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INFORMATION CONDITIONS, COMMUNICATION AND GENERAL EQUILIBRIUM*†

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It is shown that if the information sets of one game are a refinement of the information sets of the other, then the set of pure strategy equilibrium points of the game with less information is contained within the set of pure strategy equilibrium points of the game with more information.

1. Introduction. In this paper we demonstrate a relationship between games which have identical moves but different levels of information. Specifically, it is shown that if the information sets of one game are a refinement of the information sets of the other, then the set of pure strategy equilibrium points of the game with less information is contained within the set of pure strategy equilibrium points of the game with more information.

This result is proved in §2. In §3 we discuss the implications of the result for strategic market games, i.e., for games with an economic structure and specific price formation mechanisms [1], [6], [7].

2. Equilibrium points and information in games in extensive form. Let Γ and $\hat{\Gamma}$ be games in extensive form [3] on the player-set $N = \{1, \ldots, n\}$. We will say that $\hat{\Gamma}$ is a *refinement* of Γ (or Γ is a *coarsening* of $\hat{\Gamma}$), and denote it by $\Gamma < \hat{\Gamma}$, if Γ is obtainable from Γ *only*[1] by forming partitions of the information sets in Γ.

Let $\Gamma < \hat{\Gamma}$. Though the strategy sets S^i and $\hat{S}^i$ of i in Γ and $\hat{\Gamma}$ are formally different, there is a natural inclusion $S^i \subset \hat{S}^i$. A strategy s^i in S^i is identified with $\hat{s}^i$ in $\hat{S}^i$, where $\hat{s}^i$ is as follows: the move chosen by $\hat{s}^i$ at any information set $\hat{I}^i_t$ of i in $\hat{\Gamma}$ is the same as the move chosen by s^i at I^i_j, where I^i_j is the *unique* information set of i in Γ for which $\hat{I}^i_t \subset I^i_j$.

For any game Γ we will denote by $\eta(\Gamma)$ the set of all its *pure-strategy* noncooperative equilibrium points.

Our aim in this section[2] is the following straightforward but striking result.

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[1] All the elements of the extensive game (the tree, player partitions, etc.) are held fixed, and only the information sets are varied.

[2] We assume here that Γ has no chance moves. (See, however, Remark 2.) Also, while the length of Γ is finite, we permit the number of moves at any node to be infinite, in contrast with [3]–indeed this is required to be so for the application in §3. (The length of Γ is the supremum of lengths of all plays, i.e., paths from the initial move (root) to a terminal node.)

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Proposition. *Suppose* $\Gamma < \hat{\Gamma}$. *Then* $\eta(\Gamma) \subset \eta(\hat{\Gamma})$.

Proof. If $\eta(\Gamma) = \emptyset$ there is nothing to prove. Let $s = (s^1, \ldots, s^n)$ be in $\eta(\Gamma)$. Put $\hat{s} = (\hat{s}^1, \ldots, \hat{s}^n)$ where $\hat{s}^i$ corresponds to s^i as described earlier. We will show that $\hat{s}$ is in $\eta(\hat{\Gamma})$.

First observe that s and $\hat{s}$ lead to the same play P. Now if $\hat{s}$ is not in $\eta(\Gamma)$ there is some player i who can improve his payoff by deviating from $\hat{s}^i$ to $\hat{\mathbf{s}}^i$, provided that the others keep their strategies fixed according to $\hat{s}$. Take the path P defined by $(\hat{s}^1, \ldots, \hat{s}^{i-1}, \hat{s}^i, \hat{s}^{i+1}, \ldots, \hat{s}^n)$, and let $\hat{I}^i_1, \ldots, \hat{I}^i_l$ be the information sets of i in $\hat{\Gamma}$ which contain nodes of **P**. Consider $I^i_{j1}, \ldots, I^i_{jl}$ where I^i_{jt} is the (unique) information set of i in Γ such that $\hat{I}^i_t \subset I^i_{jt}$. Since no two nodes of a play can lie in the same information set, the sets $\{I^i_{jt} : t = 1, \ldots, l\}$ are all disjoint. Construct the strategy $\mathbf{s}^i$ for i in the game Γ as follows: the choice made by $\mathbf{s}^i$ at I^i_{jt} is the same as the choice made by $\hat{\mathbf{s}}^i$ at $\hat{I}^i_t$; the choice made by s^i at information sets other than $I^i_{j1}, \ldots, I^i_{jl}$ is arbitrary. Clearly the play defined by $(s^1, \ldots, s^{i-1}, \mathbf{s}^i, s^{i+1}, \ldots, s^n)$ is also **P**. Thus s is not in $\eta(\Gamma)$ since i can improve his payoff by deviating to $\mathbf{s}^i$, a contradiction. ∎

