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Nash Equilibria of Market Games: Finiteness and Inefficiency*

PRADEEP DUBEY

Cowles Foundation for Research in Economics, Yale University, New Haven, Connecticut 06520

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1. INTRODUCTION

In this paper we make good the promise [6, Sect. 9] that, under appropriate conditions, the Nash Equilibria of finite-player strategic market games are generically inefficient. Our conditions are: (a) the dimension of each trader's strategy set is at most $l - 1$, where l is the number of commodities, (b) the mapping from strategies to net trades is sufficiently smooth, (c) so are the traders' preferences. For concreteness we work out a specific model—the so-called "sell-all" model, explored in [9]. But the same proof can easily be adapted to the general case.

Condition (c) is quite standard and not particularly restrictive (e.g., see [7]); Condition (b) is necessary and without it our conclusion may well fail. Indeed see [8, 5] for models with price-cutting strategies where the Nash and Walras Equilibria coincide. For a discussion of (c), see again Section 9 of [6].

Needless to say we take our cue from the analogous finiteness results obtained for the Walras Equilibria of markets by Debreu [2], Smale [11], and others. For a similar analysis of strategic games in general, see [4].

For any integer r, let $I_r = \{1, 2, \ldots, r\}$, and let Ω^r be the nonnegative orthant of the Euclidean space of dimension r. Let I_n be the set of traders and I_{k+1} the set of commodities in which they trade. Each trader $i \in I_n$ is characterized by an initial endowment $a^i \in \Omega^{k+1}$ and a utility function $u^i : \Omega^{k+1} \to R$. (Here a_j^i is the quantity of commodity j held by trader i.) We assume that $\sum_{i=1}^n a^i > 0$ and that $a_{k+1}^i > 0$ for all[1] $i \in I_n$.

To recast the market as a game in strategic form, we single out the $(k+1)$st commodity as a money. There are k trading posts, one for each of the other commodities. Traders are required to put up all of their first k commodities

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[1] If $a_{k+1}^i = 0$, then trader i is a "strategic dummy" in our game and can be ignored.

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for sale in these trading posts, and use their endowment of commodity money for bidding on them. The strategy set S^i of trader i consists of bids on the k trading posts, but he is constrained to bid within a^i_{k+1}:

$$S^i = \left\{b^i \in \Omega^k : \sum_{j \in I_k} b_j{}^i \leqslant a^i_{k+1}\right\}.$$

Given a choice of strategies $b = (b^1, \ldots, b^n)$, $b^i \in S^i$, prices $p(b) \in \Omega^k$ are formed in the trading posts and the markets cleared, with the final bundle $x^i(\mathrm{b})$ accruing to i, according to the rules:

$$\begin{aligned} p_j(b) &= \frac{\bar{b}_j}{\bar{a}_j} && \left(\text{where } \bar{b}_j = \sum_{i \in I_n} b_j{}^i, \text{ etc.}\right) \\ x_j{}^i(b) &= \frac{b_j{}^i}{p_j(b)} && \text{if} \quad p_j(b) > 0 \\ &= 0 && \text{if} \quad p_j(b) = 0 \end{aligned}$$

for $j = 1, \ldots, k$; and

$$x^i_{k+1}(b) = a^i_{k+1} - \sum_{j=1}^{k} b_j{}^i + \sum_{j=1}^{k} p_j(b)\, a_j{}^i.$$

(We may interpret $x_j{}^i(b) = 0$ to be a confiscation of goods in the absence of any bid. Consider a $\hat{b} = (\hat{b}^1, \ldots, \hat{b}^n) \in S = S' \times \cdots \times S^n$. $\hat{b}$ is called a *Nash Equilibrium* (N.E.) of this game if, for all $i \in I_n$,

$$u^i(x^i(\hat{b})) \geqslant u^i(x^i(\hat{b} \mid b^i)), \qquad b^i \in S^i,$$

where $(\hat{b} \mid b^i)$ is the same as $\hat{b}$ but with $\hat{b}^i$ replaced by b^i; it is called an *efficient* point of the game if there is no $b \in S$ such that

$$\begin{aligned} u^i(x^i(b)) &\geqslant u^i(x^i(\hat{b})) && \text{for all} \quad i \in I_n, \\ u^l(x^l(b)) &> u^l(x^l(\hat{b})) && \text{for some} \quad l \in I_n. \end{aligned}$$

2. FINITENESS AND INEFFICIENCY

In our analysis we will, for convenience (see Remark 3, however), hold the traders' endowments fixed and vary their utilities only. Put $Q = \{x \in \Omega^{k+1} : x_j \leqslant \bar{a}_j\}$. Let U denote the linear space of all C^2 functions[2] from Q to R, endowed with the C^2-norm.[3] Thus we may think of a market game as

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[2] That is, those which can be extended to a C^2 function on R^{k+1}.

[3] That is, $\| u \| = \sup\{\| u(x)\|, \| Du(x)\|, \| D^2u(x)\| : x \in Q\}$, where $\| \ \|$ denotes the maximum norm.

given by a point $u = (u^1, \ldots, u^n) \in (U)^n$. Denote the set of its N.E. by $\eta(u)$, and the set of its efficient points by $\epsilon(u)$.

