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VALUE THEORY WITHOUT EFFICIENCY*†

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A semivalue is a symmetric positive linear operator on a space of games, which leaves the additive games fixed. Such an operator satisfies all of the axioms defining the Shapley value, with the possible exception of the efficiency axiom. The class of semivalues is completely characterized for the space of finite-player games, and for the space pNA of nonatomic games.

0. Introduction. Recently, attention has been focused on generalizations and analogues of the Shapley value that do not enjoy the efficiency, or Pareto optimality, property (e.g., [2], [4], [9], [11]). This has stemmed (partly) from the search for value functions that describe the prospects of playing different roles in a game (instead of describing fair division, in which case efficiency is a natural requirement). The purpose of this paper is to treat the subject from an axiomatic viewpoint, i.e., to characterize the class of operators that is obtained by omitting the efficiency axiom from the axioms defining the Shapley value. We consider both finite-player and nonatomic games. In the finite case, a complete solution is given; in the nonatomic case, a complete solution is given for the important space pNA.

1. The finite case. Let U be an infinite set, the universe of players. A *game* on U is a set function $v: 2^U \to R$ with $v(\emptyset) = 0$. We interpret the members of U as players and the members of 2^U as coalitions. A set $N \subset U$ is a *support* of v if, for each $S \subset U$, $v(S) = v(S \cap N)$. A *finite game* is a game which has a finite support. We denote by G the vector space of all finite games, and by G^N the subspace of G consisting of games with support N. Let AG (respectively, AG^N) be the subspace of G (respectively, G^N) of additive games. (Note that for N finite, AG^N is isomorphic to R^N, the Euclidean space of dimension $|N|$ whose axes are indexed by the elements of N. For convenience we shall often use R^N for AG^N.)

Given a permutation Θ of U (i.e., a 1-1 mapping from U onto itself) define the game $\Theta_* v$ by $(\Theta_* v)(S) = v(\Theta S)$. Finally define v to be monotonic if $v(S) \geqslant v(T)$ whenever $S \supset T$.

A *semivalue* on G is a function $\psi: G \to AG$ such that:

(1) ψ is linear;

(2) $\psi\Theta_* = \Theta_*\psi$, for each permutation Θ of U;

(3) if v is monotonic, then ψv is monotonic;

(4) if $v \in AG$, then $\psi v = v$.

These are the linearity, symmetry, monotonicity and projection axioms ([1, pp. 15–16]). The projection axiom is an easy consequence of the more familiar dummy axiom, which says that if i is a dummy player in v (i.e., $v(S \cup i) = v(S) + v(i)$

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whenever $i \notin S$) then $(\psi v)(i) = v(i)$. (We conventionally omit the braces when indicating one-element sets.) The quantity $(\psi v)(i)$, for $i \in U$, is a measure (according to ψ) of worth of the prospect of having role i in the game v.

Let ξ be a probability measure on [0, 1]. For any $i \in U$ and any $v \in G$ with finite support N, define $\psi_\xi v \in AG$ by

$$(\psi_\xi v)(i) = \sum_{S \subset N \setminus i} p_s^n [v(S \cup i) - v(S)], \tag{1.1}$$

where

$$p_s^n = \int_0^1 t^s (1-t)^{n-s-1} d\xi(t).$$

(The symbols n and s generically denote the cardinalities of the sets N and S.) Note that the right-hand side of (1.1) is independent of the choice of N, so the definition makes sense.

To interpret (1.1), choose t in [0, 1] at random in accordance with ξ, and construct a random coalition S by letting each player other than i join S with probability t, independently of the other players. Then $(\psi_\xi v)(i)$ is i's expected contribution to S.

We now come to our characterization of semivalues on G.

THEOREM 1(a). *For each probability measure ξ on* [0, 1], *ψ_ξ is a semivalue. Moreover, every semivalue on G is of this form, and the mapping $\xi \to \psi_\xi$ is* 1-1.

To prove this theorem, we first characterize the semivalues on the vector space of games on a fixed finite-player set. Then we proceed with two different proofs which shed light on Theorem 1(a) from different viewpoints. Let $N \subset U$ be a finite set. A *semivalue* on G^N is a function $\psi^N : G^N \to AG^N$ satisfying (1), (2^N), (3), and (4), where (2^N) requires that $\psi^N \Theta_* = \Theta_* \psi^N$ for every N-preserving permutation Θ of U.

