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# AN EXAMPLE OF A MULTI-OBJECT AUCTION GAME*

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Concurrent auctions of several objects are traditionally analyzed as if they were independent single-object auctions. Such an approximation may be very crude if bidders have budgetary restrictions, capacity constraints, or, in general, have non-linear utility functions. This paper presents a very simple multi-object auction for which explicit equilibrium strategies can be calculated; these equilibrium strategies have several qualitative characteristics arising from the multi-object nature of the example and therefore not present in typical single-object auctions. In particular, these results suggest that the observed variance in bids for offshore oil leases is not solely due to bidder uncertainty, but also to the method of auction currently employed.
(BIDDING)

## 1. Introduction

The auction is a common market mechanism. Public agencies, private institutions, and individuals alike often procure, sell, or allocate goods, services and resources through auctions. A typical buyer or seller is often involved with the simultaneous sale of several items. Offshore oil lease sales, the letting of defense contracts, and the procurement of supplies are examples of multi-object auctions; in a similar vein, an individual may participate in simultaneous auctions for several unrelated goods or services.

Of the approximately 500 works listed by Stark and Rothkopf [2] in their bibliography of research into auctions and competitive bidding, fewer than two dozen are explicitly concerned with multi-object auction models. Most of these deal with the sale of identical objects, such as U. S. Treasury bills; in such auctions, the number of objects won by each bidder is important, but the identities of the objects are not. When the objects have distinguishing characteristics, it is traditional to treat their simultaneous sale at auction as a collection of independent auctions. However, such an approach may be inappropriate if the bidders have nonlinear utility functions (such as might, for example, be associated with budgetary or capacity constraints). In such a case, the value of "winning" a particular object depends on what other objects are also won. Thus, in general, the value for a set of objects is not simply the sum of the values of the individual objects.

This paper considers the effects of nonadditivity on equilibrium bidding strategies and on the efficiency of auctions. A very simple example of a multi-object auction game with quite severe capacity constraints is examined. Equilibrium strategies are calculated for this example.

The equilibrium strategies illustrate several possible aspects of multi-object auctions. If the objects are sold at auction simultaneously, the resulting allocation may have only a small fraction of the maximum possible social value, where social value is defined as the sum of all bidders' profits and all sellers' revenues (this social value is maximized by any Pareto-optimal allocation of the objects). Under equilibrium bidding in simultaneous auctions, bidders may bid more aggressively for some objects than for other objects of equal value; although a bidder may be able to use only a

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certain numer of the objects, he is willing to take additional objects if they come at bargain prices. We conclude that it may not be appropriate to analyze the bidding on one of a number of simultaneously auctioned objects as if its sale were independent of the sale of the remaining objects.

## 2. An Example of a Multi-object Auction

There are $n$ husband-wife couples, each in the market for a used dresser. Each couple wants only one dresser; they are willing to pay \$100 for any single dresser, but consider additional dressers to be valueless. During the upcoming weekend, there will be garage sales at $n$ different locations; at each, a dresser will be sold by means of a progressive auction.

How should a couple go about shopping for a dresser? They could go to one randomly-selected garage sale, and participate in the auction there in the hope of winning the dresser. Of course, they risk the chance of returning home without a dresser. Alternatively, the husband and wife could participate in two separate sales (or, with the aid of children or other representatives, in more than two). This would increase their chance of coming home with at least one dresser, but would expose them to the risk of coming home with more than one, all but the first of which are worthless.

Implicit in this example is the assumption that no secondary market exists for the disposal of excess purchases, or for the sale of unpurchased dressers. Were we dealing with perishable commodities, the assumption would clearly be justified; alternatively, we may simply assume that the cost of participation in a secondary sale is prohibitive.

## 3. Equilibrium Strategies

If each couple goes to a randomly-selected sale, then the dominant strategy for each couple is to start bidding as low as possible (say, at zero), and to be willing to raise their bid to as much as \$100 if there is competition. Thus, a particular dresser may remain unsold (if no one shows up at that sale), be sold for an arbitrarily small amount (if exactly one couple shows up), or be sold for \$100 (if more than one couple shows up).

