Efficiency Properties of Strategic Market Games: An Axiomatic Approach*

PRADEEP DUBEY

Yale University, New Haven, Connecticut 06520

ANDREU MAS-COLELL

University of California, Berkeley, California 94720

AND

MARTIN SHUBIK

Yale University, New Haven, Connecticut 06520

Received July 23, 1979

The paper investigates the conditions under which an abstractly given market game will have the property that if there is a continuum of traders then every noncooperative equilibrium is Walrasian. In other words, we look for a general axiomatization of Cournot's well-known result. Besides some convexity, continuity, and nondegeneracy hypotheses, the crucial axioms are: anonymity (i.e., the names of traders are irrelevant to the market) and aggregation (i.e., the net trade received by a trader depends only on his own action and the mean action of all traders). It is also shown that the same axioms do not guarantee efficiency if there is only a finite number of traders. Some examples are discussed and a notion of strict noncooperative equilibrium for anonymous games is introduced.

1. INTRODUCTION

In this paper we shall attempt an axiomatization of a central theme of economic theory, namely, the idea that the noncooperative equilibria of

* This paper brings together and supersedes Dubey and Shubik (1978) and Mas-Colell (1978). We are indebted for discussion, suggestions, and amendments to K. Arrow, W. Heller, L. Hurwicz, Ch. Kahn, E. Maskin, A. Postlewaite, and W. Thomson. The usual caveat applies. A. Mas-Colell's research was supported by NSF Grant SOC76-19700A01 and SOC77-06000. The latter grant was at the Institute for Mathematical Studies in the Social Sciences, Stanford University. The work of Dubey and Shubik relates to Department of the Navy Contract N00014-77-C-0518, issued by the Office of Naval Research under contract authority N00047-006.
market-like economies with many relatively small traders are Walrasian (hence efficient) or, in other words, that price-taking behavior is, in a mass market, the natural consequence of "message taking" behavior. The classical partial equilibrium reference is, of course, Cournot's (1838). Two modern general-equilibrium version of Cournot's model are represented by Shubik (1973) [see also Shapley (1976), Shapley and Shubik (1977), Jaynes, Okuno, and Schmeidler (1978)] and by Gabszewicz--Vial (1972) [see also Hart (1979) and Novshek--Sonnenschein (1979)]. The work reported here develops from the Shubik line.

We take as our conceptual starting point the notion of Strategic Market Game, to be understood as a complete specification of trading rules and transaction constraints with respect to which a noncooperative equilibria is reached. This is to be contrasted with the theory of the core of a market, [which originated in Edgeworth (1881); see Hildenbrand (1974) and his references] where final outcomes are completely unconstrained by institutional mechanisms of trade.

The game being played in real economic life is quite complex. It is clear that there is not a unique, somehow given, way to formalize it as a strategic market game. So it is sound research strategy to build a gallery of models emphasizing different important features, their fine structure reflecting the host of institutional features of the economy not usually taken into account by the economic theorist, but of importance in the understanding of marketing and transaction technologies. Nevertheless, experience with different models indicates that a number of facts are quite robust to changes of specification. Among them: the noncooperative equilibria of markets with a finite number of traders tend (this is a suitably vague verb) to be inefficient, and the noncooperative equilibria of markets with many relatively small participants tend to be (almost) efficient. Thus, it seems justified in order to bring out the robustness of these phenomena to engage in axiomatics.

We proceed by sidestepping the modeling problem and assuming (admittedly, this is quite a lot to assume) that a strategic market game is given to us in an abstract manner. We then try to identify general principles yielding the efficiency results. Our axiomatization will be fairly transparent and, from the mathematical point of view, trivial. The basic principles turn out to be:

(i) convexity: traders have available a convex set of strategies.
(ii) anonymity: from the point of view of the market, only the message sent by the trader matters.
(iii) continuity: of outcomes with respect to strategies.
(iv) aggregation: the trading possibilities of any player are influenced by the messages of the other players only through the mean of those messages (and not, say, through the variance).
(v) nondegeneracy: it must be possible for individual players to influence to a substantial extent their trading possibilities in the market.

With the given axioms we first prove some inefficiency results for the finite number of traders case and then establish efficiency for the large number of small traders case. As in Aumann (1964) for "large number of small traders," we shall adopt the idealization of the continuum. We do this merely for convenience and in order to sharpen the essential facts. We will, however, devote one section to the asymptotic, in contrast to limit, theory. It is fortunate that the consideration of a large number of small traders (in the limit a continuum) has simplifying implications for noncooperative analysis. Indeed, under general conditions, if individual players are, in whichever is the relevant sense, very small relative to the size of the game, the only sensible expectations (at "equilibrium" or at "disequilibrium" plays) of individual players is that the rest of the players will not react (or react very little) to changes of their actions. Thus, in contrast to games among few, simple Cournot-Nash equilibrium emerges as the natural noncooperative solution. Our treatment of Cournot-Nash equilibrium in the continuum builds on a paper of Schmeidler (1973) which deserves to be better known.

There are clear similarities and connections between our formal structure and the extensive literature growing out of Hurwicz's (1960) seminal contribution on the designing of resource allocation mechanism. But it is important to keep in mind that the spirit of our work is very different. In particular, we do not attach normative significance to our results. For us, the market mechanism is given, and we merely analyze its properties.

The paper is organized as follows: Sections 2-5 present the basic model. Section 6 enumerates the axioms. Section 7 looks at the continuum as a limit of the finite but large situation. Section 8 discusses a nondegeneracy issue for equilibria. Sections 9 and 10 present efficiency results for the finite and continuum cases, respectively. Section 11 looks at examples; and, finally, Section 12 defines and discusses a type of noncooperative equilibria of special relevance to market situations.

2. Commodities, Agents, Assignments

The commodity space is $R^l$.

We let $I$, the set of agents names, be either a finite set equipped with the counting measure or the interval $[0, 1]$ with Lebesgue measure (denoted $\lambda$). This will allow us to treat simultaneously the finite and the continuum cases.

An assignment is an integrable map $x: I \rightarrow R^l$ such that $\int x(t) \, dt = 0$. Note that we are concerned only with net trades.
3. Economics

We now look at the agents in \( I \) as the traders of an economy. A trader is characterized by:

(i) a closed, convex, nonempty set \( X \subset R^1 \) of net trades, interpreted as the set of contracts the trader can honor, and

(ii) a complete and transitive preferences relation \( \succeq \subset X \times X \). Every \( X \) is assumed to satisfy free disposability, i.e., if \( x \in X \), then \( x + R^1 \subset X \). Preferences \( \succeq \) are taken to be continuous, convex, and monotone. The set of characteristics is \( \mathcal{A} = \{ (X_t, \succeq) \} \). As in Hildenbrand (1974), one can view \( \mathcal{A} \) as a measurable space by endowing it with the \( \sigma \)-field derived from the closed convergence topology on the closed subsets of \( R^1 \).

