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# AN ELEMENTARY PROOF OF THE NO-RETRACTION THEOREM

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**1.** The no-retraction theorem [1], [6], [7, p. 271] asserts the nonexistence of a continuous function $f$ mapping the unit ball $B \subset R^n$ into its boundary $S$, such that $f(x) = x$ for all $x \in S$. This theorem is usually proved by means of either combinatorial arguments, homology theory, differential forms, or methods from differential topology, see [1], [5], [6], [10]. The proof given in [3], while analytic and entirely elementary, uses a homotopy in order to get the desired contradiction via $(n+1)$-dimensional integration. We offer here a self-contained proof, inspired

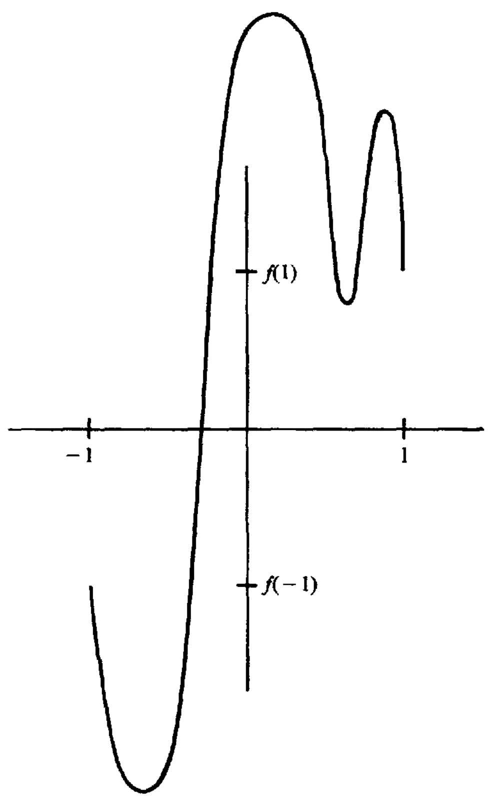


FIG. 1

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by the one in [3], of the "no differentiable retraction" theorem, a proof which employs only "engineering type" Advanced Calculus concepts. No homotopy is involved in our argument.

To motivate our method, consider the following *one*-dimensional no differentiable retraction theorem:

*There exists no differentiable* $f: [-1, 1] \to \{-1, 1\}$, *such that* $f(1) = 1, f(-1) = -1$.

(Of course, you do not need the differentiability assumption, as this theorem is just the intermediate value theorem. But having no higher dimensional analogue of the latter theorem—other than the no-retraction theorem itself—we prefer to suggest a proof which we will then generalize to $n$ dimensions.)

*Proof.* Clearly, $f'(t) \equiv 0$ for $-1 \leqslant t \leqslant 1$, since otherwise the range of $f$ would contain an interval. But

$$0 = \int_{-1}^{1} f'(t)\,dt = f(1) - f(-1) = 1 - (-1) = 2 \neq 0, \tag{1}$$

a contradiction.

In the $n$-dimensional case, it is only the evaluation of the integral which is slightly more complicated; the idea is that we replace $f'$ by the Jacobian determinant in the evaluation of the (signed) measure of the image (of $B$ under $f$) and that this measure is determined only by the behavior of $f$ at the boundary $S$. As we see in Fig. 1, the (signed) measure of the image of an interval $[a, b]$ under the map $f$ is equal to $f(b) - f(a)$; as the interval $[f(a), f(b)]$ is "assumed" once, the other regions and the other times this interval gets "covered" cancel each other out. (In Fig. 1 we exhibit the special case $a = -1$, $b = 1$, $f(a) = a$, $f(b) = b$.) The same is true in $n$ variables: The signed measure of the image of $B$ under a map $f$, such that $f$ is a diffeomorphism on $S$, is equal to the measure of the set bounded by $f(S)$ (i.e., the bounded component of $R^n$ whose boundary is $f(S)$)—other regions canceling each other out. (Degree theory makes this idea precise, but our integration argument avoids any counting.) The fundamental Theorem of Calculus will be replaced in $n$-dimensions by Gauss's theorem—see, e.g., [2], where it is argued that Gauss's theorem could be regarded as a proper analogue of the fundamental theorem of Calculus.