Alternatively, let each husband and wife separately attend a randomly-chosen pair of sales, where the husband has instructions to bid up to some amount $\$100P$ at his auction and the wife has instructions to bid up to some amount $\$100Q$ at her auction. For the sake of simplicity, we consider only the limiting case which arises when the number $n$ of couples and of garage sales is large. (All subsequent results will be for this limiting case; derivations and computations are outlined in the appendix.)

This auction game has a symmetric Nash equilibrium point, in which the equilibrium strategy for a couple is to choose $P$ at random, according to the distribution $F(p) = \Pr(P \leqslant \$100p) = -1 - \ln(1-p)$, where $1 - 1/e \leqslant p \leqslant 1 - 1/e^2$, and to let $Q$ be determined by $Q = 1 - 1/(e^2(1-P))$ (so that $0 \leqslant Q \leqslant 1 - 1/e$). Notice that in this case, the equilibrium strategies require randomization. Some dressers may remain unsold, some may be sold for a pittance and some for a substantial amount; a couple may end up with no dresser, exactly one, or perhaps more than one dresser.

## 4. Efficiency of the Allocation

A number of efficiency measures may be considered for auctions. For illustrative purposes, we will use "social value": the sum of all buyers' profits and all sellers' revenues. This quantity is maximized by a Pareto-optimal allocation; thus, social value serves as a measure of how close an allocation is to being Pareto optimal.

In the example, the social value of an outcome is equal to \$100 times the number of dressers, times the probability that any given couple wins at least one dresser. Thus, the Pareto-optimal allocation, in which each of the couples receives exactly one dresser, has a social value of \$100$n$. If each couple goes to a single randomly-chosen sale, then at the dominant strategy equilibrium, the probability that a couple receives a dresser is about two-thirds; in the limiting case, the probability is $1 - 1/e$. On the other hand, if each couple participates in two randomly-selected sales, the probability (at equilibrium) of a couple winning at least one dresser increases to about four-fifths (more precisely, $1 - 2/e^2$).

Not only is the scheme in which a husband and wife go together to a single auction less efficient than that in which they attend two different auctions, but (disregarding the additional cost of having the couple attend two rather than just one auction) the second scheme is also preferable in terms of several other measures. The probability that a dresser is sold is $1 - 1/e$ for the first scheme, and $1 - 1/e^2$ for the second. The expected revenue to the seller under the two schemes is $\$100(1 - 2/e)$ and $\$100(1 - 5/e^2)$, respectively; the expected profit to the buyers is $\$100/e$ and $\$300/e^2$, respectively. Each of these three measures is greater for the second scheme than for the first.

We suggest that inefficiencies similar to those observed in this example may occur quite generally when multi-object sales are conducted as simultaneous single-object auctions. Although it is assumed in the example that each bidder knows the true value of each object and that each object is sold at a price just barely greater than the second highest bid on the object, it seems unlikely that these assumptions are the source of the inefficiencies. Indeed, the existence of an active after-auction market for offshore oil leases attests to existing inefficiencies. (A Congressional study [4] notes that in OCS sale #40, Conoco apparently won substantially more leases than desired, and promptly resold a fraction of them to Gulf. The economic frictions associated with the after-market are a cost paid by society as a whole. In addition, the government has recently expressed concern about the incentives for collusion created by the existence of a secondary market.)

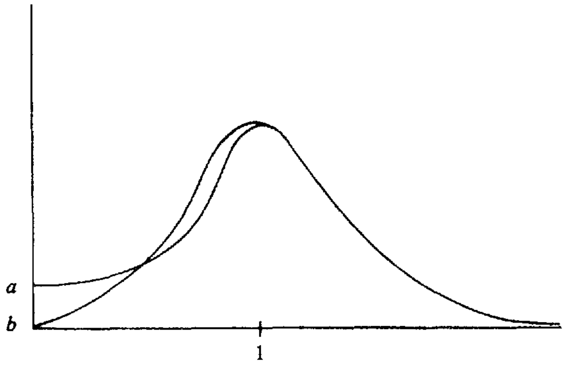
## 5. Strategic Variance

The vast majority of published research considers only single-object auction models; often there is at least an implicit assumption that if an individual is participating in several simultaneous auctions for identical objects then his behavior in any auction will be independent of his behavior in the other auctions. The equilibrium strategies for the example indicate that this assumption is not in general correct. The existence of capacity constraints or budgetary restrictions may result in different bids being made at different auctions even though the objects sold in the different auctions are identical.