We define an economy à la Aumann (1963), i.e., as a (measurable) map
\( \delta: I \rightarrow \mathcal{A}. \) We denote \( \delta(t) \) by \( (X_t, \succeq_t) \).

An assignment \( x: I \rightarrow R^1 \) is feasible if \( x(t) \in X_t \) for a.e. \( t \in I \).

The existence of two types of trade constraints is a basic distinction in our treatment. We have the public constraints embodied in the fact that the sum of net trades must be zero (i.e., markets should clear) and the private constraints requiring that individual net trades belong to the individual trading sets. As a matter of interpretation we take that the private constraints are not publicly observable.

A feasible assignment \( x \) is efficient if there is no other feasible assignment \( x' \) such that \( x'(t) \succeq_t x(t) \) for a.e. \( t \in I \) and \( x'(t) \succeq_t x(t) \) for a set of \( t \)'s of positive measure.

A feasible assignment \( x \) is Walrasian if there is \( p \in R^1 \) such that for a.e. \( t \in I \), \( x(t) \succeq_t \maximal_{x \in X_t} \{ x \in X_t : p \cdot x \leq 0 \} \). It is, of course, well known that, with the hypothesis made, a Walrasian assignment is efficient.

Remark 1. Note that (i) individual production (i.e., firms with a single owner) is encompassed by the model, (ii) consumption or technological externalities are not permitted.

4. Strategic Markets Games

We now look at the set of agents \( I \) as a set of players. There is given a nonempty set \( S \) called the states, action, or strategy space. We assume that \( S \) is a subset of a separable Banach space. It will be conceptually simpler if the reader thinks of \( S \) as lying in a finite-dimensional space, i.e., a subset of some \( R^m \).

A play is an integrable function \( s: I \rightarrow S \).
A strategic market game is specified by the set of players $I$, the strategy space $S$ and an outcome rule $\Phi$, which associates with every play $s$ an assignment $\Phi(s): I \rightarrow R^I$. If $\Phi$ is understood, we write $x_s$ for $\Phi(s)$.

For the measure theoretic treatment to be sensible, we should require that the outcome rule satisfies the following property: if $[s] = [s']$, then $[x_s] = [x_{s'}]$. The symbol $[\,\,]$ denotes the equivalence class of the function (i.e., all functions which are a.e. equal to the given function). Informally speaking, this property, which is nonvacuous only for the case $I = [0, 1]$, has the implication that (again for $I = [0, 1]$) individual actions do not have a macroscopic influence on market outcomes (i.e., pecuniary externalities are absent), which is a well-known heuristic requirement for a competitive process to yield efficient outcomes.

Let us illustrate with some examples. They will be discussed with more detail in Section 11, from where, especially for Example 1, a clearer understanding of the underlying economics will be gained.

**Example 1.** [Shubik (1973)]. There are two commodities. The strategy space is $R^2$. A message $(s_1, s_2)$ is to be interpreted as follows: $s_1$ (resp. $s_2$) is the amount of commodity 1 (resp. 2) a trader proposes to deliver to the market. Let $s$ be a play. Suppose first that $\int s_1, \int s_2 > 0$. Then the outcome is given by

$$x_s(t) = \left( \frac{\int s_1}{\int s_2} s_1(t) - s_1(t), \frac{\int s_2}{\int s_1} s_2(t) - s_2(t) \right).$$

If either $\int s_1 = 0$ or $\int s_2 = 0$, we put $x_s = 0$.

In the next two examples the strategies are whole demand-supply functions.

**Example 2.** Again $I = 2$. The strategy set is $S = \{ f \in C([0, r]) : f(0) \geq 0$, $f(r) \leq 0 \}$, where $r$ is some large number and for $p \in [0, r]$ one interprets $f(p)$ as a proposal to buy ("sell" if negative) $f(p)$ units of the second commodity in exchange for $pf(p)$ units of the first commodity. Let $s$ be a play. Define

$$p = \max \left\{ p \in [0, r] : \int s(p) = 0 \right\}.$$

Then the outcome is given by

$$x_s(t) = (-p s(t)(p), s(t)(p)).$$

This example shows that there is substance in the implicit hypothesis that to every play $s$ there corresponds a unique assignment. The rule for choosing one among the possibly many price equilibria is completely arbitrary. In the next example, the equilibrium is naturally unique.
Example 3. Identical to Example 2 except that now \( S = \{ f \in C([0, r]) : f(0) \geq 0, f(r) \leq 0 \text{ and } f \text{ is decreasing} \} \).

Remark 2. In the case where \( S \) is infinite dimensional (which shall not be emphasized), "integrable" is taken to mean Pettis integrable [see, for example, Yosida (1971)].

Remark 3. We could in the definition of a play require simply that \( s \) be measurable, thus allowing for nonintegrable plays. But not much would be gained, since in fact we will have a need to integrate.

Remark 4. In the present approach the outcomes of a play are not necessarily feasible assignments. This is in keeping with the modelling of market institutions as blind to private characteristics. Any player is free to send any message to the market although the outcome of a play can be realized only if almost every trader can honor the proposed contract without going bankrupt. Thus, our strategies are more in the nature of proposals than actual actions, and to that extent the use of the term "game" may be a little abusive. A modeling via Debreu's Generalized Games (1952) would have been possible. The present situation can be contrasted with the well-known price tatonnement. Here public constraints are always satisfied, but private ones need not be. There private constraints do hold, but not necessarily the public ones (i.e., at given arbitrary prices, there may be excess demand or supply).

Remark 5. The concept of strategic market games introduced in this section is formally related to the concept of allocation mechanism growing out of Hurwicz's (1960) approach to the design of resource allocation methods. [See, for example, Hurwicz (1979) and Schmeidler (1978)].

5. Noncooperative Equilibria

For fixed \( I \) let an economy \( \mathcal{E} : I \rightarrow A \) and a strategic market game \( \Phi \) with message space \( S \) be given.

**Definition.** A play \( s \) is a Cournot–Nash (CN) equilibrium if \( x_s \) is feasible and for almost every \( t \in I \), \( x_s(t) \) is \( \succeq t \) maximal on \( X_t \cap \{ x_{s'}(t) : \text{s}'(t') = s(t') \text{ for all } t' \neq t \} \).

The assignment \( x_s \) corresponding to a CN play will be called a CN assignment.

Since we will very quickly specialize our model, we shall not dwell on a discussion of this definition.
Remark 6. For the continuum, i.e., \( I = [0, 1] \), the definition of CN equilibrium has taken inspiration in Schmeidler’s (1973) work on the non-cooperative theory of games with a continuum of players.