Remark 1. This proposition is not true for mixed strategies. Consider the game of matching pennies, where Player 1 wins if they match. If both players move simultaneously the payoff matrix is 2×2 (Figure 1a); each player has two moves, which coincide with his strategies. The only noncooperative equilibrium is in mixed strategies where each player uses a mixture of $(\frac{1}{2}, \frac{1}{2})$ over his two pure strategies. The expected payoff to each is zero. A refinement of the information sets in this game is given if we assume that Player 2 is informed of Player 1's move before Player 2 is called upon to move.

In Figure 1b we observe that the noncooperative equilibrium in the 2×4 matrix game is given where Player 2 uses his strategy C. The notation (1,2; 2,1) can be read as the sentence: "If Player 1 chooses his first strategy choose move 2; if he chooses his second strategy choose move 1."

	1	1
1	1	-1
2	-1	1

(a)

	A (1,1;2,1)	*B* (1,1; 2,2)	*C* (1,2;2,1)	*D* (1,2;2,2)
1	1	1	-1	-1
2	-1	1	-1	1

(b)

Figure 1

Remark 2. If there are chance moves in the game, and if we vary information of the traders regarding *each others' moves only, while their information about chance moves is held fixed and identical,* then the proposition continues to hold (with the obvious modifications in the proof). However, the following example shows that the result breaks down outside of this case.[3] Consider the game in Figure 2 where players' information about chance moves is kept fixed but not identical. The game is zero-sum, the stated payoff being that to Player 1. All the nodes not enclosed explicitly by information sets form singleton information sets. The two games differ only in 1's information about 2's move. An equilibrium point in the coarse game is for 1 to go right in both cases, whereas 2 goes left on the left node and right on the right node; the payoff is 3. This is not an equilibrium point in the refined game because 1 will always

[3] We are grateful to R. J. Aumann for this example which improved upon an earlier example of ours.