In this particular example, the equilibrium strategies for the case of husband and wife attending separate auctions have a simple interpretation. Each couple desires a dresser, and thus bids aggressively on one dresser. However, to cover themselves in case they do not win with their aggressive bid, they also submit a second, less aggressive bid. By bidding on two objects, the chances of winning at least one are improved; if both husband and wife obtain a dresser, then the second one is obtained at a relatively low price.

One might view the problem from the sellers' side. Suppose that they are unaware that the buyers consider all dressers equally valuable; assume all the auctions are actually conducted as second-price sealed-bid auctions. Any seller receiving at least two bids could plot the distribution of the sizes of bids received. To achieve a common scale, each seller might divide all bids on his dresser by the average bid on his dresser; the seller, unaware of the relative values of the dresser, might consider the average bid an appropriate indication. Finally, the sellers might pool their resulting data to obtain a composite distribution (with an undetermined scale factor) of what fraction the typical bid is of the estimated value. The average composite distribution appears as Figure 1, which also contains the graph of a lognormal density function with mean and variance (of the underlying normal distribution) of zero and 0.09 respectively; the similarity between the two distributions might be of interest to those familiar with the oftmade observation that bid fractions tend to be lognormally distributed, and to those who might contemplate the use of this observation to infer that true values (or bidders' estimates) are lognormally distributed.

We suggest that some of the variance in bids observed in off-shore oil lease auctions may be due to strategic, rather than informational, factors.



[a] Average distribution of the ratio between a bid and the average bid on the corresponding object.
[b] Lognormal distribution with mean = 0 and variance = 0.09 for the underlying normal distribution.

FIGURE 1

## 6. Conclusion

In this paper, several aspects unique to multi-object auctions have been examined in terms of a specific example. It is apparent that conducting multi-object sales as simultaneous single-object auctions may, in general, be quite inefficient. In addition, analyzing simultaneous single-object auctions as symmetric replicas of one single-object auction ignores the possibility that individuals may bid differently in different auctions; even if identical objects are for sale in separate auctions, different bids can arise from the bidders' capacity constraints or budgetary restrictions, or from other nonlinearities in their utility functions. Although the example studied is quite specific and uses the second-price sealed-bid auction mechanism, the phenomena discussed appear to be somewhat independent of the specific example and should be of concern whenever bidders with non-linear utilities compete for more than one object in simultaneous auctions.

## Appendix

Throughout this appendix it is assumed that $n$, the number of objects and of bidders, is sufficiently large that the numbers of bidders in various situations can be approximated by Poisson-distributed random variables. It is also assumed that bids are raised continuously (rather than in discrete steps). This allows us to represent each auction by a sealed-bid procedure, in which the high bidder pays a price equal to the maximum of the remaining bids [3]. In the case of ties, the dresser being bid upon is awarded at random to one of the tied high bidders.

*Case* 1. Each couple attends a single randomly-chosen auction.

It is easy to verify that an expected-profit-maximizing strategy for any couple is to bid \$100, regardless of what the other couples do; this is the unique maximizing strategy if some other couples also occasionally bid \$100. The probability that any given dresser is sold is $1 - 1/e$, the probability that at least one couple attends the sale; this is also the probability that any given couple obtains a dresser.

The seller's revenue is \$100 for each dresser on which at least two couples bid. Thus the expected revenue per dresser is $\$100(1 - 2/e)$. The expected profit to any couple is \$100 times the probability that the couple faces no competition, and is thus $\$100/e$.

*Case* 2. Each husband and wife participate in a randomly-chosen pair of auctions.

Consider a particular couple, and assume that each other couple follows the strategy indicated in the text: enter bids of $\$100P$ and $\$100(1 - 1/e^2(1 - P))$ at two auctions, where $P$ is chosen according to the distribution $F(p) = -1 - \ln(1 - p)$, with $1 - 1/e \leqslant p \leqslant 1 - 1/e^2$. Assume that the selected couple enters bids of $\$100r$ and $\$100s$, with $r \geqslant s$. Clearly it is not necessary to consider the case in which $r > 1 - 1/e^2$; a bid of $1 - 1/e^2$ will have essentially the same effect. Assume $r \geqslant 1 - 1/e$. Due to the dominance of the "truthful" bidding strategy in progressive auctions, (that is, if the expected value to a bidder of the object being auctioned is independent of its value to the other bidders, then his optimal bid is equal to this value, and is independent of the bids of the others). It is optimal to then choose $s$ so that $\$100s$ equals the expected value of the dresser being bid upon, given that a bid of $\$100r$ has been entered elsewhere. Hence, $s$ should satisfy

