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SYSTEMS DEFENSE GAMES: COLONEL BLOTTO, COMMAND AND CONTROL*

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ABSTRACT

The classical "Colonel Blotto" games of force allocation are generalized to include situations in which there are complementarities among the targets being defended. The complementarities are represented by means of a system "characteristic function," and a valuation technique from the theory of cooperative games is seen to indicate the optimal allocations of defense and attack forces. Cost trade-offs between systems defense and alternative measures, such as the hardening of targets, are discussed, and a simple example is analyzed in order to indicate the potential of this approach.

1. COLONEL BLOTTO GAMES

The first example of what has come to be called a "Colonel Blotto game" was apparently given by Borel [3]. He discussed the case of a defender attempting to protect several locations against an aggressor. A typical objective of the aggressor was to maximize the expected number of locations captured.

Games involving this type of objective were subsequently studied by Tukey [11] and others (for example, Gross [7], Blackett [2], Dresher [4], Beale and Heselden [1]). As defined by Beale and Heselden, a (Colonel) *Blotto game* is a zero-sum game involving two opposing players, I and II, and n independent battlefields. I has A units of force to distribute among the battlefields, and II has B units. Each player must distribute his forces without knowing his opponent's distribution. If I sends x_k units and II sends y_k units to the kth battlefield, there is a payoff $P_k(x_k, y_k)$ to I as a result of the ensuing battle; the payoff for the game as a whole is the sum of the payoffs at the individual battlefields.

In this paper we consider a generalization of the classical Blotto game. This generalization gives regard to the important class of military problems wherein there exist complementaries among the points being defended. In such cases, the final status of the competitors is not determined merely by totalling individual target values, but depends on the relative value of

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capturing (or neutralizing) various configurations of targets. Our generalization includes the classical Blotto games, as well as, for example, games in which the aggressor's objective is to maximize the probability of capturing a majority of the targets.

By considering complementarities among targets, we are in a position to study the defense of networks. For the purposes of increased reliability and security, redundancy is often intentionally incorporated into telephone and electrical power grids, early warning networks, and command and control systems. It is natural to ask how well protected such systems are from a disabling attack. Furthermore, it is of interest to consider cost trade-offs between built-in redundancy and extrinsic defense. In order to pursue these issues, we first introduce some terminology from cooperative game theory.

2. SYSTEMS PERFORMANCE AND THE CHARACTERISTIC FUNCTION

An n-person game in coalitional form is described by a *characteristic function* $v(\cdot)$ defined for all subsets of the set N of "players." When one is considering networks (or battlefields, or strategically important facilities), $v(S)$ may be interpreted as the value remaining in the system if only the set of nodes S is held. The characteristic function captures in a general setting the many types of complementarity which can exist among the various combinations of points in the network. (In traditional cooperative game theory it is frequently assumed that the characteristic function is superadditive; that is, if S and T are disjoint then $v(S) + v(T) \leqslant v(S \cup T)$. However, in the context of strategic systems this assumption may not be reasonable. If one is protecting a network of doomsday devices, for example, the characteristic function might assign a value of 1 to every nonempty set.)

There are many different "solutions" which have been suggested by game theorists for games in coalitional form. They reflect various aspects of the cooperative dealings among players with different goals. We note in particular the value solutions, which can be given an interpretation in terms of the military problem of allocating forces to a system of n nodes. In order to give this interpretation in detail we must reformulate the original n-person game as a two-person noncooperative game.

3. THE NONCOOPERATIVE GAME

We recast the given game as if it were a zero-sum game played between two opponents, a defender and an attacker. The n players in the original game are regarded as nodes (or individual targets) in a strategic network that the defender is trying to protect and the attacker is trying to destroy.

Let A and B be the respective amounts of strategic resources (troops, for example, or antiballistic and ballistic missiles) held by the defender and the attacker. The defender may choose any nonnegative allocation $x = (x_1, \ldots, x_n)$ of resources, subject to the constraint that $\Sigma\, x_i = A$. Similarly, the attacker may choose any allocation $y = (y_1, \ldots, y_n)$ for which $\Sigma\, y_i = B$. Let $f_j(x_j,y_j)$ be the function (yet to be specified) which indicates the outcome of the battle at point j. A natural interpretation which we take at this time is that $f_j(x_j,y_j)$ is the probability that the defender retains point j.