6. Axioms

Let the strategic market game \( \mathcal{Q} \) on a set of players \( I \) be given.

We shall restrict our consideration to strategic market games whose outcomes depend only on the strategies played and not on the names of the players. More precisely: (i) if two traders choose the same action, they get assigned the same net trade, and (ii) for a given fixed action of a trader, the net trade assigned depends only on the distribution of the strategies of all traders. Those are particular conditions, but they embody the anonymity principles of well-functioning markets: the trading possibilities of any particular economic agent are limited only by their availability of commodities to trade.

Our wish to impose an anonymity axiom explains two aspects of our treatment of strategic market games: the strategy sets of different players were assumed equal and the outcome of a play was not required to satisfy the private feasibility constraints.

Let \( \mathcal{M} \) be the set of probability measures on \( S \) with finite mean. To every \( s : I \to S \) there corresponds an \( \nu_s = \lambda \circ s^{-1} \in \mathcal{M} \).

**Anonymity Axiom.** There is a function \( G : S \times \mathcal{M} \to R^I \) such that, for all plays \( s \), \( x_s(t) = G(s(t), \nu_s) \) for all \( t \in I \).

Clearly, for \( I \) finite, the function \( G \) is not unique, while for \( I = [0, 1] \), it is trivially so.

Under the Anonymity Axiom, a play \( s : I \to S \) is a CN equilibrium if \( G(\cdot, \nu_s) : I \to R^I \) is a feasible assignment and, for a.e. \( t \in I \), \( G(s(t), \nu_s) \) is \( \succeq_t \) maximal on \( X_t \cap \{G(s, \nu_s) : s \in S\} \).

A second, important axiom is:

**Continuity Axiom.** The function \( G \) given by the Anonymity Axiom is jointly continuous when \( \mathcal{M} \) is endowed with the topology of the weak convergence [see (Hildenbrand, 1974, p. 48), for a definition].

Under the continuity axiom the uniqueness of \( G \) in the continuum case (i.e., \( I = [0, 1] \)) can be sharpened to: if for all \( \nu \in \mathcal{M} \), \( G \) and \( G' \) are continuous and coincide on \( \text{supp}(\nu) \), then \( G = G' \).

A third axiom will be:

**Convexity Axiom.** The strategy set \( S \) is convex.

Given the previous axioms, our key hypothesis is:
Aggregation Axiom. For any \( v, v' \in \mathcal{M} \) if \( \int idv = \int idv' \), then \( G(, v) = G(, v') \). The symbol \( i \) denotes the identity map.

In words: the net trade proposed to a player by the strategic market game does only depend on the message sent by this player and the mean message of all players. Particular as this condition is, we would argue that it reflects well the aggregate nature of the impact of demand and supply in organized markets. To be sure, it is quite possible that an economic trading situation may be most succinctly and naturally described in a way which does not satisfy the Aggregation Axiom. The point is, however, whether or not it can be described in some way for which it does, even if it perhaps involves a much expanded, but still convex, strategy space. For example, a typical competitive problem among firms may involve as individual strategies, price and quality (suppose there are only \( m \) possible qualities). The concept of mean quality is then irrelevant to the equilibrium problem. It may not even be defined. So, the Aggregation Axiom is not satisfied. On the other hand, if we view as the set of messages the much larger set of supply (and demand) functions for the \( m \) quality goods (with arguments the \( m \) vector of prices), then the Aggregation Axiom will hold.

We will postpone a more detailed discussion of the role of the different axioms until after the statement of the efficiency results in Section 10.

If the four axioms of this section are satisfied, it will be simpler to describe the strategic market game \( \Phi \) by the continuous function \( F : S \times S \to \mathbb{R}^l \) such that \( G(s, v) = F(s, \int idv) \).

7. The Continuum as a Model for a Large Number of Participants

Suppose we have a continuum of traders \( I = [0, 1] \), an economy \( \mathcal{E} : I \to \mathcal{A} \) and a strategic market game \( \Phi \), with player set \( I \), which satisfies the Anonymity, Continuity, Convexity, and Aggregation Axioms of the previous section, i.e., the strategic market game can simply be defined by a continuous function \( F : S \times S \to \mathbb{R}^l \). In which sense does this amount to a model for an economy and strategic market game with a large number of participants? We address this question now. Given the limited nature of the strategic games we deal with (anonymous, aggregable), this is not a difficult problem. Even so we shall not attempt to obtain the most general possible results.

Suppose we are given a sequence of trader sets \( I_n, \#I_n < \infty \), economies \( \mathcal{E}_n : I_n \to \mathcal{A} \), and strategic market games \( \Phi_n \) with player set \( I_n \) and a common convex strategy space \( S \). We assume that \( \#I_n \to \infty \).

As in Hildenbrand (1974), we say that \( \mathcal{E}_n \to \mathcal{E} \) if the distribution of characteristics induced by \( \mathcal{E}_n \) converges weakly to the distribution of characteristics of \( \mathcal{E} \).

For the sequence of strategic market games we assume: (i) there is a
function $H : S \times \hat{S} \to R^1$, where $\hat{S}$ is the cone spanned by $S$, such that for all $n$, plays $s : I_n \to S$ and $t \in I_n$, $\Phi_n(s)(t) = H(s(t), \sum_{\text{int} s(t))}$ and (ii) the function $H$ is homogeneous of degree zero in its second arguments, i.e., $H(s, \lambda b) = H(s, b)$ for all $\lambda > 0$. This could be called the Homogeneity Axiom.

We say that $\mathcal{E}_n, \Phi_n$ converges to $\mathcal{E}, \Phi$ if (i) $\mathcal{E}_n \to \mathcal{E}$ and

(ii) $F = H | S \times S$. Since $H, F$ remain fixed, we identify $(\mathcal{E}_n, \Phi_n)$ with $\mathcal{E}_n$.

A comment to this definition may be useful. The requirement that $H$ be common to all terms of the sequence is in order since we want to formalize the idea that the number of participants in some fixed market institutions increases. For the same reason, i.e., the market institutions do not depend on the number of participants, the function $H$ is given in an normalized way. Thus the homogeneity axiom emerges as fundamental since it is this axiom which allows us to replace sums by averages and therefore to define a meaningful continuum model embodying the notion of the negligibility of individual participants. The sequences generated as in Examples 1–3 satisfy the axiom.