Let $p^n = (p_0^n, \ldots, p_{n-1}^n)$ be a vector such that

$$\sum_{s=0}^{n-1} \binom{n-1}{s} p_s^n = 1$$

and $p^n \geqslant 0$. Define $\psi_{p^n}^N : G^N \to AG^N$ by

$$(\psi_{p^n}^N v)(i) = \sum_{S \subset N \setminus i} p_s^n [v(S \cup i) - v(S)] \tag{1.2}$$

for all $i \in N$ and $v \in G^N$.

LEMMA. *For each vector p^n, $\psi_{p^n}^N$ is a semivalue on G^N. Moreover, every semivalue on G^N is of this form, and the mapping $p^n \to \psi_{p^n}^N$ is* 1-1.

PROOF. It is straightforward to verify that each $\psi_{p^n}^N$ is indeed a semivalue. Without loss of generality take $N = \{1, \ldots, n\}$. For any nonempty $S \subset N$, define the game $v_S \in G^N$ by $v_S(T) = 1$ if $T \supset S$, $v_S(T) = 0$ otherwise. Suppose $i \notin S$, and consider any semivalue ψ^N. By the monotonicity axiom (3), $\psi_i^N(v_S) \geqslant 0$. Set $w = -v_S + \sum_{j \in S} v_{\{j\}}$. This game is monotonic. Hence $0 \leqslant \psi_i^N(w) = -\psi_i^N(v_S)$; this follows from (1), (4), and the fact that each $v_{\{j\}} \in AG^N$. Consequently $\psi_i^N(v_S) = 0$. Consider the vector space of symmetric linear functions from G^N to AG^N, and let F be the subspace spanned by the set of semivalues. It is well known (see, for example, Appendix A of [1]) that $\{v_S : \emptyset \neq S \subset N\}$ is a basis for G^N; therefore, every element $f \in F$ is uniquely determined by its values on the games in this basis. Due to the symmetry axiom (2^N) and the argument just given, it is, in fact, sufficient to specify $f_1(v)$ for every

$v \in \{v_{S(k)} : 1 \leqslant k \leqslant n\}$, where $S(k) = \{1, \ldots, k\}$. It follows that the dimension of F is at most n.

For each $0 \leqslant k \leqslant n-1$, let $\psi_{(k)} = \psi_p^N$, as defined by (1.2) when $p_k = \binom{n-1}{k}^{-1}$ and $p_l = 0$ for all $l \neq k$. It is clear that the set $\{\psi_{(0)}, \ldots, \psi_{(n-1)}\}$ is linearly independent in F. Thus, this set is a basis for F.

Consider any semivalue $\psi^N \in F$. It can be uniquely written as $\psi^N = c_0\psi_{(0)} + \ldots + c_{n-1}\psi_{(n-1)}$. Therefore we must only show that $\sum_{s=0}^{n-1} c_s = 1$ and $c = (c_0, \ldots, c_{n-1}) \geqslant 0$; the desired result will then follow upon taking $p_s^n = \binom{n-1}{s}^{-1} c_s$, yielding $\psi^N = \psi_{p^n}^N$. Suppose some $c_k < 0$. Consider $w \in G^N$ defined by $w(T) = 1$ if $|T| > k$, $w(T) = 0$ otherwise. Then for any $i \in N$, $(\psi^N w)(i) = c_k(\psi_{(k)} w)(i) = c_k < 0$; this contradicts the monotonicity axiom (3). Next consider $v_{\{1\}} \in G^N$. By the projection axiom (4), we must have $(\psi^N v_{\{1\}})(1) = \sum_{s=0}^{n-1} c_s = v_{\{1\}}(1) = 1$. ∎

Proof of Theorem 1(a). It is straightforward to verify that each ψ_ξ is a semivalue. Consider any semivalue ψ. For each finite $N \subset U$, ψ induces a semivalue ψ^N on G^N. From the preceding lemma we know that each ψ^N has the form

$$(\psi^N v)(i) = \sum_{S \subset N \setminus i} p_s^N [v(S \cup i) - v(S)]$$

where all $p_s^N \geqslant 0$ and $\sum_{s=0}^{n-1} \binom{n-1}{s} p_s^N = 1$. Furthermore, it is a simple consequence of the symmetry axiom that there is a collection of constants $\{p_s^m : s = 0, \ldots, m-1; m = 1, 2, \ldots\}$ such that for all $N \subset U$ with $|N| = m$, $p_s^N = p_s^m$.