We will focus our attention on certain open sets of $(U)^n$. Let e be the unit vector in Ω^{k+1}, i.e., $e_j = 1$ for each j. Take two positive numbers σ and σ', $\sigma < \sigma'$. Define $U(\sigma, \sigma') = \{u \in U : \sigma e < Du < \sigma' e\}$. The manifold of games which we will consider will be open sets of the type $(U(\sigma, \sigma'))^n$. Finally let $S' = \{(b^1, \ldots, b^n) \in S : b^i$ is a vertex of S^i for at least one $i \in I_n\}$. We are now ready to state our main result:

THEOREM. *Fix* σ *and* σ', $0 < \sigma < \sigma'$. *There is an open dense set* E *of* $(U(\sigma, \sigma'))^n$ *such that, for any* $u \in E$,

(a) $\eta(u)$ *is a finite set,*

(b) $\eta(u) \cap \epsilon(u) \subset S'$.

Remark 1. To obtain generic inefficiency, we could, for instance, confine our attention to $u = (u^1, \ldots, u^n) \in (U(\sigma, \sigma'))^n$ such that:[4]

> for any i in I_n, there exist at least two distinct $j_1(i)$ and $j_2(i)$ in I_k with the property ${u^i}^{-1}(r) \cap \{x \in \Omega^{k+1} \colon x_{j_1(i)} = 0\} = \varnothing$ and ${u^i}^{-1}(r) \cap \{x \in \Omega^{k+1} \colon x_{j_2(i)} = 0\} = \varnothing$, whenever $r > u^i(0)$.

Such u form an open set $\tilde{E}$ in $(U(\sigma, \sigma'))^n$, and clearly imply $\eta(u) \cap S' = \varnothing$. Thus the theorem could be restated with $\tilde{E}$ in place of $(U(\sigma, \sigma'))^n$ and "$\eta(u) \cap \epsilon(u) = \varnothing$" in place of (b).

3. PROOF OF THEOREM

LEMMA 1. *Fix* σ *and* σ', $0 < \sigma < \sigma'$. *Then there exists a* $\mu > 0$ *such that for any* $u \in (U(\sigma, \sigma'))^n$,

$$\eta(u) \subset S_\mu = \{b \in S : \bar{b}_j > \mu \quad \textit{and} \quad b_j^i < \bar{b}_j \quad \textit{for all} \quad i \in I_n, j \in I_k\}.$$

Proof. Let $u \in (U(\sigma, \sigma'))^n$ and $b = (b^1, \ldots, b^n) \in \eta(u)$. Then clearly $\bar{b}_j > 0$ for $j \in I_k$. Otherwise, if $\bar{b}_j = 0$ for some j, then any trader could bid an arbitrarily small ϵ on the jth trading post (if necessary by reducing some other bid). By this change of strategy, he acquires $\bar{a}_j$, while his other holdings change by amounts that go to zero with ϵ. Hence for small enough ϵ this improves his payoff, a contradiction.

Next, it is also clear that $b_j^i < \bar{b}_j$ for all i and j. For if $b_j^i = \bar{b}_j$, then i could reduce b_j^i and improve his payoff, a contradiction.

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[4] Intuitively this says that each trader "sufficiently desires" at least two commodities.

To establish the lower bound on $\bar{b}_j$ we consider two cases.

Case I. $\sum_{r=1}^{k} b_r{}^h < a_{k+1}^h$ for all $h \in I_n$. Let i be such that $b_j{}^i/\bar{b}_j < 1/2$. Now if i bids ϵ more on j, his increase in payoff for small ϵ is approximately:

$$\epsilon \cdot \left[\frac{\partial u^i}{\partial x_j}(x^i(b)) \cdot \frac{\bar{a}_j}{\bar{b}_j}\left(\frac{\bar{b}_j - b_j{}^i}{\bar{b}_j}\right) - \frac{\partial u^i}{\partial x_{k+1}}(x^i(b))\left(1 - \frac{a_j{}^i}{\bar{a}_j}\right)\right]$$
$$\geqslant \epsilon \cdot \left\{\frac{\sigma}{2} \cdot \frac{\bar{a}_j}{\bar{b}_j} - \sigma'\left[1 - \frac{a_j{}^i}{\bar{a}_j}\right]\right\}.$$

For b to be a N.E. we must have $a_j{}^i < \bar{a}_j$, and

$$\frac{\sigma\bar{a}_j}{2\bar{b}_j} \leqslant \sigma'\left[1 - \frac{a_j{}^i}{\bar{a}_j}\right],$$

i.e.,

$$\boxed{\bar{b}_j \geqslant \frac{\sigma\bar{a}_j}{2\sigma'\left[1 - \frac{a_j{}^i}{\bar{a}_j}\right]}}\ .$$