If assertions about CN equilibria in the continuum are to be relevant, then it should at least be true that the “limits” of CN equilibria for $\mathcal{E}$ are CN equilibria for the limit $\mathcal{E}$. But what is a limit of CN equilibria? We borrow the notion of equilibrium distribution introduced by Hart, Hildenbrand, and Kohlberg (1974). Given a (Borel) distribution $\mu$ on $\mathcal{A} \times \mathcal{S}$, let $\mu_d, \mu_s$ be the respective marginals. We say that $\mu$ is a CN distribution for $\mathcal{E}_n$ (resp. $\mathcal{E}$) if (i) $\mu_d$ is the characteristic distribution of $\mathcal{E}_n$ (resp. $\mathcal{E}$) and (ii) for all $(X, \succeq, s) \in \text{supp}(\mu)$, $F(s, \int \mu_d)$ is $\succeq$-maximal on $X \cap \{F(s', \int \mu_d) : \forall s' \in S\}$, where $i$ is the identity map on $S$ and $\mu'$ is the distribution obtained by transferring a total mass of $1/n$ from $(X, \succeq, s)$ to $(X, \succeq, s')$ (resp. $\mu = \mu'$). Note that the concept of CN distribution captures everything essential in the definition of CN equilibria in the sense that if two CN equilibria have the same distribution, then they are identical up to a relabeling of the traders' names.

Then we have:

**Proposition 1.** Assume that for all $n$ and $t \in I_n$, $X_t = R^1$. Let $\mu_n$ be a CN distribution for $\mathcal{E}_n$ and $\mu_n \to \mu$ weakly. Then $\mu$ is a CN distribution for $\mathcal{E}$.

**Proof.** If $\mu_n \to \mu$, then $\mu_{n,d} \to \mu_d$ and $\mu_{n,s} \to \mu_s$. So the first condition in the definition of CN distribution is satisfied for $\mu$. Now let $(\succeq, s) \in \text{supp}(\mu)$ (we can forget about $X$ as $X = R^1$). Take $(\succeq, s) \to (\succeq, s)$, $(\succeq, s) \in \text{supp}(\mu_n)$. Call $x = F(s, \int \mu_d)$ and for arbitrary $s' \in S$ let $y = F(s', \int \mu_d)$. By the continuity of $F$, $x_n \to x$ and $y_n \to y$, where $x_n = F(s_n, \int \mu_d)$ and
Proposition 1 is a closed graph property for the CN equilibrium correspondence (defined from distribution on $\mathcal{A}$ to distribution on $\mathcal{A} \times S$). It is well known that in order to prove results of this sort it is essential that constraint sets correspondences be continuous. To guarantee this we take $X_t \equiv R^2$ in Proposition 1. Of course, more general conditions could be given.

A stronger result than Proposition 1 would be the upper hemicontinuity of the equilibrium correspondence. Only then can we assert that the equilibria of large economies are near the equilibria of the continuum. Upper hemicontinuity obtains under the hypothesis of Proposition 1 and, for example, the very strong assumption that $S$ be compact (this is trivial to prove). If upper hemicontinuity fails (as it may well be the case in the context, for example, of an Example 3), then the behavior of the equilibria in arbitrarily large economies may differ markedly from the continuum situation. See Thompson (1978) for examples where precisely this happens. This is an important point which does surely deserve closer investigation.

8. PROPER AND FULL COURNOT–NASH EQUILIBRIA

Our aim is to establish efficiency properties for CN equilibria. It is clear that, informally speaking, a necessary condition for efficiency is that individual net trades be responsive enough to individual actions. Indeed, if all markets are closed, every play is a CN equilibrium. It will be useful to distinguish two types of not totally unrelated local rigidities. The first would arise if in some markets there were quantitative limits on trades and some trades had reached them. The second would arise if some markets were effectively closed at the CN equilibrium. This motivates the following definitions.

Let the agents set $I$, economy $\mathcal{A}$ and strategic market game $\mathcal{D}$ be given. Suppose that $s$ is a CN equilibrium. For every $t$ let $B_t = \{s'(t') : s'(t') = s(t') \text{ for all } t' \neq t\}$.

We say that the CN equilibrium $s$ is proper if for some $n \leq l$ and a.e. $t \in I$, a neighborhood of $x_t(t)$ on $B_t$ is homeomorphic to $R^n$. If we can take $n \geq l - 1$, we say that the equilibrium is full.

Note that the notion of proper equilibrium is compatible with traders being at the boundary of their private trading sets.

We shall see in Section 11 that it is quite possible, and even typical, that for the same economy a strategic market game exhibit both full and nonfull CN equilibria. Improper equilibria would appear to be more pathological $n$ "flexible prices" models, but they would be typical in rationing models.
9. ON THE EFFICIENCY OF THE CN NET TRADE ALLOCATIONS IN THE FINITE NUMBER OF TRADERS CASE

Let $F : S \times S \rightarrow R^l$ represent a strategic market game satisfying the four axioms of Section 7. It is fairly clear from the usual examples that if $F$ acts on an economy with a finite number of traders, there is no reason to expect the CN equilibrium allocations to be efficient. We shall here illustrate this point by proving, under a number of simplifying but reasonable hypotheses, that given $F$ and $n \geq 2$, there are economies with $n$ traders exhibiting nonefficient full CN equilibria (which, further, cannot be perturbed away).

Let $I$ be a traders index set with $\# I = n$. Given a play $s : I \rightarrow S$ and $t \in I$, it is convenient to define: $s_t = \frac{1}{n} (n - 1) \sum_{t' \neq t} s_{t'}$. Let also $H : S \times S \rightarrow R^l$ be given by $H(s, s') = F(s, (1/n)((n - 1) s' + s))$. Then $B_s = H(S, s')$ is the public budget of any $t \in I$ at any play $s$ with $s' = \frac{1}{n} (n - 1) \sum_{t' \neq t} s_{t'}$.

We shall make the following hypothesis:

(I) $S$ is open and $H$ is $C^1$.

(II) For all $s$, $s'$, $D_s H(s, s') = I - 1$ and $R^1_s \cap \text{Range} D_s H(s, s') \neq \emptyset$.

(III) For some $s$ and $s'$, $D_s s' H(s, s') = I$.

These are very reasonable hypotheses. Assumption (II) asserts that it is not possible by an infinitesimal displacement to get something from nothing, but that, subject to this fact, $D_s H(s, s')$ is maximal, i.e., $I - 1$. This is a clear requirement if there is to be any hope that the allocations induced by the CN equilibria of the strategic market game be efficient. Assumption (III) asserts that at some combination of actions the strategic market game allows, if cooperation prevails, for (local) unrestricted variation of an individual allocation (i.e., cooperation makes it possible for some individual to get something from nothing). It is satisfied if, for example, $H(S \times S)$ contains an open set.

PROPOSITION 2. If $H$ satisfies (I), (II), and (III), there is an economy $\delta : I \rightarrow \Delta$ and a play $s : I \rightarrow S$ such that:

(i) $s$ is a CN equilibrium,

(ii) the net trade allocation of $s$ is not efficient.