Consider the collection of games $\{\hat{v}_S^N\}$, where $\hat{v}_S^N$ in G^N is defined for any $S \subset N \subset U$ by $\hat{v}_S^N(T) = 1$ if $T \supsetneqq S$, $\hat{v}_S^N$ (T) = 0 otherwise. For any $i \in N \setminus S$,

$$\psi^N(\hat{v}_S^N)(i) = p_s^N = p_s^n.$$

For any given player $d \in U \setminus N$, the game $\hat{v}_S^N$ can be viewed as a game in $G^{N \cup d}$. It is easily shown that for any $i \in N \setminus S$,

$$\psi^{N \cup d}(\hat{v}_S^N)(i) = p_s^{N \cup d} + p_{s+1}^{N \cup d} = p_s^{n+1} + p_{s+1}^{n+1}.$$

Since ψ^N and $\psi^{N \cup d}$ are restrictions of the same operator ψ, it follows that for any $i \in N \setminus S$,

$$\psi^N(\hat{v}_S^N)(i) = p_s^n = p_s^{n+1} + p_{s+1}^{n+1} = \psi^{N \cup d}(\hat{v}_S^N)(i). \tag{1.3}$$

For notational ease, set $\alpha_n = p_n^{n+1}$ (for $n = 0, 1, 2, \ldots$). Obviously, $\{p_s^n\}$ determines $\{\alpha_n\}_{n=0}^{\infty}$. Moreover, using (1.3) it can be shown by induction that for any $0 \leqq s \leqq n$,

$$p_s^{n+1} = (-1)^{n-s}\left[\alpha_n - \binom{n-s}{1}\alpha_{n-1} + \binom{n-s}{2}\alpha_{n-2} + \cdots + (-1)^{n-s}\alpha_s\right]$$
$$= (-1)^{n-s}\Delta^{n-s}\alpha_n,$$

where Δ is the standard "backwards difference" operator. Consequently, we see that every sequence $\{\alpha_n\}$ of real numbers uniquely defines a collection $\{p_s^n\}$. It can be shown by direct summation that, for each n, the numbers $\{\binom{n-1}{s} p_s^n\}_{s=0}^{n-1}$ add to α_0. Therefore, the collection $\{p_s^n\}$ will define a semivalue if and only if $\alpha_0 = 1$ and all $p_s^n \geqq 0$.

It is well known (for example, Theorem 4.6 of [6]) that a sequence $\{\alpha_n\}$ (with $\alpha_0 = 1$) and the successive differences $(-1)^k\Delta^k\alpha_n$ of all orders are nonnegative if and only if $\alpha_0, \alpha_1, \ldots$ are the moments of a uniquely-determined probability distribution ξ on

[0, 1]. In this case, since each $\alpha_n = \int_0^1 t^n d\xi(t)$, it follows that each

$$p_s^{n+1} = \int_0^1 \left[t^s - \binom{n-s}{1} t^{s+1} + \cdots + (-1)^{n-s} t^n \right] d\xi(t)$$
$$= \int_0^1 t^s (1-t)^{n-s} d\xi(t). \quad ∎$$