Case II. $\sum_{r=1}^{k} b_r{}^h = a_{k+1}^h$ for some $h \in I_n$. Then there is a $r^* \in I_k$ such that $b_{r^*}^h \geqslant a_{k+1}^h/k$. If $r^* = j$, then

$$\boxed{\bar{b} \geqslant \frac{a_{k+1}^h}{k}}\ .$$

If $r^* \neq j$, choose $i \in I_n$ to ensure that $b_j{}^i/\bar{b}_j < 1/2$. Now if i reduces his bid on r^* by ϵ and increases his bid on j by ϵ, then his increase in payoff for small ϵ is approximately:

$$\epsilon \cdot \left\{\frac{\partial u^i}{\partial x_j}(x^i(b)) \cdot \frac{\bar{a}_j}{\bar{b}_j}\left(\frac{\bar{b}_j - b_j{}^i}{\bar{b}_j}\right) - \frac{\partial u^i}{\partial x_{r^*}}(x^i(b)) \cdot \frac{\bar{a}_{r^*}}{\bar{b}_{r^*}}\left(\frac{\bar{b}_{r^*} - b_{r^*}^i}{\bar{b}_{r^*}}\right)\right.$$
$$\left. - \frac{\partial u^i}{\partial x_{k+1}}(x^i(b))\left(\frac{a_{r^*}^i}{\bar{a}_{r^*}} - \frac{a_j{}^i}{\bar{a}_j}\right)\right\}.$$

Now note

$$\frac{\partial u^i}{\partial x_{r^*}}(x^i(b)) \cdot \frac{\bar{a}_{r^*}}{\bar{b}_{r^*}}\left(\frac{\bar{b}_{r^*} - b_{r^*}^i}{\bar{b}_{r^*}}\right) \leqslant \sigma' \frac{\bar{a}_{r^*}k}{a_{k+1}^h},$$

$$\frac{\partial u^i}{\partial x_{k+1}}(x^i(b))\left[\frac{a_{r^*}^i}{\bar{a}_{r^*}} - \frac{a_j{}^i}{\bar{a}_j}\right] \leqslant \sigma' \frac{a_{r^*}^i}{\bar{a}_{r^*}},$$

$$\frac{\partial u^i}{\partial x_j}(x^i(b))\frac{\bar{a}_j}{\bar{b}_j}\left(\frac{\bar{b}_j - b_j{}^i}{\bar{b}_j}\right) \geqslant \frac{\sigma\bar{a}_j}{2\bar{b}_j}.$$

Hence the increase in i's payoff is at least:

$$\epsilon \cdot \left\{\frac{\sigma\bar{a}_j}{2\bar{b}_j} - \sigma' \left[\frac{a^i_{r*}}{\bar{a}_{r*}} + \frac{\bar{a}_{r*}k}{a^h_{k+1}}\right]\right\}.$$

This must be nonpositive, from which we get

$$\boxed{\bar{b}_j \geqslant \frac{\sigma\bar{a}_j}{2\sigma' \left[\frac{a^j_{r*}}{\bar{a}_{r*}} + \frac{\bar{a}_{r*}k}{a^h_{k+1}}\right]}}\ .$$

Let

$$M^1 = \min\{\bar{a}_j : j \in I_k\}$$

$$M^2 = \min\{a^i_{k+1} : i \in I_n\}$$

$$M^3 = \max\left\{1 - \frac{a_j^i}{\bar{a}_j} : i \in I_n\,,\ j \in I_k\,,\ a_j^i < \bar{a}_j\right\}$$

$$M^4 = \max\left\{\frac{a_j^i}{\bar{a}_j} + \frac{k\bar{a}_j}{a^h_{k+1}} : i \in I_n\,,\ h \in I_n\,,\ j \in I_k\right\}.$$

Set

$$\mu = \min\left[\frac{M^1}{2M^3\sigma'}, \frac{M^1}{2M^4\sigma'}, \frac{M^2}{k}\right]. \qquad \text{Q.E.D.}$$

Lemma 1 enables us to steer clear of the discontinuity in the payoff functions at $\bar{b}_j = 0$ for $j \in I_k$. This is important—see Remark 5—because in order to show the openness of E in the theorem we need to be able to extend the payoff functions smoothly in a neighborhood of S_μ.

Given Lemma 1, it suffices to prove the

AUXILIARY THEOREM. *There is an open dense set E of $(U(\sigma, \sigma'))^n$ such that for any $u \in E$*

(a) *$\eta(u) \cap S_\mu$ is a finite set*

(b) *$\eta(u) \cap \epsilon(u) \cap S_\mu \subset S'$.*

We proceed to prove the Auxiliary Theorem through a sequence of lemmas. Unfortunately, first we need to introduce some fairly cumbersome notation

V^i = the set of all the $k + 1$ vertices of S^i,
$\tilde{V}^i$ = the set of all nonempty subsets of V^i,
$\tilde{V} = \tilde{V}^1 \times \cdots \times \tilde{V}^n$.

For any $T^i \in \tilde{V}^i$, $T_0{}^i = T^i/\{0^i\}$, where 0^i is the zero vertex of S^i.
For any $T = \{T^1, \ldots, T^n\} \in \tilde{V}$ and $i \in N$:

$$\begin{aligned}
S^i(T) &= \text{convex hull of } T^i,\\
\dot{S}^i(T) &= \text{relative interior of } S^i(T),\\
\dot{S}(T) &= \dot{S}^1(T) \times \cdots \times \dot{S}^n(T),\\
\tilde{S}(T) &= \dot{S}(T) \cap S_\mu,\\
S_j{}^i &= \left\{ b \in S\colon a_j{}^i(\bar{b}_j - b_j{}^i) + \bar{a}_j b_j{}^i \left(\frac{a_j{}^i}{\bar{a}_j} - 1\right) = 0 \right\},\\
\tilde{S}_j{}^i(T) &= \tilde{S}(T) \cap S_j{}^i,\\
N(T) &= \{ i \in N\colon |\, T^i \,| > 1 \},\\
\hat{T} &= \bigcup \{ T_0{}^i\colon i \in N(T) \},\\
t^i &= |\, T_0{}^i \,|,\\
\hat{t} &= \sum_{i \in N(T)} t^i,\\
t &= |\, N(T) |,\\
R^{N\hat{T}} &= \text{Euclidean space of dimension } t\hat{t} \text{ whose axes}\\
&\ \text{are indexed by pairs } (i, j) \in N(T) \times \hat{T}.
\end{aligned}$$