In Section 2 we will carry out the proof in $n$-dimensions. At the end of Section 2 we exhibit a condensed version of our proof, using the theory of differential forms. In Section 3 we outline the well-known derivation of the Brouwer fixed-point theorem from the no-retraction theorem. An analytic proof of the Brouwer fixed-point theorem has appeared recently [8]. We feel that our proof is more motivated.

I am very much indebted to Professor H. Scarf for encouraging discussion concerning this work.

**2.** We prove in the present section the following:

"NO DIFFERENTIABLE RETRACTION" THEOREM. *There exists no twice differentiable map $f$ of the unit ball $B$ in $R^n$ into its boundary $S$, such that $f(x) = x$ for all $x \in S$.*

*Proof.* Let $f$ be such a retraction, $f(x) = (f_1(x), \ldots, f_n(x))$. Let $J(x)$ denote the Jacobian determinant of $f$ at $x$. Expanding $J(x)$ by the first column, we get

$$J(x) = \sum_{i=1}^{n} (-1)^{i+1} \frac{\partial f_1}{\partial x_i} E_i(x) \tag{2}$$

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Answers to the questions on p. 256.

1. A. Weil, this MONTHLY, 61 (1954) 36.
2. Oliver Wendell Holmes, *The Autocrat of the Breakfast Table*, 1857, Part 1.

where $E_i(x)$ is the determinant of the matrix obtained from the matrix

$$M(x) = \begin{bmatrix} \frac{\partial f_2}{\partial x_1}, \ldots, \frac{\partial f_n}{\partial x_1} \\ \vdots \\ \frac{\partial f_2}{\partial x_n}, \ldots, \frac{\partial f_n}{\partial x_n} \end{bmatrix} \tag{3}$$

by omitting the $i$th row. Note that $J(x)$ vanishes identically on $B$, as the $n$ scalar functions $f_1, \ldots, f_n$ satisfy the functional relation $\sum_{i=1}^n f_i^2(x) \equiv 1$. (Note that we use here only the easy part of the vanishing Jacobian theorem.) Integrating $J(x)$ over $B$, we find, using the rule for differentiation of a product and (2), that

$$\begin{aligned} 0 &= \int_B \cdots \int J(x)\, dx_1 \cdots dx_n \\ &= \int_B \cdots \int \sum_{i=1}^n (-1)^{i+1} \frac{\partial}{\partial x_i}(f_1 E_i)\, dx_1 \cdots dx_n + \int_B \cdots \int \sum_{i=1}^n (-1)^i f_1 \frac{\partial E_i}{\partial x_i} dx_1 \cdots dx_n. \end{aligned} \tag{4}$$

According to a well-known theorem of Jacobi [4], [9] (used in the proof of the Brouwer fixed-point theorem in [3]),

$$\sum_{i=1}^n (-1)^i \frac{\partial E_i}{\partial x_i}(x) \equiv 0. \tag{5}$$

(If $n = 2$, then (5) reduces to the equality of the mixed derivatives.) To prove (5), let $c_{i,j}(x)$, $i \neq j$, denote the determinant of the matrix obtained from $M(x)$ by omitting the $i$th row and replacing the row

$$\left(\frac{\partial f_2}{\partial x_j}, \ldots, \frac{\partial f_n}{\partial x_j}\right) \text{ by } \left(\frac{\partial^2 f_2}{\partial x_i \partial x_j}, \ldots, \frac{\partial^2 f_n}{\partial x_i \partial x_j}\right).$$