ALTERNATIVE PROOF OF THEOREM 1(a). It suffices to establish that ψ is of the form ψ_ξ for a unique probability measure ξ on [0, 1]. Let $i \in U$ be fixed. For each finite subset N of $U \backslash i$, ψ induces a semivalue on $G^{N \cup i}$, and hence, by Lemma 1, induces a probability measure c_N on the space whose elements are subsets of N, such that $c_N(S) = p_s^{n+1}$. If $N \subset L$, then by considering the natural embedding of $G^{N \cup i}$ into $G^{L \cup i}$, we have $c_N(S) = \sum c_L(T)$, where the summation runs over all T for which $S \subset T \subset L$ and $T \cap N = S$. Let $\{N_k\}$ be an increasing sequence of finite subsets of $U \backslash i$. The measures on the subsets of the various N_k are "consistent," and therefore by Kolmogorov's consistency theorem ([7, p. 94]), there is a sequence of (0, 1)-valued random variables $\{Y_j : j \in \cup N_k\}$ such that $\text{Prob}(\{j \in N_k : Y_j = 1\} = S) = c_{N_k}(S) = p_s^{n_k+1}$. Thus, $\{Y_j\}$ is an exchangeable sequence of random variables. De Finetti's theorem ([5, §9.6.1]) asserts that the distribution of every exchangeable infinite sequence of random variables is a unique mixture of distributions of sequences of independent identically-distributed random variables. As $\text{Prob}(Y_j = 0 \text{ or } 1) = 1$, there exists a unique probability measure ξ on [0, 1] such that for every finite sequence $\{\epsilon_j : j \in N\}$ of 0's and 1's, $\text{Prob}(Y_j = \epsilon_j \text{ for all } j \in N) = \int_0^1 t^{\Sigma \epsilon_j}(1-t)^{n - \Sigma \epsilon_j} d\xi(t) = c_N(\{j : \epsilon_j = 1\})$.

It is obvious from the axiom of symmetry that the mixing measure depends neither on the particular player i, nor on the sequence N_k, and thus ξ is uniquely determined by ψ alone. ∎

This alternative proof provides another view of the theorem. Let $\{X_i : i \in U\}$ be a family of independent identically-distributed random variables distributed uniformly on [0, 1]. If $v \in G$ and $t \in [0, 1]$, define the random variable $\Delta v(t)$ by $\Delta v(t) = v(\{i : X_i \leqslant t\}) - v(\{i : X_i < t\})$. We then have the following restatement of Theorem 1(a):

THEOREM 1(a′). *For each probability measure ξ on* [0, 1] *there is a semivalue ψ_ξ on G defined by*

$$(\psi_\xi v)(i) = \int_0^1 E(\Delta v(t) \mid X_i = t) d\xi(t).$$

Moreover, each semivalue on G is of this form and the mapping $\xi \to \psi_\xi$ is 1-1.

The *Shapley value* [10] is defined as $\phi = \psi_\lambda$, where λ denotes Lebesgue measure on [0, 1]. This is the only semivalue which has the efficiency property: for every $N \subset U$ and $v \in G^N$, $\phi v(N) = v(N)$. Define the *variation norm* of a game $v \in G$ with support N, as $\|v\| = \inf(v_+(N) + v_-(N))$, where the infimum is taken over all pairs v_+, v_- of monotonic games for which $v = v_+ - v_-$. With respect to this norm on G, the Shapley value is a continuous linear operator of norm 1. (For any monotonic $v_+, v_- \in G^N$ such that $v = v_+ - v_-$, $\|\phi v\| = \sum |\phi v(i)| \leqslant \sum(\phi v_+(i) + \phi v_-(i)) = v_+(N) + v_-(N)$; hence $\|\phi v\| \leqslant \|v\|$. But for any monotonic $v \in G^N$, $\|\phi v\| = v(N) = \|v\|$.)

We shall characterize the class of continuous semivalues on G. Let W be the subset of $L_\infty(0, 1)$ of all nonnegative functions g with $\int_0^1 g(t)\,dt = 1$.

THEOREM 1(b). *For each $g \in W$, the operator $\psi_g : G \to AG$ defined by*

$$\psi_g v(i) = \int_0^1 E(\Delta v(t) \mid X_i = t) g(t)\,dt$$

is a continuous semivalue. Moreover, every continuous semivalue on G is of this form. The map $g \to \psi_g$ is a linear isometry (that is, $\|\psi_g\| = \|g\|_{L_\infty}$).

PROOF. Consider any $g \in W$ and define $\xi = \int g d\lambda$. By Theorem 1(a′), $\psi_g = \psi$ is a semivalue. For any $v \in G^N$, and monotonic games v_+, v_- with $v = v_+ - v_-$,

$$\begin{aligned}\|\psi_g v\| &= \sum|\psi_g v(i)| \leqslant \sum|\psi_g v_+(i)| + \sum|\psi_g v_-(i)| \\ &\leqslant \|g\|(\sum|\phi v_+(i)| + \sum|\phi v_-(i)|) \\ &= \|g\|(v_+(N) + v_-(N));\end{aligned}$$

therefore, $\|\psi_g v\| \leqslant \|g\|\,\|v\|$. Hence ψ_g is continuous, and $\|\psi_g\| \leqslant \|g\|$.