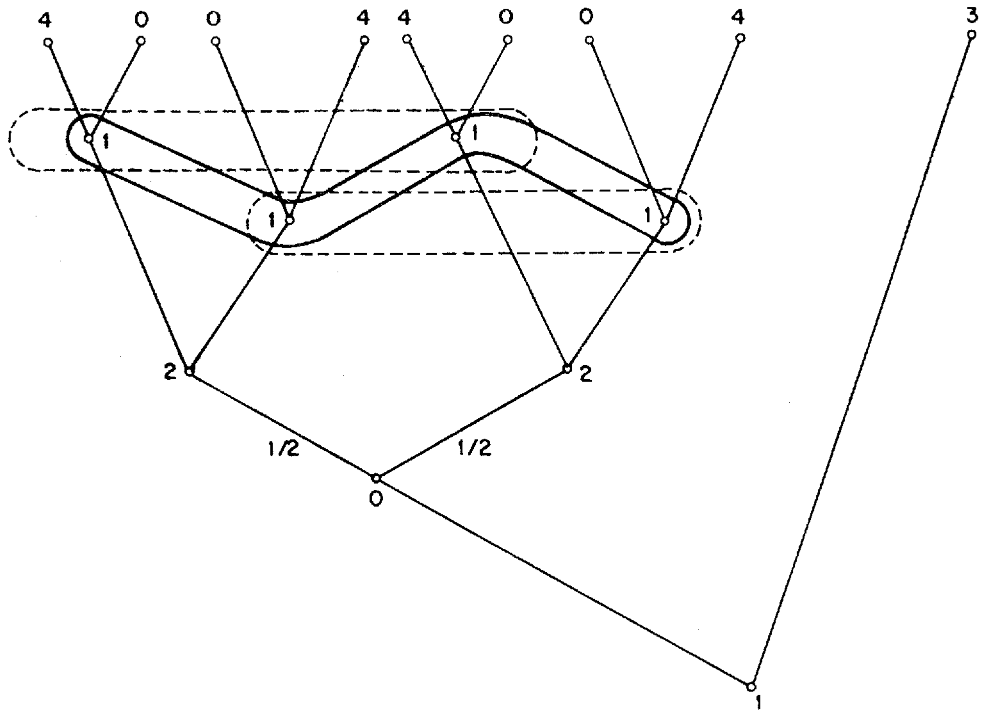

FIGURE 2

know what 2 has done and so can get 4. When the players' information about chance moves is allowed to vary, then a counterexample is even easier; for example, see §5.2 of Ponssard's paper [2]. This is similar to Hirschleifer's famous mutual crop insurance example [4], where reliable weather forecasts may actually be harmful; i.e. adding information may destroy beneficial equilibrium points.

REMARK 3. The result also breaks down if one substitutes "perfect equilibrium" [5] for "equilibrium," as shown by the example in Figure 3. In the coarse game, a perfect equilibrium point is for both players always to go left; in the refined game this is still an equilibrium point, but it is not perfect. In fact, here the inclusion goes in the opposite direction: the refined game has fewer perfect equilibria than the coarse one. It might be conjectured that this is always the case, but that is not true either. Indeed, consider a 2×2 matrix game with no pure strategy equilibria. Now give one player

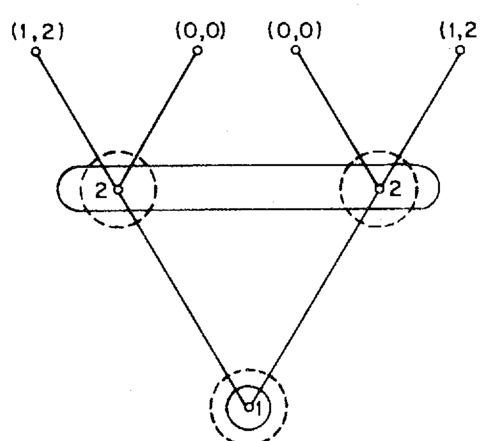

FIGURE 3

perfect information (converting it to a 2×4 matrix game). This always has at least one perfect equilibrium.

3. Equilibrium points in market games. One approach to the competitive equilibrium of a market is to represent a market as a game with a trading method fully defined and to show that the competitive equilibria may be obtained as noncooperative equilibria of this game. There are many ways of doing this, (e.g., [1], [6], [7], [8]). A problem with these models is that it is not clear to what extent the conclusions depend on the precise details of the informational structure of the game (i.e. who knows what at which stage).

For example, [1] describes a bid-offer game, which we call Γ^*. Each player decides how much of his endowment of each good to offer for sale, and how much of each good he wants to buy (his *bid* for that good). In Γ^*, all bids and offers are made simultaneously, so that really nobody knows anything about other players' moves at the time he makes his. In reality the various players' bids and offers may be made at different times, so that players may have partial information about other bids and offers at the time they make theirs. We will call such games *information variants* of Γ^*.

The game Γ^* has many representations in extensive form, all of them equivalent. What is common to them all is that whenever a player is called upon to make a move, he knows *nothing* about what has previously been done; each of them is an extensive game in its coarsest form.[4] We have shown that–in an appropriate sense and under appropriate conditions–the N. E.'s of Γ^* "converge" to the C. E.'s of the market if the market itself "approaches" a nonatomic market. In this sense, the N. E.'s of Γ^* may be considered "associated" with the C. E.'s of the market.

Any information variant Γ of Γ^* may be obtained by refining information sets in some representation of Γ^*. But then for all such Γ, it follows from our proposition that $\eta(\Gamma^*) \subset \eta(\Gamma)$. In words: the *C. E.–associated noncooperative equilibria are the only noncooperative equilibria that are common to all information variants of the market game.*

Similar results hold for any of the noncooperative game models of markets mentioned above.

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[4]Strictly speaking, players in an extensive game in its coarsest form do not even remember what they themselves previously did. However, it can be shown that the pure strategy N. E.'s do not change if such a game is modified by having each player remember his own previous moves.

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