Proof. Let $s$ and $s'$ be as in Assumption (III). With $I = \{1, \ldots, n\}$, we consider the play $s(1) = s, s(j) = s'$ for $2 \leq j \leq n$. We let $s_i = s(i), s_1 = s_1$. By (II), for every $i$ there is a unique $p_i \geq 0$, $\|p_i\| = 1$, such that $p_i \cdot D_s H(s_1, s_i) = 0$. Let $x : I \rightarrow R^l$ be the net trade induced by $s$. If $x$ is to be
efficient for any economy which satisfies our hypothesis and makes \( s \) into a CN equilibrium, then \( p_i \) must be the same for all \( j \). Indeed, for every \( i \), we can take \( X_i = R^1 \) and choose monotone preferences \( \succeq_i \) representable by a \( C^0 \) utility function and such that \( x(i) \) is \( \succeq_i \) maximal on \( B_{s'} \). By Assumption (I) this can be done (it is easy to verify that at any \( s' \) and \( x \in B_{s'} \), \( B_{s'} \cap (\{x\} + R_{s'}) = \{x\} \). See Fig. 1), and it makes \( s \) into a CN equilibrium. By

the usual efficiency conditions, if \( x \) is efficient, then the normalized gradients of utility functions, which equal the \( p_i \)'s must be the same.

So, let \( p_1 = \cdots = p_n = p \). Differentiating \( \sum_{i=1}^{n} H(s_i, s'_i) \) with respect to \( s'_i \), we get \( D_s H(s_i, s'_i) + \sum_{i' \neq i} D_{s'} H(s_i, s'_i) = 0 \) for all \( j \). Therefore, \( \sum_{i \neq j} p \cdot D_s H(s_i, s'_i) = 0 \) for all \( j \). As \( p \cdot D_{s'} H(s_i, s'_i) - p \cdot D_{s'} H(s_i, s'_j) = \sum_{i \neq j} p \cdot D_{s'} H(s_i, s'_i) - \sum_{i \neq j} p \cdot D_{s'} H(s_i, s'_j) = 0 \) for all \( j, j' \), we conclude that \( p \cdot D_{s'} H(s_i, s'_i) = 0 \) for all \( i \). In particular, \( p \cdot D_{s'} H(s_i, s'_i) = p \cdot D_{s'} H(s, s') = 0 \). But then \( p \cdot D_{s} H(s, s') = 0 \), which implies rank \( D_{s, s'} H(s, s') \leq l - 1 \) and contradicts Assumption (III). Therefore not all \( p_i \)'s are the same, and we are done. 

Remark 7. Proposition 2 is a weak result as it asserts only the existence of an economy exhibiting a nonefficient CN equilibrium. We would surmise that the general picture is as follows:

Postulate Hypothesis (II) for all \( s, s' \). Then restricting ourselves to the class of smooth economies, it would be the case that, but for some exceptional economies (i.e., "generically"), the set of efficient net allocations is a codimension one smooth manifold in the space of net trades allocations and the set of CN allocations induced by the strategic market game is also a smooth manifold. Further both manifolds intersect transversally. This means that
if a particular noncooperative theory of market allocation is deterministic in spirit (i.e., generically the set of CN equilibria is finite), then but for coincidental cases, no CN allocation will be efficient. This we have verified for the case \( \dim S = l - 1 \) [see Dubey (1979a), where a particular model is thoroughly analyzed]. More generally, the efficient CN allocations will form a lower-dimensional submanifold of the CN manifold. The investigation of this topic, which does require the methods of differential topology, we have not pursued at much length although we think its clarification is of importance. For a study of the generic properties of CN equilibria in a purely game theoretic context, see Dubey (1978).

Remark 8. Without the Continuity Axiom the conclusions of Proposition 2 may well fail [see, for example, Dubey (1979b) and Schmeidler (1978)]. The same is true for the Aggregation Axiom [see Hurwicz (1979) model, where the CN equilibria are efficient, but outcomes depend on the squares of the messages].

10. **On the Efficiency of the CN Net Trade Allocations in the Continuum of Traders Case**

We now take up an examination of efficiency of CN allocations in the continuum of traders-players situation. We will be led to qualitatively different results from those in the finite case.

From now on \( I = [0, 1] \), and we have given a strategic market game satisfying the Anonymity, Continuity, Convexity, and Aggregation Axioms. So, the mechanism is summarized by function \( F: S \times S \rightarrow \mathbb{R}^l \). After stating our main results, we will discuss the role of each of the axioms.

The first obvious consequence of being in the continuum of agents is that all traders face the same public budget set, as no single trader can affect the average action and anonymity holds. For every \( s' \in S \) let \( B_{s'} = F(S, s') \). Then given a play \( s: I \rightarrow S \), the common public budget set is \( B_{\bar{s}} \) where \( \bar{s} = \int s \). Of course, a play \( s \) is a CN equilibrium if for a.e. \( t \in I \) \( F(s(t), \bar{s}) \) is \( \succeq_t \) maximal on \( B_{\bar{s}} \cap X_t \).

The second and fairly obvious consequence of being in the continuum of agents is that for every \( s' \in S \), \( B_{s'} \) is in fact a convex set. Aggregation is the crucial axiom here.

**Proposition 3.** Under the Anonymity, Continuity, Convexity, and Aggregation Axioms, \( B_{s'} \) is a convex set for every \( s' \).

**Proof.** Let \( s' = c \) be given. By the continuity axiom, we can assume without loss of generality that \( c \) belongs to the relative interior of \( S \). We proceed to show that \( F(\cdot, c): S \rightarrow \mathbb{R}^l \) is then linear. Let \( f_c: S \rightarrow \mathbb{R} \) be any
of the coordinate functions of \( F(s, c) \). We know that if \( s \) is a play with \( \int s = c \), then \( \int (f_s \circ s) \, dt = 0 \). If \( f_s \), which is continuous, is not linear, then an easy argument shows the existence of \( s_1, s_2 \in S \) and \( 0 < \alpha < 1 \) such that

\[ \alpha f_s(s_1) + (1 - \alpha) f_s(s_2) \neq f_s(\alpha s_1 + (1 - \alpha) s_2). \]

Let \( c : I \to S \) be a play with \( \int c = c \) and such \( \lambda(c^{-1}(s_1)) > 0 \), \( \lambda(c^{-1}(s_2)) > 0 \). Such a \( c \) exists because \( c \) belongs to the relative interior of \( S \). Now let \( c' \) be as \( c \) except that for a small \( \epsilon > 0 \) an amount of mass equal to \( \alpha \epsilon \) (resp. \( (1 - \alpha) \epsilon \)) is transferred from \( \{s_1\} \) (resp. \( \{s_2\} \)) to \( \alpha s_1 + (1 - \alpha) s_2 \). We still have \( \int c' = c \). Therefore, \( \int (f_s \circ c') \, dt \neq \int (f_s \circ c) \, dt \). But this is impossible because \( \int (f_s \circ c) \, dt = 0 = \int (f_s \circ c') \, dt \).