For any $v \in R^{N\hat{T}}$, $v_j{}^i$ will be its (i, j)th component. Also for any $L \subset \hat{T}$, $v_L{}^i$ will be the vector in R^L (whose axes are indexed by elements of L) with components $\{v_j{}^i : j \in L\}$.

Note that there is a natural correspondence between elements of $V_0{}^i$ and the variables $\{b_j{}^i : 1 \leqslant j \leqslant k\}$. Thus, without confusion, we will speak sometimes of the variable x_l, $l \in \hat{T}$.

We construct a mapping[5]

$$_T\psi : (U(\sigma, \sigma'))^n \times \tilde{S}(T) \to R^{N\hat{T}},$$

which will enable us to study $\eta(u)$ and $\epsilon(u)$. Letting $u = (u^1, \ldots, u^n) \in (U(\sigma, \sigma'))^n$ $b \in \tilde{S}(T)$, define (for $i \in N(T)$, $j \in \hat{T}$)

$$_T\psi^i(u, b) = \left(\frac{\partial u^i}{\partial x_l}\right)(b), \qquad i \in N(T), \quad l \in \hat{T}.$$

It is easy to compute that:

$$_T\psi_j{}^i(u, b) = \frac{\partial u^i}{\partial x_j}(x^i(b)) \cdot \left[\frac{\bar{a}_j(\bar{b}_j - b_j{}^i)}{(\bar{b}_j)^2}\right] + \frac{\partial u^i}{\partial x_{k+1}}(x^i(b)) \left[\frac{a_j{}^i}{\bar{a}_j} - 1\right]$$

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[5] Assuming $\tilde{S}(T)$ is not empty.

if $j \in T_0{}^i$; and

$$_T\psi_j{}^i(u, b) = -\frac{\partial u^i}{\partial x_j}(x^i(b)) \cdot \left[\frac{\bar{a}_j b_j{}^i}{(\bar{b}_j)^2}\right] + \frac{\partial u^i}{\partial x_{k+1}}(x^i(b)) \left[\frac{a_j{}^i}{\bar{a}_j}\right]$$

if $j \notin T_0{}^i$.

Using $_T\psi$ we now define some other mappings. First, define the subspace $R^{\hat{T}}$ of $R^{N\hat{T}}$ by $R^{\hat{T}} = \{v \in R^{N\hat{T}} : v_j{}^i = 0 \text{ if } j \notin T_0{}^i\}$. Then define $_T\tilde{\psi} : (U(\sigma, \sigma'))^n \times S(T) \to R^{\hat{T}}$ by setting $_T\tilde{\psi}_j{}^i = 0$ if $j \notin T_0{}^i$. Finally define

$$_T\varnothing : (U(\sigma, \sigma'))^n \times \tilde{S}(T) \to R^{N\hat{T}} \times \tilde{S}(T)$$

and

$$_T\tilde{\varnothing} : (U(\sigma, \sigma'))^n \times \tilde{S}(T) \to R^{\hat{T}} \times \tilde{S}(T)$$

by

$$\begin{aligned} _T\varnothing(u, b) &= (_T\psi(u, b), b), \\ _T\tilde{\varnothing}(u, b) &= (_T\tilde{\psi}(u, b), b). \end{aligned}$$

We next need to define two subsets of $R^{N\hat{T}}$. To this end, first let

$$\begin{aligned} T_a &= \{i \in N(T) : 0^i \in T^i\}, \\ T_b &= \{i \in N(T); 0^i \notin T^i\}. \end{aligned}$$

Then let

$$\begin{aligned} \Delta^1(T) = \{v \in R^{N\hat{T}} : \ & \text{the projections of the } v^i,\ i \in N(T), \\ & \text{on } S(T) \text{ are linearly dependent}\}, \\ \Delta^2(T) = \{v \in R^{N\hat{T}} : \ & v^i_{T_0{}^i} = 0 \quad \text{for} \quad i \in T_a\,; \\ & v_j{}^i = v_l{}^i \quad \text{for} \quad i \in T_b\,, j \in T_0{}^i, l \in T_0{}^i\}. \end{aligned}$$

LEMMA 2. *Let M be any submanifold of $R^{\hat{T}} \times \tilde{S}(T)$. Then $_T\tilde{\varnothing}$ is transversal*[6] *to $M((_T\tilde{\varnothing} \pitchfork M))$.*

Proof. Consider any $(u, b) \in (U(\sigma, \sigma'))^n \times \tilde{S}(T)$ such that $_T\tilde{\varnothing}(u, b) = y \in M$. Take any $v \in R^{\hat{T}}$ and any $w \in$ Tangent space of $\tilde{S}(T)$. We will show that there is a differentiable path $\{(_\tau u, {}_\tau b)\}_{\tau=0}^{1}$ in[7] $(U)^n \times \tilde{S}(T)$ such that

$$(_0u, {}_0b) = (u, b),$$

$$\frac{d}{d\tau}[_T\tilde{\varnothing}(_\tau u, {}_\tau b)]\Big|_{\tau=0} = (v, w).$$

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[6] For the definition of "transversal," see Appendix.