Next, consider any continuous semivalue ψ_ξ. Select any (relatively) open interval $J \subset [0,1]$. Fix a player $i \in U$, and for each $k > 0$, select $N_k \subset U$ such that $i \in N_k$ and $|N_k| = k$. Let $v_k \in G^{N_k}$ be defined by $v_k(S) = \lambda([0, s/n] \cap J)$. By the law of large numbers, $\lim_k n_k \psi_\xi v_k(i) = \xi(J)$. Therefore, by the symmetry of v_k, $\lim_k \|\psi_\xi v_k\| = \lim_k \sum_{j \in N_k} |\psi_\xi v_k(j)| = \lim_k n_k \psi_\xi v_k(i) = \xi(J)$, while each $\|v_k\| = \lambda(J)$. Hence $\|\psi_\xi\| \geqslant \xi(J)/\lambda(J)$. The continuity of ψ_ξ implies that $\|\psi_\xi\|$ is finite. Consequently, $M \equiv \sup\{\xi(J)/\lambda(J) : J \text{ is an interval in } [0,1]\} < \infty$, and the Radon-Nikodym derivative $d\xi/d\lambda = g$ is in W. Therefore $\psi_\xi = \psi_g$, and $\|\psi_g\| \geqslant M = \|g\|$. ∎

2. The infinite case. All definitions and notation are according to [1]. Let $(I, \mathcal{C})$ be a measure space isomorphic to $([0,1], \mathfrak{B})$, where $\mathfrak{B}$ is the σ-field of Borel subsets of $[0,1]$. The members of I are called *players*, the members of $\mathcal{C}$ *coalitions*, and set functions are called *games*. Let BV be the space of bounded-variation set functions on $(I, \mathcal{C})$. The space of all bounded, finitely-additive set functions is denoted FA, and its subspace of all nonatomic measures is denoted NA. Denote by $\mathcal{G}$ the group of automorphisms of $(I, \mathcal{C})$. For each $\Theta \in \mathcal{G}$, $\Theta_* : \mathrm{BV} \to \mathrm{BV}$ is defined by $\Theta_* v(S) = v(\Theta S)$. If $Q \subset \mathrm{BV}$ then Q^+ denotes the subset of Q of all monotonic set functions. A subset Q of BV is *symmetric* if for each $\Theta \in \mathcal{G}$, $\Theta_* Q \subset Q$. An operator $\psi: Q \to \mathrm{BV}$ is called *positive* if $\psi(Q^+) \subset \mathrm{BV}^+$, and *symmetric* if for each $\Theta \in \mathcal{G}$, $\Theta_* \psi = \psi \Theta_*$.

Let Q be a linear symmetric subspace of BV. A *semivalue* on Q is an operator ψ from Q into FA such that:

(1) ψ is linear,

(2) ψ is symmetric,

(3) ψ is positive,

(4) if $v \in Q \cap \mathrm{FA}$ then $\psi v = v$.

We will characterize the semivalues on pNA, the closed subspace of BV spanned by all powers of NA^+ measures. This space plays an important role in the theory of nonatomic games, and contains many games of interest. For example, pNA contains all "vector measure games" satisfying appropriate differentiability conditions, i.e., all set functions of the form $f \circ \mu$, where $\mu = (\mu_1, \ldots, \mu_n)$ is a nonatomic finite-dimensional vector measure and f is an appropriately differentiable real-valued function defined on the range of μ, with $f(0) = 0$. As our main theorem in this section uses notations and terminology related to the "extension" of a game, we restate here relevant definitions and results from [1]. $\mathcal{I}$ denotes the family of all measurable functions from $(I, \mathcal{C})$ to $([0,1], \mathfrak{B})$. There is a partial order on $\mathcal{I}$: $f \geqslant g$ if $f(s) \geqslant g(s)$ for all $s \in I$. A real valued function w on $\mathcal{I}$ with $w(0) = 0$ is called an *ideal set function*; it is called *monotonic* if $f \geqslant g$ implies $w(f) \geqslant w(g)$. The characteristic function of a member S of $\mathcal{C}$ is denoted χ_S. We will sometimes denote χ_S by S and $t \cdot \chi_I$ by t.