Hence, \( F(s, c) \) is linear and since \( S \) is convex, \( B_c = F(S, c) \) is convex.

For all \( s' \) we also have \( 0 \in B_{s'} \), since the constant play \( s'(t) = s' \) should yield, by anonymity and aggregation, the null set trade. Let \( L_{s'} \) be the linear subspace spanned by \( B_{s'} \). Then we have:

**Proposition 4.** Under the Anonymity, Continuity, Convexity, and Aggregation Axioms, if \( s : I \to S \) is a proper CN play, then for a.e. \( t \), \( F(s(t), s') \) is \( \succeq_t \) maximal on \( L_{s'} \cap X_t \), where \( s' = \int s \).

**Proof.** The proof is trivial. Since \( s \) is a CN equilibrium, it is true that for a.e. \( t \), \( F(s(t), s') \) is \( \succeq_t \)-maximal on \( B_{s'} \cap X_t \). As the CN equilibrium is proper and \( B_{s'} \) is a convex set containing \( 0 \), we have that for a.e. \( t \in I \), \( F(s(t), s') \) belongs to the relative interior of \( B_{s'} \). Now suppose that for some of those \( t \) we had \( x \in L_{s'} \cap X_t \), \( x >_t F(s(t), s') \). As preferences are convex, we could take \( x \) in \( B_{s'} \), but this is impossible. Hence the proposition is proved.

Proposition 4 could be interpreted as saying that proper CN equilibria yield Walrasian allocations relative to the set of markets which, given the equilibrium strategies, are open. This is particularly clear if \( L_{s'} \) belongs to a coordinate subspace. It can well happen (see Section 11) that the set of open markets is different at different CN equilibria.

One could ask about the efficiency properties of proper CN equilibria. Taking a cue from the previous paragraph, one could assert the efficiency of the equilibrium allocations relative to a properly defined subset of net trades allocations. It is not clear, however, if this would amount to much, as the really interesting property would be the efficiency of the CN net trade allocation relative to the net trade allocations which are feasible within the given strategic market game. Unfortunately, it is easy to convince oneself that proper CN equilibria are not necessarily efficient in this sense.

So, we are left with the following corollary of Proposition 5, which, being the main result of this paper, we call a theorem:
Theorem 5. Under the Anonymity, Continuity, Convexity, and Aggregation Axioms, every full CN equilibrium allocation is Walrasian, hence efficient.

Remark 9. Let’s restrict ourselves to strategic market games on \( I = [0, 1] \) satisfying the Anonymity and Continuity Axioms. It will be convenient and admissible to identify a game with the continuous function \( G : S \times \mathcal{M} \to R^1 \). As we are in the continuum case, the public budget set is common to all traders. So, if the game \( G \) is understood, we let \( B_v = G(S, v) \) for any \( v \in \mathcal{M} \).

We show now how given any interval \( T \subset R \), continuous function \( g : T \to R \), and nondegenerate probability measure \( \tilde{v} \in \mathcal{M} \) satisfying \( \int \tilde{v} \, dv = 0 \) and \( \int g \tilde{v} \, dv = 0 \), we can construct an anonymous strategic market game \( G \) for a two-commodity world with \( I = [0, 1] \) and \( B_v = \text{Graph } g \).

Indeed, it suffices to define

\[
G(s, v) = \left( \int s \, dv \right) \int s \, dv \, d\tilde{v} \times \left( s - \frac{1}{\int s \, dv} \int s \, dv, g(s) - \frac{1}{\int s \, dv} \int g \, dv \right).
\]

If \( \int s \, dv = 0 \) put \( G(s, v) = 0 \). The Continuity and Convexity Axioms are satisfied, but the Aggregation one is not. Also, if \( g(0) = 0 \), then \( 0 \in B_v \) for all \( v \in \mathcal{M} \). Obviously, this \( G \) does not have any particular economic interpretation, which is not surprising, given the failure of aggregation.

It is therefore possible to have strategic market games satisfying the Anonymity, Continuity and Convexity Axioms and generating CN equilibria as the ones depicted in Figs. 2a, b, c, d. In every case, the economy is formed by two equal weight types with \( X = \mathbb{R}^2 \) and the indicated preferences. The play distribution is in all cases \( \nu(x_1) = \nu(x_2) = \frac{1}{2} \).

Figure 2a exhibits an improper equilibria. Figure 2b gives a proper, but not full equilibria. Both equilibria are inefficient. It is obvious that both pictures could be sustained by strategic market games satisfying the Aggregation Axiom. Figures 2c and 2d provide examples of full equilibria failing to be efficient on account of the lack of aggregation.

 Remark 10. It is important to notice the interplay between the Aggregation axiom and the convexity hypothesis on the strategy space \( S \). It is the latter which gives force to the first. Indeed, it is always formally possible, by expanding \( S \), to satisfy approximately the aggregation axiom. But the expanded \( S \) may fail to be convex. For example, if \( S = [0, 1] \) and the outcome net trades depend on plays through the mean and the variance of the message distribution, we could extend the message space to \( S' = (s, s') : s \in [0, 1] \cap [0, 1]^\mathbb{F} \). Then, the Aggregation Axiom is satisfied, but the Convexity Axiom is not.
Remark 11. Under the hypothesis made, the Theorem concludes that CN allocations are not just efficient, but in fact Walrasian. Essentially, this is a consequence of the Anonymity Axiom. Under quite general circumstances, it is the case that if the allocation derived from an anonymous continuum strategic market game is efficient then it is Walrasian. See Hammond (1979) for this point. Ch. Kahn (1979) has succeeded in formulating and proving an analog of the Theorem without the Anonymity Axiom. He gets that CN allocations are still efficient but not necessarily Walrasian. A common price system and linear budget sets still obtain, but lump-sum transfers at equilibrium are possible.

Remark 12. The Continuity Axiom is troublesome. It is basic to our approach since, as we saw in Section 7, we need something like it if the continuum model is to be of any use as a representation of a finite but large situation. But there is no denying that there are sensible economic models for which this axiom poses problems. As an instance, the outcome rule may be naturally multivalued (as in Example 2; more generally, this will tend to be the case if the outcome is itself the result of some equilibrium process). Or think of price competition along strict Bertrand lines, or on retaliation strategies, etc. See Green (1979) for a discussion and analysis of the continuity problem in a different (dynamic) context.