Applying the rule for differentiating determinants we see that $\partial E_i / \partial x_i = \sum_{j \neq i} c_{i,j}$. The equality of the mixed derivatives $\partial^2 f_k / \partial x_i \partial x_j = \partial^2 f_k / \partial x_j \partial x_i$ implies that $c_{j,i} = (-1)^{j-i-1} c_{i,j}$ as the row of the second-order derivatives get shifted $j - i - 1$ rows when one passes from $c_{i,j}$ to $c_{j,i}$ if $i < j$, and $i - j - 1$ rows otherwise. Hence

$$\begin{aligned} \sum_{i=1}^n (-1)^i \frac{\partial E_i}{\partial x_i} &= \sum_{i=1}^n (-1)^i \left[\sum_{j<i} c_{i,j} + \sum_{j>i} c_{i,j}\right] \\ &= \sum_{j<i} (-1)^i c_{i,j} + \sum_{j>i} (-1)^i (-1)^{j-i-1} c_{j,i} = 0. \end{aligned}$$

Substituting (5) in (4), we find that a contradiction would follow once we prove that

$$I = \int_B \cdots \int \sum_{i=1}^n (-1)^{i+1} \frac{\partial}{\partial x_i}(f_1 E_i)\, dx_1 \cdots dx_n \neq 0. \tag{6}$$

We transform $I$ into a surface integral according to Gauss's divergence theorem [2], [11], applied to the vector field whose $i$th component is $(-1)^{i+1} f_1(x) E_i(x)$. We denote by $d\sigma$ the surface element ($(n-1)$-dimensional volume) on the unit sphere $S$, and utilize the fact that the outward unit normal of $S$ coincides with $x = (x_1, \ldots, x_n)$. Hence

$$I = \int_S \cdots \int f_1(x) \sum_{i=1}^n (-1)^{i+1} x_i E_i(x)\, d\sigma. \tag{7}$$

In order to calculate $I$, observe that $f_i(x) \equiv x_i$ on $S$, $1 \leqslant i \leqslant n$. Hence $\operatorname{grad} f_i - \operatorname{grad} x_i$ is perpendicular to $S$ there. Thus there exist scalars $\lambda_i$ (depending on $x$) such that $\operatorname{grad} f_i(x) =$

$\operatorname{grad} x_i + \lambda_i x$, and the matrix $M$ can be written as

$$\begin{bmatrix} \lambda_2 x_1 & ,\dots, & \lambda_n x_1 \\ 1+\lambda_2 x_2 & & \\ \vdots & & \vdots \\ \lambda_2 x_n & & 1+\lambda_n x_n \end{bmatrix}.$$

The sum $\sum_{i=1}^{n}(-1)^{i+1}x_i E_i(x)$ is equal to the determinant

$$\begin{vmatrix} x_1 & \lambda_2 x_1 & \cdots & \lambda_n x_1 \\ x_2 & 1+\lambda_2 x_2 & & \lambda_n x_2 \\ \vdots & & & \vdots \\ x_n & \lambda_2 x_n & & 1+\lambda_n x_n \end{vmatrix} = \begin{vmatrix} x_1 & 0 & \cdots & 0 \\ x_2 & 1 & & 0 \\ \vdots & & & \vdots \\ x_n & 0 & & 1 \end{vmatrix} = x_1.$$

Moreover, $f_1(x) = x_1$ on $S$. Inserting these results in (7), we get the result $I = \int_S \cdots \int x_1^2 d\sigma > 0$, contradicting (4). (It is easy to compute that $I = (1/n)\int_S \cdots \int d\sigma = \operatorname{Vol}(B)$.)

Note that the passage from the volume integral (4) to the surface integral (7) corresponds to the two left equalities in (1) and that the evaluation of $I$ as being equal to $\operatorname{Vol}(B)$ corresponds to the fact that the one-dimensional volume of the interval $[-1, 1]$ is equal to 2, the right-hand side of (1).