[7] Clearly, for sufficiently small τ, this path will lie in $(U(\sigma, \sigma'))^n \times \tilde{S}(T)$.

To do this, let $\{{}_\tau b\}_{\tau=0}^{1}$ be a path in $\tilde{S}(T)$ such that[8]

$$ {}_0 b = 0, $$

$$ \frac{d}{d\tau}\,{}_\tau b \Big|_{\tau=0} = w $$

and then let ${}_\tau u = \{{}_\tau u^i : i \in N(T)\}_{\tau=0}^{1}$ be given by:

$$ {}_\tau u^i(x) = u^i(x) + \sum_{j \in T_i^0} \tau \alpha_j{}^i x_j \,, $$

where

$$ \begin{aligned} \alpha_j{}^i &= \frac{1}{A_j{}^i}\,[v_j{}^i - A_j{}^i L_j{}^i - B_j{}^i Q_j{}^i - D_j{}^i], \\ A_j{}^i &= \frac{\bar{a}_j(\bar{b}_j - b_j{}^i)}{(\bar{b}_j)^2}\,, \\ L_j{}^i &= \lim_{\tau \to 0^+} \frac{d}{d\tau}\Big[\frac{\partial u^i}{\partial x_j}\,(x({}_\tau b))\Big], \\ B_j{}^i &= \lim_{\tau \to 0^+} \Big[\frac{\partial u^i}{\partial x_j}\,(x({}_\tau b))\Big], \\ Q_j{}^i &= \lim_{\tau \to 0^+} \frac{d}{d\tau}\Big[\frac{\bar{a}_j({}_\tau\bar{b}_j - {}_\tau b_j{}^i)}{({}_\tau\bar{b}_j)^2}\Big], \\ D_j{}^i &= \Big(\frac{a_j{}^i}{\bar{a}_j} - 1\Big) \lim_{\tau \to 0^+} \frac{d}{d\tau}\Big[\frac{\partial u^i}{\partial x_{k+1}}\,(x({}_\tau b))\Big]. \end{aligned} $$

($A_j{}^i \neq 0$ since $b \in S''$, and these limits exist because u^i is C^2.) It is straightforward to verify that this path has all the requisite properties.

To complete the proof of the lemma, we need to establish that $(T_{(u,b)}\,{}_T\tilde{\varnothing})^{-1}$ $(T_y M)$ splits. Since we have established that the derivative map is surjective, this follows from the finite dimensionality of the range $T_y M$.

Q.E.D.

LEMMA 3. *There exists an open dense set* $E(T)$ *in* $(U(\sigma, \sigma'))^n$ *such that, for any* $u \in E(T)$,

(a) $\eta(u) \cap \tilde{S}(T)$ is a finite set,

(b) $\eta(u) \cap \tilde{S}_j{}^i(T)$ *is empty (for* $i \in N(T), j \in T^i$*).*

Proof. For any fixed u, denote by ${}_T^u\tilde{\varnothing}$ the mapping from $\tilde{S}(T)$ to $R^{\hat{T}} \times \tilde{S}(T)$ given by ${}_T^u\tilde{\varnothing}(b) = {}_T\tilde{\varnothing}(u, b)$. Consider the submanifold $(\Delta^2(T) \cap R^{\hat{T}}) \times \tilde{S}(T)$ of $R^{\hat{T}} \times \tilde{S}(T)$, and call it Z. By the Transversal Density and Openness

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[8] Clearly such a path exists.

Theorems,[9] there is an open dense set $\hat{E}(T)$ of $(U(\sigma, \sigma'))^n$ such that for each $u \in \hat{E}(T)$, ${}^{u}_{T}\tilde{\varnothing} \pitchfork Z$. But (letting $t_b = |T_b|$) codim $Z = \hat{t} - t_b = \dim \tilde{S}(T)$. Hence, for such u, ${}^{u}_{T}\tilde{\varnothing}^{-1}(Z)$ has dimension zero. Being bounded, it must be a finite set.[10]

Next let $Z_j^i = (\Delta^2(T) \cap R^{\hat{T}}) \times \tilde{S}_j^i(T)$. Again there is an open dense set $E_j^i(T)$ of $(U(\sigma, \sigma'))^n$ such that for any $u \in E_j^i(T)$, ${}^{u}_{T}\tilde{\varnothing} \pitchfork Z_j^i$. But codim $Z_j^i > \hat{t} - t_b = \dim \tilde{S}(T)$. Hence, for such u, ${}^{u}_{T}\tilde{\varnothing}^{-1}(Z_j^i) = \varnothing$.

Let $E(T)$ be the intersection of $\hat{E}(T)$ and all the $E_j^i(T)$. Since $\eta(u) \cap \tilde{S}(T) \subset {}^{u}_{T}\tilde{\varnothing}^{-1}(Z)$ and $\eta(u) \cap \tilde{S}_j^i(T) \subset {}^{u}_{T}\tilde{\varnothing}^{-1}(Z_j^i)$, we have proved the lemma.