$$\begin{aligned} s &= 1 - \Pr(\$100r \text{ wins elsewhere}) \\ &= 1 - \sum_{k=0}^{\infty} \Pr\big(\$100r \geqslant \text{maximum of other bids at same location} \mid \\ &\qquad\qquad \text{there are } k \text{ other bids in } \left[1 - 1/e, 1 - 1/e^2 1\right]\big). \end{aligned}$$

$$\begin{aligned} &\Pr\left(k \text{ other bids in } \left[1 - 1/e, 1 - 1/e^2\right]\right) \\ &\quad = 1 - \sum_{k=0}^{\infty} \left[F(r)\right]^k \cdot e^{-1}/k! \\ &\quad = 1 - 1/\left(e^2(1 - r)\right). \end{aligned}$$

If $r < 1 - 1/e$, it can be shown that the optimal value of $s$ is greater than $r$. Hence, the couple should use a bid pair $\{r, 1 - 1/(e^2(1 - r))\}$, with $1 - 1/e \leqslant r \leqslant 1 - 1/e^2$.

For any such bid pair, the expected profit is $\$300/e^2$, and does not depend on $r$. Since the expected profit is independent of $r$, any mixture of bid pairs (and in particular, the mixture prescribed by $F(\cdot)$) is an optimal response to the others' strategies; hence, the suggested strategies are in equilibrium. The associated statistics cited in the text can be derived in a straightforward manner.

The equilibrium situation just derived involves the use of the same mixed strategy by all couples. Alternatively, assume that each couple employs a pure strategy of the $\{p, 1 - 1/e^2(1 - p)\}$-type, and that the values of $p$ are distributed across the population according to $F(\cdot)$. It is clear that the resulting situation is also in equilibrium, and leads to the same distribution of observed bids.[1]

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# ERRATUM TO "AN EXAMPLE OF A MULTI-OBJECT AUCTION GAME"

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(AUCTIONS; COMPETITIVE BIDDING)

A number of galley revisions were not implemented prior to the publication of [1]. The principal purpose of the revisions listed below is to increase the clarity and notational consistency of the paper; on matters of substance, the original text is correct as its stands. Reprints of the paper, together with these revisions, are available from the Cowles Foundation for Research in Economics, Yale University, New Haven, Connecticut 06520.

| Page | Line | Uncorrected text | Correction |
|---|---|---|---|
| 1272 | 3 | non-linear | nonadditive |
| 1272 | 10 | (BIDDING) | (AUCTIONS; COMPETITIVE BIDDING) |
| 1272 | 26 | nonlinear | nonadditive |
| 1272 | 27 | (such . . . constraints). | such . . . constraints. |
| 1272 | − 2 | Yale University | University of Illinois |
| 1273 | 1 | numer | number |
| 1273 | − 12 | point, | point (that is, a situation in which no couple can gain from a unilateral change of strategy), |
| 1273 | − 10 | $\Pr(P \leqslant \$100p)$ | $\Pr(P \leqslant p)$ |
| 1274 | 7 | probability | proportion |
| 1274 | 8 | a couple winning | couples winning |
| 1274 | 18 | We suggest | The most important observation is that both schemes fall substantially short of the efficiency of the Pareto-optimal allocation. |
| 1275 | 2 | relative values of the | specific value of his |
| 1275 | 3 | indication. | estimate. |
| 1275 | 6 | as Figure 1, | in Figure 1, |
| 1275 | − 11 | nonlinearities | nonadditivities |
| 1275 | − 8 | nonlinear | nonadditive |
| 1276 | 19 | auctions, ( | auctions ( |
| 1276 | 22 | others). It | others), it |
| 1276 | 27 | $e^2 1]$). | $e^2]$) |
| 1276 | 28 | $\Pr(k$ other bids | $\cdot \Pr(k$ other bids |

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1. ENGELBRECHT-WIGGANS, R. AND WEBER, R. J., "An Example of a Multi-Object Auction Game," *Management Sci.*, Vol. 25, No. 12 (1979), pp. 1272–1277.

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