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The ALEP Definition of Complementarity and Least Concave Utility Functions*

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The use of least concave utility functions describing a given concavifiable preference relation is suggested for determining the complementary vis-à-vis substitute nature of a pair of commodities.

Let $\succsim$ be a preference relation defined on an open convex subset K of R^n such that $\succsim$ is representable by a twice differentiable concave utility function u. The commodities i and j, $i \neq j$, are said to be ALEP complementary at x if $\partial^2 u(x)/\partial x_i \, \partial x_j > 0$. (This definition is due to Auspitz and Lieben [2, p. 482], and was adopted by Edgeworth [6, p. 117n] and Pareto [11, Chap. IV and Appendix, Sects. 12–13, pp. 505–507]—hence the term ALEP definition.) It is well known that this definition was criticized by Allen [1, p. 171n], Hicks and Allen [9, p. 60n], and Hicks [8, pp. 42–45] on the grounds that the condition $\partial^2 u(x)/\partial x_i \, \partial x_j$ is not invariant under monotone (increasing) transformations of the utility function u and so is not an intrinsic (ordinal) property of the preference ordering $\succsim$. In fact, this condition is not invariant even under the more restricted class of strictly monotone transformations preserving the concavity (or even the strict concavity) of the utility function. Quite the other way around: the more concave the utility function representing the given preference relation $\succsim$, the less apt are i and j to be ALEP complementary at x, at least if $\succsim$ is monotone (compare also [12]). For, let F be a twice-differentiable function of a single variable with $F'(t) > 0$ for all t, and set $v(x) = F(u(x))$. Denoting partial derivatives by subindices, we find that

$$v_{ij}(x) = F'(u) \left[u_{ij}(x) + \frac{F''(u)}{F'(u)} u_i(x) \, u_j(x)\right]. \tag{1}$$

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If u is assumed to be sufficiently monotone so as to satisfy $u_i(x) > 0$ for all $1 \leqslant i \leqslant n$, then, choosing $F''(u(x))$ sufficiently negative, we can make $v_{ij}(x)$ negative even if $u_{ij}(x) > 0$.

We have seen thus that in the interesting cases every pair of commodities can be made to be ALEP substitutes at any given point x, by a suitable choice of a concave utility function. On the other hand, it was proved recently by Chipman [3] that ALEP complementarity does affect, in an invariant way, the behavior of the demand function (it was observed earlier by Georgeseu–Roegen [7] that complementarity influences the shape of the indifference curves of $\gtrsim$). In view of all this, we are led to consider the class of least concave utility functions for the given preference ordering $\gtrsim$. Least concave (alias minimally concave) utility functions were introduced by de Finetti [5]; their existence was proved by Debreu [4], and several ways of computing them were given by the author in [10]. The concave utility function $u(x)$ representing the preference relation $\gtrsim$ is said to be least concave if every concave utility function $v(x)$ representing $\gtrsim$ is given by $v(x) = F(u(x))$, where F is a strictly monotone and concave function of a single variable. (Note that least concave utility functions are unique up to translations and multiplications by positive constants.) We read off (1) that if a least concave utility function u is twice differentiable, if i and j are (ALEP) complementary at a certain x when a certain twice-differentiable utility v is considered, and if the preference relation $\gtrsim$ is monotone, then $u_{ij}(x)$ must also be positive. If, however, $u_{ij}(x) < 0$, then for no concave utility function (representing $\gtrsim$) is it going to happen that $v_{ij}(x) > 0$. (Note that the sign of $u_{ij}(x)$ is the same for all least concave utility functions representing $\gtrsim$.)

These facts suggest that we consider the sign of the mixed derivatives of least concave utility functions (if the latter are sufficiently smooth) in defining complementarity, and adopt the following.

DEFINITION. The commodities i and j, $i \neq j$, are said to be complementary at x if $u_{ij}(x) > 0$ for a least concave utility function u representing the preference relation $\gtrsim$ of the consumer, and are said to be substitutes if $u_{ij}(x) < 0$.

Remarks. (i) If the least concave utility function is not twice differentiable at x, one could still define i and j to be complementary if

$$\liminf_{h,k\to 0}\, [u(x + he_i + ke_j) + u(x) - u(x + he_i) - u(x + ke_j)]/(hk) > 0, \quad (2)$$

where e_i and e_j are the unit vectors in the ith and jth directions, respectively. Similarly, i and j are substitutes if

$$\limsup_{h,k\to 0}\, [u(x + he_i + ke_j) + u(x) - u(x + he_i) - u(x + ke_j)]/(hk) < 0. \quad (3)$$

If $u_{ij}(x) = 0$ then i and j are said to be (ALEP) independent. Note that if u is not twice differentiable, then it might happen that the lim inf in (2) is negative while the lim sup in (3) is positive. Such a case may be termed indeterminate. It is doubtful, however, whether this case is of importance in the economic applications.

(ii) The following example might illustrate the ideas presented here. Let $K = \{(x_1, x_2): x_1 > 0, x_2 > 0\}$ and let $(x_1, x_2) \gtrsim (y_1, y_2)$ if and only if $x_1 x_2 \geqslant y_1 y_2$. Commodities 1 and 2 appear to be independent (everywhere) if one chooses the separable (concave) utility function $\ln x_1 + \ln x_2$, and appear to be substitutes everywhere if the concave utility function $-1/(x_1 x_2)$ is chosen instead. Considering the least concave utility function $(x_1 x_2)^{1/2}$, we see that commodities 1 and 2 are indeed complementary everywhere, according to the definition suggested here.

(iii) The argument of this note lends further credibility to the selection of least concave utility functions as cardinal utilities (compare [4, 10]).

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