Q.E.D.

In Lemmas 4, 5, and 6 we will assume that $N(T) = N$. Consider $M(T) = \tilde{S}(T)/\cup\{\tilde{S}_j^i(T) : i \in N, j \in T^i\}$. Then $M(T)$ is a manifold of dimension $\hat{t} - t_b$.

LEMMA 4. *Consider any* $M(T)$, *and the mapping* ${}_T\varnothing : (U(\sigma, \sigma'))^n \times M(T) \rightarrow R^{N\hat{T}} \times M(T)$. *Then, for any* $b \in M(T)$, *and* $i \in N$,

(a) $L_T^i(b) = \{{}_T\varnothing^i(u, b) : u \in (U(\sigma, \sigma'))^n\}$ *is a manifold of dimension* $t^i + 1$.

(b) $L_T^i(b) \cap R^{T_0^i}$ is an open set in $R^{T_0^i}$ (where $R^{T_0^i} = \{v \in R^{N\hat{T}}: v_j^l \neq 0$ *if, and only if,* $l = i$ *and* $j \in T_0^i\}$).

Proof. (a) Take any $j \in T_0^i$ (w.l.o.g. $j = 1$) and consider the $(k + 1) \times (k + 1)$ matrix:

$$\begin{array}{|c|c|c|c|c|c|c|}
\hline
\dfrac{\bar{a}_1(\bar{b}_1 - b_1^i)}{(\bar{b}_1)^2} & 0 & 0 & \cdots & 0 & 0 & \dfrac{a_1^i}{\bar{a}_1} - 1 \\
\hline
-\left(\dfrac{\bar{a}_1 b_1^i}{(\bar{b}_1)^2}\right) & 0 & 0 & \cdots & 0 & 0 & \dfrac{a_1^i}{\bar{a}_1} \\
\hline
0 & -\left(\dfrac{\bar{a}_2 b_2^i}{(\bar{b}_2)^2}\right) & 0 & \cdots & 0 & 0 & \dfrac{a_2^i}{\bar{a}_2} \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
0 & 0 & 0 & \cdots & 0 & -\left(\dfrac{\bar{a}_k b_k^i}{(\bar{b}_k)^2}\right) & \dfrac{a_k^i}{\bar{a}_k} \\
\hline
\end{array}$$

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[9] See Appendix.

[10] This sentence is made rigorous in Remark 5.

Since $b \notin S_j{}^i(T)$ for any $j \in T_0{}^i$,

$$\det \begin{vmatrix} \dfrac{\bar{a}_1(\bar{b}_1 - b_1{}^i)}{(\bar{b}_1)^2} & \dfrac{a_1{}^i}{\bar{a}_1} - 1 \\ -\left(\dfrac{\bar{a}_1 b_1{}^i}{(\bar{b}_1)^2}\right) & \dfrac{a_1{}^i}{\bar{a}_1} \end{vmatrix} \neq 0$$

from which it is easily deduced that the rank of the matrix is $t^i + 1$. To prove (a) note (i) the mapping from $(U(\sigma, \sigma'))^n$ to $R^{N\hat{T}}$ given by $_T\varnothing^i(u, b)$ for fixed b is linear, (ii) the image of the mapping is obtained by linear combinations[11] of $(t^i + 1)$ nonzero vectors each with $\hat{t}$ components, (iii) the matrix displayed above (after removal of columns j, $j \notin T_0{}^i$) is a submatrix of the matrix (which, for future reference, will be denoted by $C_T{}^i(b)$) of these vectors.

(b) This is obvious.

Q.E.D.

By Lemma 4, $_T\varnothing((U(\sigma, \sigma'^n)) \times M(T))$ is a manifold $\tilde{M}(T)$ in $R^{N\hat{T}} \times M(T)$ (of dimension $\hat{t} + n + \hat{t} - t_b$). From now on, view the range of the mapping $_T\varnothing$ as $\tilde{M}(T)$.

LEMMA 5. *Consider* $_T\varnothing : (U(\sigma, \sigma'))^n \times M(T) \to \tilde{M}(T)$. *Let* M' *be any submanifold of* $\tilde{M}(T)$. *Then* $_T\varnothing \pitchfork M'$.

Proof. This is along the same lines as the proof of Lemma 2. Let $_T\varnothing(u, b) = (v, b) \in \tilde{M}(T)$. Consider any differentiable path $({}_\tau v, {}_\tau b)$ in $\tilde{M}(T)$ for $\tau \in [0, 1]$, with $({}_0 v, {}_0 b) = (v, b)$. We will show that there is a differentiable path $({}_\tau u, {}_\tau b')$ in [12]$(U(\sigma, \sigma'))^n \times M(T)$ such that $_T\varnothing({}_\tau u, {}_\tau b') = ({}_\tau v, {}_\tau b)$ for $\tau \in [0, 1]$, and $({}_0 u, {}_0 b) = (u, b)$.