At any rate, if continuity fails the properties of equilibria for both the finite and the finite but large models may be very different from the conclusions of this paper. We may have models where equilibria are always efficient, even in the finite case (Bertrand-like models, for example) and models where there is no tendency for the equilibria to become efficient even if the economy is very large [see Green (1979)].

Once we place ourselves in the continuum, the Continuity Axiom plays
only a minor role in the proofs of efficiency of a CN play $s$. It is obvious from the proofs that it could be replaced by either of the hypotheses ‘$s \in \text{Int} \, S$’ or ‘$F$ is continuous on $S \times \{s\}$’.

Remark 13. It is easy to convince oneself that we cannot dispense with the convexity of preferences. It is this hypothesis which, for each trader, makes a local optimum (and this is all we can prove for a CN equilibrium with our axioms) into a global maximizer in the budget set.

11. Some Examples

We shall now examine some specific examples in the direction of the ones introduced by Shubik (1973). See also Shapley (1976) and Shapley and Shubik (1977).

The space of traders is $I$, finite or infinite.

As a first step, let there be only two commodities, $h = 1, 2$.

The simplest model calls for strategies which are nothing more than the single act of sending messages promising delivery of a certain amount of goods to either side of a single market, i.e., $S \rightarrow \mathbb{R}^2$ where by $(s_1, s_2) \in S$ we understand a commitment to supply $s_h$, $h = 1, 2$, of good $h$.

Let $s : I \rightarrow S$ be a play. Which rules should we specify for the disbursement of $\bar{s}_1 = \int s_1, \bar{s}_2 = \int s_2$? Or, in other words, which conditions shall we impose on the outcome $x : I \rightarrow \mathbb{R}^2$ of the play? Consider the following three (postulated to be valid for every possible play and players space):

(i) Suppose we create a fictitious player space $[0, 2]$—endowed with Lebesgue measure—and a play $\hat{s} : [0, 2] \rightarrow S$ defined by letting $\hat{s}(r) = (s_1(r), 0)$ if $r \leq 1$ and $\hat{s}(r) = (0, s_2(r - 1))$ if $1 \leq r \leq 2$. Let $\hat{x}$ be the outcome of this play. Then $x(t) = \hat{x}(t) + \hat{x}(t + 1)$. In words, there is complete separation of buying and selling and we can, without loss of generality, assume that $I = I_1 \cup I_2$ where

$$I_1 = \{t \in I : s_h(t) = 0\}, I_2 = \{t \in I : s_h(t) = 0\}.$$

(ii) If $t \in I_h$, $h = 1, 2$, then $x_h(t) = -s_h(t)$, i.e., the supply of a commodity offered to the market is always taken.

(iii) With the convention that $0/0$ qualifies as any number, the ratio $x_1(t)/x_2(t)$ (resp. $x_2(t)/x_1(t)$) is independent of $t \in I_2$ (resp. $t \in I_1$). This can be interpreted as a kind of anonymity cum arbitrage-freeness.

Then, of course, the outcome is

$$x(t) = \left(\frac{s_1(t)}{s_2(t)}, -s_1(t) - s_2(t), s_1(t) - s_2(t)\right).$$
If $\delta_2 = 0$ (resp. $\delta_1 = 0$) we put the first (resp. the second) entry equal to $-s_1(t)$ (resp. $-s_2(t)$). This is nothing but Example 1 in Section 4 (with an inessential difference: to enforce that $x$ be always an allocation, we required there that $x = 0$ whenever $\delta_1 = 0$ or $\delta_2 = 0$ while here if, say $\delta_1 \neq 0$ and $\delta_2 = 0$, $x$ will not be an allocation, since $\delta_1$ is not redistributed to anyone). Note that the Anonymity, Convexity, and Aggregation Axioms are satisfied while Continuity may fail only at the point $\delta_1 = \delta_2 = 0$. It is Condition (iii) which yields anonymity and aggregation. Alternatively, we could have assumed the latter and derive the former. For example, if there was a function $F : S \times S \rightarrow R$ such that $x(t) = F(s(t), \tilde{s})$ for every possible play and players space, then, by Theorem 5, $F$ will have to be linear on the first argument and we recover Condition (iii).

A multicommodity extension of the previous example is readily available. Let there be $l$ commodities. We may define a simple market as one in which quantities of a single commodity $i$ are exchanged for another single commodity $j$. A market structure composed of simple markets can be represented by a graph where the points represent commodities and the edges simple markets. See Fig. 3. We could call the edges of the graph open markets (and the nonexisting edges closed markets). A market structure is complete if its graph is connected. A commodity is a money if in the market structure there is an edge to every other commodity. So, every market structure with a money is complete. In Fig. 3, Structure 3a is not complete, while the rest are. Structures 3c, 3d, 3e have, respectively, 1, 2, and 4 money commodities.

Given a market structure $M$ we could assume, to focus on the simplest case, that every market functions as the simple market previously discussed. Then the strategy space is $S = R^2_{\leq M}$ where for every $k \in M$ $s_{kb}, s_{ka}$ represent the quantities offered to the market $k$ of the two commodities transacted in that market. For a given play $s$ we could let $M_s \subset M$ be the active markets.
at \( s \), i.e., \( M_s = \{ k \in M : \int s_{ik} \geq 0 \} \) and \( S_s = \mathbb{R}^{M_s} \) the "active" strategy set.

Suppose \( s \) is a CN play. If we forget about nonactive strategies, i.e., imagine that our strategy set is \( S_s \) then each one of our axioms is satisfied (see end of Remark 12). So, the efficiency Theorem 5 applies if the equilibrium is full (note that the equilibrium is always proper). Now, the equilibrium will be full if and only if the market structure formed by the active markets \( M_s \) is complete. For this type of model an inactive market is no better than a closed one.

The reader familiar with Shubik's original model (1973) and with the multicommodity extensions due to Shapley (1976), Jaynes, Okuno and Schmeidler (1976), Postlewaite and Schmeidler (1978), Dubey and Shapley (1977), may wonder why we get efficiency of equilibria in the continuum case merely out of the connectedness of active markets when this was not at all the case in the previous references. The reason lies in the feasibility constraint we are presently using. We completely abstract from transactions problems and for equilibrium we only require the net trade to be individually feasible. So, it is possible to use the receipts in one market to buy in another market. If we visualize our simple markets as taking place simultaneously, this amounts to the absence of a liquidity constraint, i.e., to the presence of a perfect credit system.

Another observation is that the previous model will typically have multiplicity of equilibria. In fact if an existence theorem is available (it is, see references of the previous paragraph), we will have an equilibrium for every a priori specification of inactive markets.