REMARK 1. Only a slight modification of the argument is needed to prove the nonexistence of a differentiable retraction for a general bounded open subset $B$ of $R^n$ with a smooth boundary $S$, provided that the divergence theorem holds for $B$ and $S$ **[2]**, **[11]**. In fact, all the arguments leading to the formula (7) are valid (with no change) in the general case too. Let $\gamma = (\gamma_1, \dots, \gamma_n)$ denote the outward unit normal of $S$, and let $d\sigma$ denote the surface element on $S$. The divergence theorem implies that

$$I = \int_S \cdots \int f_1(x) \sum_{i=1}^{n} (-1)^{i+1}\gamma_i E_i(x)\, d\sigma. \tag{7'}$$

Observe that $\operatorname{grad} f_i(x) = \operatorname{grad} x_i + \lambda_i \gamma$ on $S$, $i = 1, \dots, n$. Hence $I$ can be written in the form (compare the argument following (7))

$$I = \int_S \cdots \int f_1(x)\gamma_1(x)\, d\sigma = \int_S \cdots \int x_1\gamma_1\, d\sigma. \tag{8}$$

Applying the divergence theorem once more (to the vector $(x_1, 0, \dots, 0)$) we have

$$\operatorname{Vol}(B) = \int_B \cdots \int 1\, dx_1 \cdots dx_n = \int_B \cdots \int \frac{\partial x_1}{\partial x_1} dx_1 \cdots dx_n = \int_S \cdots \int x_1\gamma_1\, d\sigma.$$

REMARK 2. The proof presented in this note can be cast in a compact form using concepts and results from the theory of differential forms. In fact, define two $(n-1)$-forms on $B$ by

$$\omega_1 = f_1\, df_2 \wedge \cdots \wedge df_n, \qquad \omega_2 = x_1\, dx_2 \wedge \cdots \wedge dx_n.$$

Then

$$d\omega_2 = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n, \qquad d\omega_1 = df_1 \wedge df_2 \wedge \cdots \wedge df_n = 0$$

(as the differentials $df_1, \dots, df_n$ are linearly dependent). Note also that the restrictions to $S$ of $\omega_1$ and $\omega_2$ coincide, as $f_i \equiv x_i$ on $S$, $1 \leqslant i \leqslant n$. By Stokes's theorem,

$$0 = \int_B d\omega_1 = \int_S \omega_1 = \int_S \omega_2 = \int_B d\omega_2 = \operatorname{Vol}(B) \neq 0. \tag{9}$$

Note the similarity between (9) and (1).

3. The Brouwer fixed-point theorem follows from the no differentiable retraction theorem in a well-known way (see, e.g., [3]). We sketch the argument for completeness. Suppose that $g : B \to B$ is a fixed-point-free continuous map. The compactness of $B$ implies that $|g(x) - x| > \varepsilon > 0$ for $x \in B$. Let $h(x)$ be a $C^2$ function such that $|h(x) - g(x)| < \varepsilon/2$ on $B$ and such that $h : B \to B$ (we can even let $h$ be a polynomial). Then $h(x) \neq x$ for $x \in B$, and let $f(x)$ denote (for $x \in B$, $X \notin S$) the unique point on $S$ such that $h(x)$, $x$ and $f(x)$ lie on the same line and $x$ is between $h(x)$ and $f(x)$ (see Fig. 2), $f(x) = x$ for $x \in S$. Then $f(x)$ is a $C^2$ retraction,

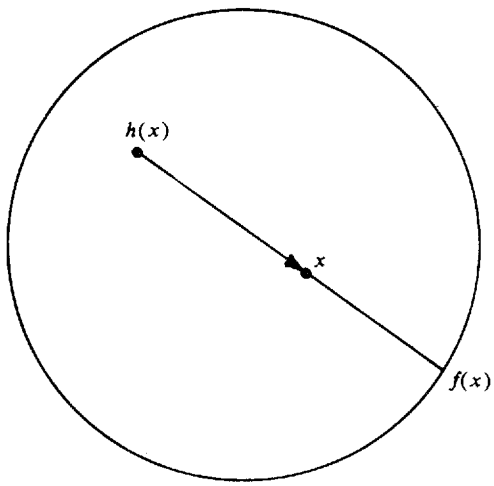


FIG. 2

contradicting the theorem of Section 2. It is not difficult to show (using a suitable approximation argument) that the no differentiable retraction theorem implies directly that there exists no continuous retraction. Using Remark 1 it follows, e.g., that there exists no continuous retraction of the torus onto its boundary.

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