Write ${}_\tau v = ({}_\tau v^1, \ldots, {}_\tau v^n)$, ${}_\tau u = ({}_\tau u^1, \ldots, {}_\tau u^n)$. Since each $C_T{}^i({}_\tau b)$ has full rank there exists a unique $(t^i + 1)$-vector, ${}_\tau\alpha^i = \{{}_\tau\alpha_j{}^i : j \in T_0{}^i \text{ and } j = k + 1\}$ such that $C_T{}^i({}_\tau b){}_\tau\alpha^i = {}_\tau v^i$. Now construct the path $({}_\tau u, {}_\tau b')$ as follows:

$${}_\tau b' = {}_\tau b,$$

$${}_\tau u^i(x) = u^i(x) + \sum_{j \in T_i{}^0} {}_\tau\delta_j{}^i x_j + {}_\tau\delta_{k+1}^i x_{k+1},$$

where

$${}_\tau\delta_j{}^i = {}_\tau\alpha_j{}^i - \frac{\partial u^i}{\partial x_j}(x^i({}_\tau b))$$

for $j \in T_0{}^i$ and $j = k + 1$.

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[11] Where the coefficients of the combination are picked from an open set in R^{t^i+1}.

[12] Again, for small enough τ, this path lies in $(U(\sigma, \sigma'))^n \times M(T)$.

It is easily checked that this path has the requisite properties. The rest of the proof is concluded as in Lemma 2. Q.E.D.

LEMMA 6. *For any T there is an open dense set $V(T)$ of $(U(\sigma, \sigma'))^n$ such that for $u \in V(T)$, $\eta(u) \cap \epsilon(u) \cap M(T)$ is empty.*

Proof. Define $M^*(T) = [R^{N\hat{T}} \cap \Delta^1(T) \cap \Delta^2(T)] \times M(T)$. Recall that $\Delta^1(T)$ is a finite union of submanifolds of $R^{N\hat{T}}$, each of which has codimension $\geqslant 1$. Piecing this with the previous lemma, we see that $M^*(T)$ is then a finite union of submanifolds of $\tilde{M}(T)$, say $M_1^*(T), \ldots, M_{r(T)}^*(T)$, with codimension $M_i^*(T) > \hat{t} - t_b$. But Lemma 5 again implies that there exist open dense sets $V_i(T)$ $(i = 1, \ldots, r(T))$ of $(U(\sigma, \sigma'))^n$ such that for any $u \in V_i(T)$, ${}^{u}_{T}\varnothing \pitchfork M_i^*(T)$. But then codim ${}^{u}_{T}\varnothing^{-1}(M_i^*(T)) =$ codim $M_i^*(T) >$ dim $M(T)$; hence ${}^{u}_{T}\varnothing^{-1}(M_i^*(T)) = \varnothing$. On the other hand, using the proposition from [10] $[\eta(u) \cap \epsilon(u) \cap M(T)] \subset {}^{u}_{T}\varnothing^{-1}(M_1^*(T) \cup \cdots \cup M_{r(T)}^*(T))$. Put $V(T) = \cap \{V_i(T) : i = 1, \ldots, r(T)\}$. Q.E.D.

Proof of Auxiliary Theorem. Put

$$E = [\cap\{E(T) : T \in \tilde{V}] \cap [\cap\{V(T) : T \in \tilde{V}, N(T) = N\}].$$

The proof follows from the observation that $\{\eta(u) \cap \tilde{S}(T) : T \in \tilde{V}\}$ constitutes a partition of $\eta(u) \cap S_\mu$. Q.E.D.

4. CONCLUDING REMARKS

Remark 2. Let $\hat{U} = \{[u^1, \ldots, u^n) \in (U(\sigma, \sigma'))^n$: each u^i is concave$\}$. Then it can be shown—see [9]—that $\eta(u) \neq \varnothing$ for any $u \in \hat{U}$. Since $\hat{U}$ contains an open set of $(U(\sigma, \sigma'))^n$—for instance, consider those u^i which are strictly concave—our theorem is not vacuous.

Remark 3. We could have let the space of games be $(U)^n \times (\text{Int}\, \Omega^{k+1})$, allowing for the endowments to vary also. However, it would become necessary to restrict this variation to ensure that the N.E. are bounded away from the zero-price strategies, as in Lemma 1. Thus, for instance, we could state the theorem with $(U(\sigma, \sigma'))^n \times (\hat{Q})^n$, where $\hat{Q}$ is the interior of a cube which itself is in the interior of Ω^{k+1}.

Remark 4. The generic inefficiency of the N.E. attenuates in this model as the player-set is increased in the direction of a nonatomic continuum. For the continuum itself, the N.E.—under appropriate conditions (see [3])—are efficient, and coincide with the Walras Equilibria of the market.