Assume that the goal of the defender is to maximize the (expected) effectiveness of the surviving configuration of targets. If the interests of the attacker are directly opposed to those of the defender, then we have at hand a two-person zero-sum game. The probability that the targets in the set S survive, while all others are destroyed, is

$$\prod_{i \in S} f_i(x_i,y_i) \prod_{j \notin S} (1 - f_j(x_j,y_j)).$$

Therefore, the expected effectiveness of the surviving collection is

$$\sum_{S \subset N} \left\{ \prod_{i \in S} f_i(x_i,y_i) \prod_{j \notin S} (1 - f_j(x_j,y_j)) \right\} v(S) ;$$

this is the defender's payoff.

If we suspend the interpretation of the functions f_j as probabilities, we find that this competitive game is indeed a direct generalization of the traditional Colonel Blotto game. Assume that the underlying characteristic function is additive, so that $v(S) = \sum_{k \in S} v(\{k\})$ for all S N. Then

$$D(x,y) = \sum_{k=1}^{n} f_k(x_k,y_k)\ v(\{k\}) .$$

By identifying $P_k(x_k,y_k)$ with $f_k(x_k,y_k) \cdot v(\{k\})$ (for example, by taking $P_k = f_k$ and $v(\{k\}) = 1$ for all $k \in N$), we can represent any desired classical Blotto game.

4. BATTLE MODELS

A listing of the various battle models which have been considered is beyond the scope of this paper. Moreover, a critical evaluation of the relative validity of these models does not appear to be available. Even Napoleon's dictum that God is on the side of the strongest battalion does not appear to be borne out when the force sizes of victors and losers in major battles are compared (for example, see Dupuy, page 89 [6]).

For the purposes of this paper we have chosen to consider a moderately general class of models in which the attacker and defender have homogenous resources. Hence, force mix problems have been set aside. Still, while it may be reasonable to assume that the probability that a target j is captured or destroyed is simply a function $f_j(x_j,y_j)$ of the resources expended in attack and defense by the two sides, the actual form of this function depends on empirical factors such as target type, physical vulnerability, troop morale, and the like.

We specifically consider outcome functions of the form

$$f(x,y) = \frac{\gamma x^m}{\gamma x^m + (1-\gamma) y^m},$$

where we set $f(0,0) = \gamma$. The parameter γ may be interpreted as an indicator of the natural defensibility of the target; if $x = y$, then $f(x,y) = \gamma$. The homogeneity of the function f allows us to concern ourselves with the ratio $k = x/y$ of defending to attacking forces, rather than with the specific amounts x and y. The parameter m reflects the importance of the relative difference in size between the attacking and defending forces.

In the limit, as m becomes large, the outcome function becomes the crudest form of "superior forces" model: the side which commits a greater force will win with certainty. If the resources of the defender and the attacker are of comparable size, in this limiting case the force-allocation game may fail to have a solution in pure strategies. (For an investigation of the degree of disparity of initial force sizes sufficient to guarantee the existence of optimal pure strategies, see Young, [13]).

On the other hand, if m is not too large, the outcome function is relatively insensitive to small changes in opposing allocations. We consider this case in the next section.

5. VALUE SOLUTIONS

Let $v(\cdot)$ be a characteristic function on N, and let $p = (p_1, \ldots, p_n)$ be a vector of probabilities (that is, each $0 \leqslant p_i \leqslant 1$). Then the $(p_1, \ldots, p_n)$-*value* of v is the n-vector $\beta = (\beta_1, \ldots, \beta_n)$ defined for all $i \in N$ by

$$\beta_i = \sum_{S \subset N \setminus i} \left\{ \prod_{j \in S} p_j \prod_{\substack{k \in N \setminus S \\ k \neq i}} (1 - p_k) \right\} [v(S \cup i) - v(S)].$$

Consider the force-allocation game based on v, in which the initial resources of the opposing sides are A and B, respectively. Assume that the outcome function at the kth target is defined by $f_k(x,y) = \gamma_k x^m / (\gamma_k x^m + (1 - \gamma_k) y^m)$. Then if both sides have optimal pure strategies, these strategies must be force allocations proportional to the $(f_1, \ldots, f_n)$ (A,B)-value of the underlying game. Furthermore, for all sufficiently small values of m, allocations proportional to the $(f_1, \ldots, f_n)$ (A,B)-value are indeed optimal.

Further details concerning these results are presented elsewhere (Shubik and Weber [9]).

6. THE COSTS OF SYSTEMS DEFENSE

"What price freedom?" is an important question, but one which political philosophers, economists, and Department of Defense budget proposers often find difficult to make precise. A model which links the value and cost of defense is presented here. (A different model is presented in Section 7, where we take the cost of defense as given but consider the possibility of trade-offs between direct defense and the physical reinforcement of individual targets.)

At an abstract level, there are four major items in the description of a defensive system: the military or societal "worth" of defense; the type, quantity, and structure of defensive forces; the cost of these forces; and the "hardness" (defensive strength) of individual targets.