It is shown in [1, Theorem G] that there is a unique monotonicity-preserving linear mapping which associates with each $v \in \mathrm{pNA}$ an ideal set function v^*, such that $(vw)^* = v^* w^*$ for all $v, w \in \mathrm{pNA}$, and $\mu^*(f) = \int_I f d\mu$ for all $\mu \in \mathrm{NA}$ and $f \in \mathcal{I}$.

Denote $\partial v^*(t,S) = (d/d\tau)v^*(t\chi_I + \tau\chi_S)|_{\tau=0}$. By Theorem H of [1] we know that for each $v \in$ pNA and each $S \in \mathcal{C}$, the derivative $\partial v^*(t,S)$ exists for almost all t in [0, 1], and is integrable over [0, 1] as a function of t.

Recall that W is the set of nonnegative functions $g \in L_\infty(0,1)$ such that $\int_0^1 g(t)\,dt = 1$.

THEOREM 2. *For each $g \in W$ the operator ψ_g : pNA → FA defined by*

$$\psi_g v(S) = \int_0^1 \partial v^*(t,S)g(t)\,dt$$

is a semivalue. Moreover, every semivalue on pNA *is of this form. The map $g \to \psi_g$ of W onto the family of semivalues on* pNA *is a linear isometry.*

PROOF. Let $g \in W$ be given. For $v \in$ pNA, Lemma 23.1 of [1] asserts that $\int_0^1 |\partial v^*(t,S)|\,dt \leqslant \|v\|$. Hence

$$|\psi_g v(S)| = \left|\int_0^1 \partial v^*(t,S)g(t)\,dt\right| \leqslant \|g\| \cdot \|v\|;$$

this proves that $\psi_g v$ is bounded. If $S, T \subset I$ with $S \cap T = \emptyset$ then $\partial v^*(t, T \cup S) = \partial v^*(t,T) + \partial v^*(t,S)$ for almost all t [1, Proposition 24.1]. Therefore $\psi_g v(S \cup T) = \psi_g v(S) + \psi_g v(T)$, which proves that ψ_g takes pNA into FA. Linearity of ψ_g follows from the linearity of the extension as well as that of the derivative. Symmetry of ψ_g follows from the fact that $\partial(\Theta_* v)^*(t,S) = \partial v^*(t,\Theta S)$ and thus $\Theta_*\psi_g v(S) = \int \partial v^*(t,\Theta S)g(t)\,dt = \int \partial(\Theta_* v)^*(t,S) \cdot g(t)\,dt = \psi_g(\Theta_* v)(S)$. Let $v \in$ pNA$^+$. Then v^* is also monotonic and $\partial v^*(t,S) \geqslant 0$; thus $\psi_g v$ is monotonic, which proves the positivity of ψ_g. Finally, any $u \in$ pNA ∩ FA is in NA (Corollary 5.3 of [1], and the continuity of the elements of the space AC ([1], p. 205), imply that u is countably additive). Hence $\partial u^*(t,S) = u(S)$ and consequently $\psi_g u = u$. This completes the proof that ψ_g is a semivalue.

Now, let ψ be a semivalue on pNA. Let μ be a fixed probability measure in NA. Each $f \in L_1(0,1)$ induces a game v_f defined by