Remark 5. We have been somewhat slipshod in parts of the proof in order to keep the main ideas clear. Now we redress this. Let $\hat{Q}$ be an open neighborhood of Q in R^{k+1} and $\hat{U}$ the manifold of all C^2 functions on $\hat{Q}$ whose restriction to Q is an element of $U(\sigma, \sigma')$. Note that if $\hat{E}$ is open (or dense) in $\hat{Q}$, then E, obtained by restricting the functions in $\hat{E}$ to the domain Q, is open (or dense) in E. This follows from the well-known fact that there is a $K > 0$ such that: each $u \in U(\sigma, \sigma')$ can be extended to a $\hat{u} \in \hat{U}$ with $\| u \| \leqslant K \| \hat{u} \|$. Next let $\hat{S}_\mu$ be an open neighborhood of S_μ in Ω^{nk} such that[13] $\hat{S}_\mu \cap \{(b^1, \ldots, b^n) \in \Omega^{nk} : \bar{b}_j = 0, \text{ some } j \in I_k\} = \varnothing$. $\hat{S}(T) = (\operatorname{Aff} \tilde{S}(T)) \cap \hat{S}_\mu$, where Aff stands for "affine hull." Then define the mappings ${}_T\psi$, ${}_T\tilde{\psi}$, ${}_T\varnothing$ and ${}_T\tilde{\varnothing}$ on $(\hat{U})^n \times \hat{S}(T)$ exactly as before.[14] By adjusting $\hat{Q}$ to be large enough, and $\hat{S}_\mu$ to be small enough, there is no problem of definition. (There would have been a problem if $\hat{S}_\mu$ was not bounded away from points where $\bar{b}_j = 0$. Hence the importance of Lemma 1.)

Lemma 2, in fact, holds—and the proof really shows this—for this bigger mapping ${}_T\varnothing : (\hat{U})^n \times \hat{S}(T) \rightarrow R^{\hat{T}} \times \hat{S}(T)$. Now the proof of Lemma 3 may be reread as follows. First we apply the Transversal Density and Openness Theorems (see Appendix) to this ${}_T\tilde{\varnothing}$ with $A \equiv (\hat{U})^n$, $X \equiv \hat{S}(T)$, $K \equiv$ Closure of $\tilde{S}(T)$, etc. This gives us the open dense set in $(\hat{U})^n$, which in turn induces the desired open dense set $\hat{E}(T)$ in $(U(\sigma, \sigma'))^n$, as explained above. Also $\eta(u) \cap \tilde{S}(T) \subset {}^{u}_{T}\varnothing^{-1}(Z) \cap K$, and the latter set is clearly finite since it is the intersection of a compact set and a 0-dimensional manifold.

Lemmas 4, 5, and 6 can similarly be reread to make the proof rigorous.

APPENDIX

We recall the results from [1] used in this paper.

"Let X and Y be C^1 manifolds, $f: X \rightarrow Y$ a C^1 map, and $W \subset Y$ a submanifold. We say that f is *transversal to W at a point* $x \in X$, in symbols: $f \pitchfork_x W$, iff, where $y = f(x)$, either $y \notin W$ or $y \in W$ and

(1) the inverse image $(T_x f)^{-1}(T_y W)$ splits, and

(2) the image $(T_x f)(T_x)$ contains a closed complement to $T_y W$ in $T_y Y$. We say f is *transversal to W*, in symbols: $f \pitchfork W$, iff $f \pitchfork_x W$ for every $x \in X$.

Let $\mathscr{A}$, X, and Y be C^r manifolds, $\mathscr{C}^r(X, Y)$ the set of C^r maps from X to Y, and $\rho : \mathscr{A} \rightarrow \mathscr{C}^r(X, Y)$ a map. For $a \in \mathscr{A}$ we write ρ_a instead of $\rho(a)$; i.e., $\rho_a : X \rightarrow Y$ is a C^r map. We say ρ is a C^r *representation* iff the evaluation map

$$\operatorname{ev}_\rho : \mathscr{A} \times X \rightarrow Y$$

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[13] This is clearly possible.

[14] I.e., by the same formulas.

given by

$$\mathrm{ev}_\rho(a, x) = \rho_a(x)$$

for $a \in \mathscr{A}$ and $x \in X$ is a C^r map from $\mathscr{A} \times X$ to Y.

TRANSVERSAL DENSITY THEOREM. *Let $\mathscr{A}$, X, Y be C^r manifolds, $\rho : \mathscr{A} \to \mathscr{C}^r(X, Y)$ a C^r representation, $W \subset Y$ a submanifold (not necessarily closed), and $\mathrm{ev}_\rho : \mathscr{A} \times X \to Y$ the evaluation map. Define $\mathscr{A}_W \subset \mathscr{A}$ by*

$$\mathscr{A}_W = \{a \in \mathscr{A} \mid \rho_a \pitchfork W\}.$$

Assume that:

(1) *X has finite dimension n and W has finite codimension q in Y;*

(2) *$\mathscr{A}$ and X are second countable;*

(3) $r > \max(0, n - q)$;

(4) $\mathrm{ev}_\rho \pitchfork W$.

Then $\mathscr{A}_W$ is residual (and hence dense) in $\mathscr{A}$.

Openness of Transversal Intersection. Let $\mathscr{A}$, X, and Y be C^1 manifolds with X finite dimensional, $W \subset Y$ a closed C^1 submanifold, $K \subset X$ a compact subset of X, and $\rho : \mathscr{A} \to C^1(X, Y)$ a C^1 pseudorepresentation. Then the subset $\mathscr{A}_{KW} \subset \mathscr{A}$ defined by

$$\mathscr{A}_{KW} = \{a \in \mathscr{A} \mid \rho_a \pitchfork_x W \quad \text{for} \quad x \in K\}$$

is open. This holds even if X is not finite dimensional, provided that ρ is a C^1 representation."

For our purposes, it is enough to note that every C^1 representation is a C^1 pseudorepresentation. Also T_yW is the tangent space to W at y; $T_xf\colon T_xX \to T_yY$ is the derivative map of f at x. See [1] for detailed definitions.

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