The model of Section 3 avoids the problem of comparing value and cost by representing value within the characteristic function and taking as given the available attack and defense forces. Thus, constraints on military resources enter only as boundary conditions on a force assignment problem, rather than as a result of taking resource costs into account in the payoff structure.

We can modify the games of Section 3 to include costs in the following manner. The defender and attacker first select force levels k_1 and k_2, incurring costs of $c_1(k_1)$ and $c_2(k_2)$. They then each assign forces, and the payoffs are given by

$$(*) \qquad P_D = v(S) - c_1(k_1), \text{ and}$$
$$P_A = w(S^c) - c_2(k_2),$$

where $v(S)$ is the worth (in monetary units) to the defender of the configuration S of surviving targets, and $w(S^c)$ is the worth to the attacker of destroying or capturing the targets in S^c. This is a two-stage nonconstant-sum game, which might be studied in terms of either equilibrium or minimax theories.

The fact that the above game formulates well as a two-stage process calls attention to the fact that the two stages are separate in both time and bureaucratic control. The problem for a defense department in dealing with the government as a whole is to select k_i, incurring the budgetary expense $c_i(k_i)$. The problem of the commander, having been presented with forces k_i, is to allocate these forces wisely.

From the viewpoint of analysis, the models of Section 3 seem worth pursuing at the level of command and control. However, it appears that the first stage of the model suggested by (*) concerns a very different aspect of decision making, and involves deep issues in the area of defense budgeting (some of these issues have been discussed by Hitch and McKean [8]).

7. THE HARDENING OF TARGETS

In order to illustrate some of the preceding considerations, we analyze a simple example. Assume that a defender seeks to protect three sites, at each of which several antiballistic missiles are siloed. If the attacker destroys any two (or all three) of the targets, the overall defensive system will collapse. The first site houses more missiles than the second, which in turn houses more than the third; although any two surviving sites will yield an adequate system, the survival of all three provides even greater security. We model this situation with a characteristic function v, which satisfies v(123) = 4; v(12) = 3; v(13) = 2; v(23) = 1; and v(S) = 0 if $|S| \leqslant 1$.

Assume that the attacker and defender possess comparable amounts of strategic resources; say, $A = B = 1$. Let the outcome of conflict at site k be represented by the function $f_k(x,y) = \gamma_k x/(\gamma_k x^m + (1-\gamma_k)y^m)$, for some moderately small value of m (that is, assume that equal forces engaged at site k will yield a result favorable to the defender with probability γ_k, and further assume that small differences in resource assignments lead to only relatively small changes in this probability). The parameter γ_k indicates the "hardness" of the target at site k (that is, its natural strength against attack). It follows, as was indicated in Section 5, that the optimal allocation of strategic forces by each side will be proportional to the $(\gamma_1,\gamma_2,\gamma_3)$-value of the game v. Hence, this allocation will be proportional to the vector

$$\beta = (2\gamma_3 + 3\gamma_2 - 2\gamma_2\gamma_3,\ 3\gamma_1 + \gamma_3 - 2\gamma_1\gamma_3,\ 2\gamma_1 + \gamma_2 - 2\gamma_1\gamma_2).$$

In particular, if we initially have $\gamma_1 = \gamma_2 = \gamma_3 = 1/2$, the optimal allocation for each side is (4/9, 3/9, 2/9).

Now, assume that additional capital is available to the defender, and may be used to harden any of the targets. Specifically, assume that an investment of Δc_k units of capital at site k will yield an increase of $(1-\gamma_k)\Delta c_k$ in the hardness of target k; that is, $\partial\gamma_k/\partial c_k = (1-\gamma_k)$. A natural question is how best to invest the additional capital.

Let the defender's allocation of forces be $x = (x_1,x_2,x_3)$, while the attacker's deployment is $y = (y_1,y_2,y_3)$. Then the value of the outcome of the competitive game, to the defender, is

$$D(x,y) = 3f_1f_2 + 2f_1f_3 + f_2f_3 - 2f_1f_2f_3,$$

where each f_k is evaluated at (x_k,y_k). The optimal strategies are $x^* = y^* = \beta/\Sigma\beta_i$. Therefore, the rate of gain from investment in the hardening of target k is

$$\frac{\partial D}{\partial c_k}(x^*,y^*) = \frac{\partial D}{\partial p_k}(x^*,y^*)\ \frac{\partial f_k}{\partial\gamma_k}\ (x^*,y^*)\ \frac{\partial\gamma_k}{\partial c_k}$$

$$= (\beta_k/\Sigma\beta_i)\cdot 1\cdot(1-\gamma_k).$$

The best investment is in the target (or targets) for which this expression is maximized. But the expression varies with the parameters γ_1, γ_2, and γ_3. Hence, if we begin with all γ_k equal, it is best to initially invest in work at the site for which β_k is maximal; this changes β as well as γ_k, after which we can determine the best target for further investment. Beginning with $\gamma_1 = \gamma_2 = \gamma_3 = 1/2$, we obtain the results indicated in the figures. (As the available capital increases without limit, the value of $D(x^*,y^*)$ approaches 4, and the three sites attract nearly equal proportions of the capital.)