$$v_f(S) = \int_0^{\mu(S)} f(t)\,dt.$$

In other words, f defines a function $F : [0,1] \to R$ by $F(s) = \int_0^s f(t)\,dt$, and $v_f = F \circ \mu$. As $f \in L_1$, F is absolutely continuous and therefore $v_f \in$ pNA. In analogy with the proof of Proposition 6.1 of [1] it follows that $\psi v_f = C(f) \cdot \mu$, where $C(f)$ is a constant independent of μ. Observe that $v_{f+g} = v_f + v_g$; thus the linearity of ψ implies that C is linear. We now proceed to show that C is continuous. Observe that $\|v_f\| = \|f\|_{L_1}$. Since pNA is internal ([1], Proposition 7.19), it is reproducing; by definition it is closed, and thus ([1], Proposition 4.15) ψ is continuous on pNA. That is, there exists a constant K with $\|\psi v\| \leqslant K\|v\|$, which in particular implies that $|C(f)| = \|C(f)\mu\| \leqslant K\|v_f\| = K\|f\|_{L_1}$. Hence $C : L_1 \to R$ is a continuous linear functional and therefore is of the form $C(f) = \int_0^1 f(t)g(t)\,dt$ for some $g \in L_\infty$. We shall show that $\psi = \psi_g$. As was shown at the beginning of the proof, ψ_g(pNA) ⊂ FA and $|\psi_g v(S)| \leqslant \|g\| \cdot \|v\|$, which implies that ψ_g is continuous. For each $f \in L_1$, $\partial v_f^*(t,S) = f(t)\mu(S)$ for almost all t, and thus $\psi_g v_f(S) = \mu(S)\int f(t)g(t)\,dt = C(f)\mu(S) = \psi v_f(S)$; therefore $\psi_g v_f = \psi v_f$. The linear symmetric subspace spanned by $\{v_f : f \in L_1\}$ is dense in pNA (it contains all powers of NA measures). The operators ψ and ψ_g are linear and symmetric and thus coincide on this subspace; as they are also continuous, they coincide on pNA. It remains for us to show that $g \in W$. For $v \in$ NA ⊂ FA ∩ pNA, it follows that $\partial v^*(t,S) = v(S)$. Thus $\psi_g v(S) = (\int_0^1 g(t)\,dt)v(S)$, which shows that $\int_0^1 g(t)\,dt = 1$. Let $B_\epsilon = \{t :$

$g(t) \leqslant -\epsilon\}$ and let f be the characteristic function of B_ϵ. Then $f \geqslant 0$ and hence v_f is monotonic. But as $\psi_g v_f(I) = \int f(t)g(t)\,dt \leqslant -\epsilon\lambda(B_\epsilon)$ (λ denotes Lebesgue measure on $[0, 1]$) and $\psi_g = \psi$ is positive, it must be that $\lambda(B_\epsilon) = 0$. As this holds for any $\epsilon > 0$, g is nonnegative. This completes the proof that any semivalue ψ is of the form ψ_g for some $g \in W$.

Now, for any $g \in W$ and $\epsilon > 0$ there exists a nonnegative $f \in L_1$ with $\|f\|_{L_1} = 1$ and $\int f(t)g(t)\,dt = \|g\| - \epsilon$. Observe that $\|v_f\| = \|f\|_{L_1} = 1$ and that $\|\psi_g v_f\| = \|g\| - \epsilon$; hence $\|\psi_g\| \geqslant \|g\|$. On the other hand, for $v \in \mathrm{pNA}^+$,

$$\|\psi_g v\| = \psi_g v(I) = \int \partial v^*(t, I) g(t)\,dt \leqslant \|g\| \cdot \int_0^1 \partial v^*(t, I)\,dt = \|g\| \cdot \|v\|.$$

In the general case, when v is not necessarily monotonic, let $\epsilon > 0$ be given. Set $v = u - w$, where u and w are in pNA^+ and $\|v\| + \epsilon \geqslant \|u\| + \|w\|$; such u and w exist because pNA is internal. Then

$$\|\psi_g v\| \leqslant \|\psi_g u\| + \|\psi_g w\| \leqslant \|g\|(\|u\| + \|w\|) \leqslant \|g\|\,(\|v\| + \epsilon),$$

and if we let $\epsilon \to 0$, $\|\psi_g v\| \leqslant \|g\| \cdot \|v\|$; this completes the proof of the equality $\|\psi_g\| = \|g\|$. ∎

3. Remarks. Continuous semivalues are diagonal. (The proof in [8] that continuous values are diagonal does not make use of the efficiency axiom and therefore the same proof works here.) Furthermore, semivalues on closed reproducing spaces are diagonal.

The semivalues here derived axiomatically on pNA can also be obtained from the complementary, asymptotic point of view [3] which links the finite-player and nonatomic approaches.

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