/u_2((x_1 + x_2)/2) < 2h'_n(x_1 - x_2)/h_n(x_1 - x_2)\), because \(u'_1((x_1 + x_2)/2)/u_1((x_1 + x_2)/2) + u'_2((x_1 + x_2)/2)/u_2((x_1 + x_2)/2) < 0\). Hence we get \((\partial/\partial x_2) \log k_2^r(x_1, x_2) < 0\).

Hence we have shown that \((x_1, x_2)\) cannot be an equilibrium point of \(J_n\) for all \(n \geq 1\). In the case where \(u'_1((x_1 + x_2)/2)/u_1((x_1 + x_2)/2) > 0\) and \(u'_2((x_1 + x_2)/2)/u_2((x_1 + x_2)/2) < 0\), we can prove similarly that \((x_1, x_2)\) is not an equilibrium point of \(J_n\) for all \(n \geq 1\).

Q.E.D.

Proof of Main Theorem. Initially let us consider the 2nd stage, and let \((q_1, q_2) \in (0, Q]^2\) be arbitrarily chosen and fixed in the 1st stage. We show that \(u_i(z) = U(Q - (q_1 + q_2)/2, z)\) and \(u_i(z) = V((q_1 + q_2)/2, M - z)\) on \((0, M]\) satisfy Assumptions (B₁)–(B₃) of Auxiliary Theorem B. \(u_1(z)\) and \(u_2(z)\) are increasing and decreasing functions by Assumption (A), respectively. Clearly, they are differentiable and concave functions by (B) and (C). It also follows from Lemma 1 that these functions satisfy (B₃). Assumption (B₄) follows from Assumptions (A) and (D). Hence we can apply Auxiliary Theorems A and B to \(\{SG(q_1, q_2, g_n)\}\), and it follows that if \((b^*_p, b^*_q) = ((q^*_p, t^*_p), (q^*_q, t^*_q))\) is a perfect equilibrium point, then

\[t^*_q(q_2, q_3) = t^*_q(q_1, q_3) = z^*,\]

where \(z^*\) is the number given in (A₄). Of course, this \(z^*\) coincides with \(m((q_1 + q_2)/2)\), which is given in Lemma 2.

Next let us consider the 1st stage. We show that \(u_i(q) = u(q) = U(Q - q, m(q))\) and \(u_i(q) = v(q) = V(q, M - m(q))\) on \((0, Q]\) satisfy the assumptions of Auxiliary Theorem C. It follows from Lemmas 1 and 2 that \(u(q) > 0\) and \(v(q) > 0\) for all \(q \in (0, Q]\). Assumptions (A₃) and (A₅) follow from Assumption (F) and Lemma 4, respectively. Further (A₁) is true by the following lemma, which will be proved in the Appendix.

Lemma 5. \(m(q)\) is a differentiable function of \(q\) on \((0, Q]\).

Hence we can apply Auxiliary Theorems A and C to \(\{RG(f_n, g_n, t^*_p, t^*_q)\}\), and it follows that if \((b^*_p, b^*_q) = ((q^*_p, t^*_p), (q^*_q, t^*_q))\) is a perfect equilibrium point, then \(q^*_q = q^*_p = q_n\), where \(q_n\) is given in Lemma 4.

Q.E.D.
**APPENDIX**

**Proof of Lemma 1.** Let \( a = q/Q \). Since \( a > 0 \), it follows from (C) and (D) that for \( m \) given in (D)

\[
U(Q - q, am) \geq aU(0, m) + (1 - a) U(w') = aU(0, m) > 0, \\
V(q, M - am) \geq aV(Q, M - m) + (1 - a) V(w') = aV(Q, M - m) > 0.
\]

It is clear that \( V(q, M) > 0 \). Since \( U(0, M) > 0 \) by (D), we have, by (A), \( U(Q - q, M) \geq U(0, M) > 0 \). For all \( m' \in [0, am] \), it holds by (A) that

\[
V(q, M - m') \geq V(q, M - am) > 0.
\]

For all \( m' \in [am, M] \), it holds that

\[
U(Q - q, m') \geq U(Q - q, am) > 0.
\]

Q.E.D.