This example illustrates several, but by no means all, of the types of computations which appear to be feasible and relevant to the study of tradeoffs in defense, in the hardening of targets, and in built-in system redundancy.

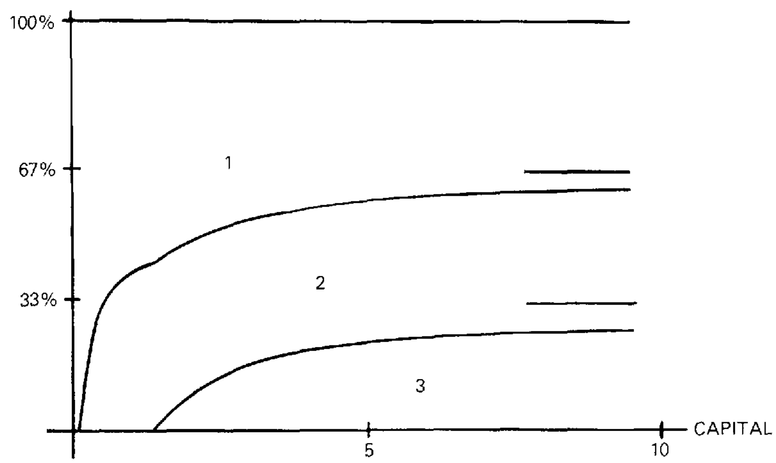

FIGURE 1. Allocation of capital to target reinforcement

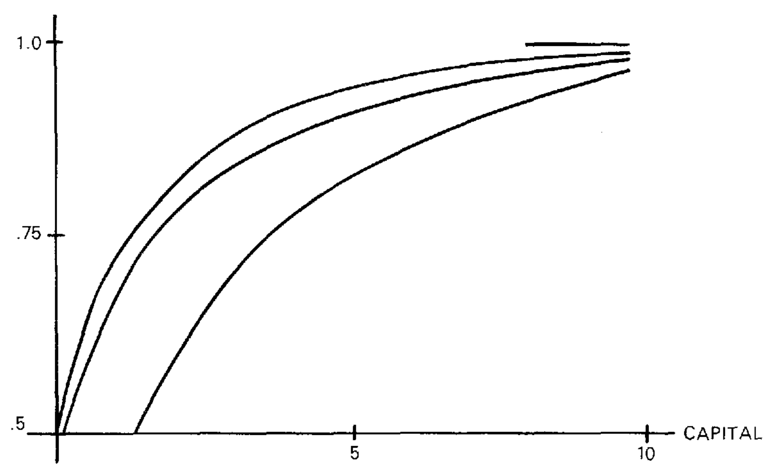

FIGURE 2. Hardness of targets: γ_1 γ_2 and γ_3

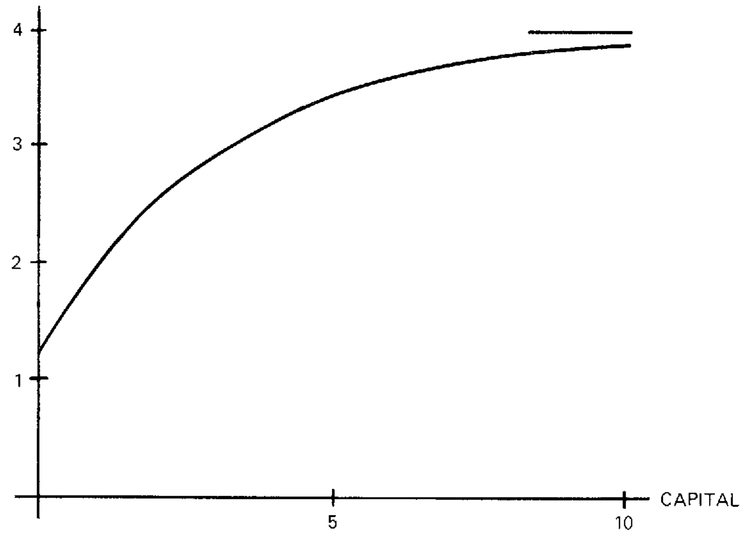

FIGURE 3. Value of game to defender: $D(x^*, y^*)$.

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