The market strategies so far considered are rather extreme. They are commitments to supply a given amount of commodity \( i \) to the market in exchange for any quantity of commodity \( j \). More generally, we could envision a supply message (interpretable, perhaps, as a decision rule) to a simple market as being a function \( s_i(x_i) \) to be read as a promise to deliver \( s_i(x_i) \) units of commodity \( i \) in exchange for \( x_j \) units of commodity \( j \). In line with Conditions (i), (ii), and (iii) for the simpler case, we could require of market clearing and disbursal rules that at the transactions outcome of a play we had that for all \( i \) and \( j \), \( s_i(x_i); x_j \) be independent of the particular supplier. Of course, this is nothing but market clearing at a price system, and if we further constrain every \( s_i \) to be nondecreasing and concave, we put ourselves in the anonymous and aggregate situation of Example 3 in Section 4. Indeed, it suffices to redefine a selling strategy as a 0-homogeneous function \( \delta_i(p_i, p_j) \) promising to deliver \( s_i(p_i, p_j) \) units of commodity \( i \) in exchange for \( p_i s_i(p_i, p_j) \) units of commodity \( j \). Clearly, every \( s_i \) strategy induces a \( \delta_i \) strategy and equilibrium prices are determined by the condition

\[ p_i \int \delta_i(p_i, p_j) = p_j \int \delta_j(p_i, p_j). \]

See Fig. 4.
The example at the beginning of this section is a limit (the Cournot case) of this latter model. It corresponds to the case \( s_i(x_i) = c \). The other limit, \( s_i(x_i) \) linear homogeneous, is the Bertrand case. Since outcomes are in this case highly discontinuous on strategies, it is not easily amenable to our analysis.

12. On the concept of strict Cournot-Nash equilibrium in continuum economies

Suppose that for \( I = [0, 1] \) we have given an economy \( \mathcal{E} : I \to \mathcal{A} \) and an anonymous trading game \( G : S \times \mathcal{M} \to \mathbb{R}^I \). From a noncooperative game theoretic standpoint, a CN play \( s : I \to S \) will be rather fragile if there were traders (strictly speaking, a nonnull set of traders) that could obtain a better net trade by pretending to be \( n \) different players (\( n \) finite but otherwise arbitrary). Indeed, in an anonymous mass market it may not be possible to prevent an individual trader from entering it several times or simply, using proxies. This observation prompts the following

**Definition.** A play \( s : I \to S \) is a strict Cournot-Nash equilibrium if \( x_s \) is feasible and for a.e. \( t \in [0, 1] \) \( G(s(t), v_s) \) is \( \succ_t \) maximal on \( X_t \cap \bigcup_{n=1}^\infty nB_{x_s} \).
From Proposition 4 it is clear that

**Proposition 6.** If the Anonymity, Continuity, Convexity, and Aggregation Axioms are satisfied, then every proper CN equilibrium is strict.

Without the Aggregation Axiom a proper CN equilibrium needs not to be strict. For example if at a CN distribution we have that $B_0 \cap R^+_{++} \neq \emptyset$ (as in the example of Fig. 2c) then by the monotonicity of preferences, the distribution cannot be a strict CN.

Without the Aggregation Axiom, is every strict Cournot–Nash equilibrium Walrasian? Not necessarily. In Fig. 5 we can see an example where the CN equilibria are proper and full but not Walrasian (the two types indicated have the same weight). By using the technique of Remark 9 we can without difficulty construct economies and strategic market games supporting the figure. In the figure, $B_t$ fails to be smooth. An obvious result is:

**Proposition 7.** Suppose that $X_t = R^4$ for a.e. $t$ and that the Anonymity Axiom holds. Let $s$ be a strict CN equilibrium. If $0 \in B_{a_{t}}$ and $B_a$ is a $(l - 1)$ manifold, then the equilibrium is Walrasian.

**Proof.** Let $T$ be the $(l - 1)$ tangent plane to $B_t$ at 0. It is easy to verify that for any $z \in T$, $\cup_{n=1}^{\infty} nB_{a_{t}}$ contains net trades arbitrarily close to $z$. Henceforth by the continuity of preferences, for a.e. $t \in [0, 1]$, $x_4(t) \geq z$ for any $z \in T$. Because of monotonicity of preferences, $T \cap R^+_{++} = \emptyset$. Hence we can pick $p \geq 0$ such that $T = \{z \in R : p \cdot z = 0\}$. With $p$ as the candidate Walrasian price, the proof proceeds from this point on as in Proposition 4. \[\blacksquare\]

As the example of Fig. 2d shows, Proposition 6 holds for strict CN equilibria but not for ordinary ones.

**Remark 14.** Given an anonymous strategic market game $G : S \times M \to R^l$ it is possible to build a new one $G^* : S^* \times M^* \to R^l$ in such a manner
that the concept of ordinary and strict CN equilibria coincide in $G^*$ and the ordinary equilibria of $G^*$ correspond to the strict equilibria of $G$.

One would proceed as follows. Let $S$ be a complete, separable subset of a Banach space. Denote by $S^*$ the set of finite, nonnegative integer valued measures on $S$. Endow $S^*$ with the weak-star topology and corresponding Borel $\sigma$-field. Let $\mathcal{M}^*$ be the (Borel) probability measures on $S^*$ with bounded mean. A play is now an integrable $s^* : I \rightarrow S^*$. The integral is defined by $\int s^*(V) = \int s^*(t)(V)$ for every Borel set $V \subset S$. It is a nonnegative finite measure on $S$ [see (Mas-Colell, 1975, Sect. 3.1) for the mathematical technicalities of this remark]. Given $s^*$ we let $\nu^* \in \mathcal{M}^*$ be the Borel probability measure induced by $s^*$ on $S^*$. An element $\nu^* \in \mathcal{M}^*$ induces a distribution $\nu$ on $A$ by letting

$$\nu(V) = \frac{1}{\int s^*(A)} \int s^*(V) \, d\nu^*.$$

Given a play $s^*$, one let $\nu^*$ be the measure in $S$ corresponding to $\nu^*$. Clearly,

$$\nu^* = \frac{1}{\int s^*(A)} \left( \int s \right).$$

Finally, one defines

$$G^*(s^*, \nu^*) = \int G(s, \nu) \, ds^*.$$

Note that in this new game the Aggregation Axiom is satisfied but $S^*$ is not convex. Also, a proper strict CN equilibrium of $G$ is an ordinary, but not necessarily proper, CN equilibrium of $G^*$. (See Fig. 5 where $B_*$ is the convex hull of $B_*).$

**References**

24. D. Schmeidler, "Walrasian Analysis via Strategic Outcome Functions," mimeograph, Tel-Aviv University, Tel-Aviv, Israel, 1978.