**Proof of Lemma 2.** For each fixed \( q \), \( \log U(Q - q, m) + \log V(q, M - m) \)

is defined for some \( m \) by Lemma 1. If \( m = 0 \) or \( m = M \), then \( U(Q - q, m) V(q, M - m) < 0 \) by (A) and (D). Hence \( \log U + \log V \) is defined on an open interval \((a, b)\) since \( U \) and \( V \) are concave functions. Since \( U(Q - q, a) V(q, M - a) = U(Q - q, b) V(M - b) = 0 \), we have

\[
\lim_{m \to a, b} (\log U + \log V) = \lim_{m \to b, a} (\log U + \log V) = -\infty.
\]

Further since \( \log U + \log V \) is a strictly concave function of \( m \), there is a unique \( m(q) \) satisfying (1.2).

Q.E.D.

**Proof of Lemma 3.** Let \( q_1, q_2 \) and \( a \) \((0 < a < 1, b = 1 - a \) and \( q_1 \neq q_2 \)) be arbitrarily chosen. Then we have, by (C) and definition of \( m(q) \),

\[
aL(q_1) + bL(q_2) \leq \log(aU(Q - q_1, m(q_1)) + bU(Q - q_2, m(q_2))) \\
+ \log(aV(q_1, M - m(q_1)) + bV(q_2, M - m(q_2))) \\
< \log U(Q - (aq_1 + bq_2), (aq_1 + bm(q_1)) \\
+ \log V(q_1, b_2, M - (am(q_1) + bm(q_2))) \\
< \log U(Q - (aq_1 + bq_2), m(aq_1 + bq_2)) \\
+ \log V(q_1, b_2, M - m(aq_1 + bq_2)) \\
= L(aq_1 + bq_2). 
\]

Q.E.D.

**Proof of Lemma 4.** Since \( L(q) \) is a strictly concave function and \( \lim_{q \to 0} L(q) = -\infty \), there is a \( q_N \) such that \( L(q_N) \geq L(q) \) for all \( q \in (0, Q] \). By (E) we have \( q_N < Q \). The uniqueness follows from the strict concavity of \( L(q) \).

If there is another allocation \(((Q - q^*, m^*), (q^*, M - m^*))\) such that
q* ≠ q, and L(q*) ≤ \log U(Q - q, m*) + \log V(q*, M - m*), then it holds by the above that \log U(Q - q, m*) + \log V(q*, M - m*) > L(q*), which is a contradiction. Q.E.D.

Proof of Lemma 5. It follows from Lemma 2 that m = m(q) if and only if

\[
\frac{U_2(Q - q, m)}{U(Q - q, m)} = \frac{V_2(q, M - m)}{V(q, M - m)}.
\]

Here U_2 = \partial U/\partial q_2 and V_2 = \partial V/\partial q_2.

Let \( f(q, m) = \frac{U_2(Q - q, m)}{U(Q - q, m)} - \frac{V_2(q, M - m)}{V(q, M - m)} \). Then \( f(q, m(q)) \equiv 0 \). Let us consider \( f(q + \Delta q, m(q) + \Delta m) \equiv 0 \). Since f has continuous partial derivatives \( \partial f/\partial q = f_1 \) and \( \partial f/\partial m = f_2 \) by (B), it holds that

\[
0 = f(q + \Delta q, m(q) + \Delta m) \\
= f(q + \Delta q, m(q) + \Delta m) - f(q + \Delta q, m(q)) \\
+ f(q + \Delta q, m(q)) - f(q, m(q)) \\
= f_2(q + \Delta q, m(q)) \cdot \Delta m + o(\Delta m) + f_1(q, m(q)) \cdot \Delta q + o(\Delta q).
\]

Hence we have

\[
\frac{\Delta m}{\Delta q} \left( f_2(q + \Delta q, m(q)) + \frac{o(\Delta m)}{\Delta m} \right) + f_1(q, m(q)) + \frac{o(\Delta q)}{\Delta q} = 0.
\]

Since \( F(m) \equiv \log U(Q - q, m) + \log V(q, M - m) \) is a strictly concave function of m, \( d^2F/dm^2 < 0 \), i.e., \( f_2 < 0 \). Hence we have \( dm(q)/dq = -f_1/f_2 \). Q.E.D.

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