

## ASYMPTOTIC SEMIVALUES AND A SHORT PROOF OF KANNAI'S THEOREM\*†

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Generalizations of the Shapley value, called semivalues, are introduced. We consider their asymptotic behavior for sequences of games that approach a nonatomic game, and show that limiting semivalues exist on  $pNA$ . In the process a short proof of Kannai's asymptotic theorem for the Shapley value is also arrived at.

**1. Introduction.** We consider the asymptotic behavior of "semivalues" (see [1], [2]) for sequences of games that approach a nonatomic game, and show that limiting semivalues exist on  $pNA$ . This complements the axiomatic approach to semivalues on  $pNA$  that is taken in [1]. In the process, a short and simple proof of Kannai's asymptotic theorem for the Shapley value is also arrived at.

**2. Asymptotic semivalues.** Let  $G_N$  be the set of all games (with sidepayments) on the player-set  $N = \{1, \dots, n\}$ , i.e.,  $G_N$  consists of all functions  $v : 2^N \rightarrow R$  such that  $v(\emptyset) = 0$ . Denote by  $R^N$  the Euclidean space of dimension  $n$  whose axes are labelled by the players in  $N$ . For any  $x \in R^N$ ,  $x_i$  is the  $i$ th component of  $x$ . Put

$\Lambda =$  the set of all probability measures on the unit interval  $[0, 1]$ ,

$\Lambda' =$  the set of all  $\lambda$  in  $\Lambda$  which are absolutely continuous with respect to the Lebesgue measure  $\mu$ , and whose Radon-Nikodym derivative  $d\lambda/d\mu$  is bounded, i.e., there is some finite  $M(\lambda) > 0$  such that  $(d\lambda/d\mu)(t) \leq M(\lambda)$  for all  $t \in [0, 1]$ , except perhaps for  $t$  in a  $\mu$ -null set.

For any  $v \in G_N$  and  $\lambda \in \Lambda$  define  ${}^\lambda\xi(v) \in R^N$  by:

$${}^\lambda\xi_i(v) = \sum_{S \subset N \setminus \{i\}} {}^\lambda P(S) [v(S \cup \{i\}) - v(S)],$$

$i \in N$ , where  ${}^\lambda P(S) = \int_0^1 t^s (1-t)^{n-1-s} d\lambda(t)$  and  $s$  is the cardinality of  $S$ . When  $\lambda$  concentrates all its weight on  $t \in [0, 1]$  we shall abbreviate  ${}^\lambda\xi$  ( ${}^\lambda P$ ) by  $'\xi$  ( $'P$ ).

${}^\lambda\xi_i(v)$  can be interpreted as the expected marginal worth of player  $i$  in the game  $v$ , when coalitions of the other players form according to  ${}^\lambda P$ . Note that for the Lebesgue measure  $\mu$ ,  ${}^\mu\xi$  is the familiar Shapley value [3]. (Also note:  ${}^{1/2}\xi$  is the not-so-familiar Banzhaf value [4].) We shall think of  ${}^\lambda\xi$  (for a fixed  $\lambda$ ) as defined on the domain  $U\{G_N : N \text{ is a finite set}\}$ , and call it the  $\lambda$ -semivalue. When convenient, we shall also think of  ${}^\lambda\xi(v)$ , for  $v \in G_N$ , as a set-function  ${}^\lambda\xi(v) : 2^N \rightarrow R$ , given by  $({}^\lambda\xi(v))(S) = \sum_{i \in S} {}^\lambda\xi_i(v)$ .

The sets of semivalues  $\{\xi : \lambda \in \Lambda\}$  and  $\{\xi : \lambda \in \Lambda'\}$  have been characterized axiomatically in [1]. Our intention here is to examine the limit behavior of  ${}^\lambda\xi(v^n)$ , where  $v^n$  is a sequence of finite games that "approaches" a nonatomic game. First we need to recall some basic definitions and results from [5].<sup>1</sup>

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<sup>1</sup> From now on, all unreferenced results will be out of [5].

Let  $\{I, \mathcal{C}\}$  be a measurable space, isomorphic to the closed unit interval with its Borel subsets. The term “set function” will mean a mapping  $v$  of  $\mathcal{C}$  into the reals such that  $v(\emptyset) = 0$ . In the interpretation, a set function is a game.  $I$  is the player set, and  $\mathcal{C}$  is the  $\sigma$ -algebra of coalitions. A set function  $v$  is *monotonic* if  $S \supset T$  implies  $v(S) \geq v(T)$ , and is of *bounded variation* if it is the difference of two monotonic set functions. The collection of all set functions of bounded variation forms a vector space over the reals and will be called  $BV$ .

Our focus in this paper will be on a particular subspace of  $BV$  called  $pNA$ , which also plays a crucial role in [5]. First let us introduce the variation norm  $\|\cdot\|$  on  $BV$  given by  $\|v\| = \inf[u(I) + w(I)]$ , where the infimum is taken over all monotonic set functions  $v$  and  $u$  such that  $v = u - w$ .  $BV$  is a Banach space with this norm (see Proposition 4.3). Denote by  $NA$  the subspace of  $BV$  consisting of nonatomic finite measures. The subspace  $pNA$  is the subset of  $BV$  spanned by all powers of measures in  $NA$ . The word “spanned” is used here in a topological linear sense, i.e., the space spanned by a subset of  $BV$  is the closure (in the variation norm) of the set of all linear combinations of elements of that subset.

Of particular interest are those games  $v$  in  $pNA$  of the form  $v = f \circ \mu$  (i.e.,  $v(S) = f(\mu(S))$  for all  $S \in \mathcal{C}$ ), where  $\mu$  is a finite-dimensional vector of measures in  $NA$ , and  $f$  is a real-valued function that is continuously differentiable<sup>2</sup> on the range of  $\mu$ , with  $f(0) = 0$ . (Note that by Lyapunov’s theorem the range of  $\mu$  is compact and convex.) For the proof that  $f \circ \mu \in pNA$  for all such  $\mu$  and  $f$  see Proposition 7.1. Let  $D$  denote the subset of  $pNA$  consisting of all games of the form  $f \circ \mu$ . Note that  $\bar{D} = pNA$  (where the bar denotes closure).

Take a  $v \in BV$ . Following Kannai [6] we consider a sequence of finite games which converge to  $v$ , and ask if their  $\lambda$ -semivalues (for fixed  $\lambda \in \Lambda$ ) converge. For the formal treatment, we quote directly from Chapter 3:

A *partition*  $\Pi$  of the underlying space  $I$  is a finite family of disjoint subsets whose union is the whole space; a partition  $\Pi_2$  is a *refinement* of another partition  $\Pi_1$  if each member of  $\Pi_2$  is a union of members of  $\Pi_1$ . A partition is called *measurable* if each of its members is measurable. A sequence  $(\Pi_1, \Pi_2, \dots)$  of partitions is said to be *decreasing* if  $\Pi_{m+1}$  is a refinement of  $\Pi_m$  for each  $m$ ; and *separating* if for all  $s, t \in I$  with  $s \neq t$ , there is an  $m$  such that  $s$  and  $t$  are in different members of  $\Pi_m$ . A decreasing separating sequence of measurable partitions is called an *admissible sequence*.

If  $v$  is a set function and  $\Pi$  a measurable partition of the underlying space, let  $v_\Pi$  be the finite game whose players are the members of  $\Pi$ , given by

$$v_\Pi(\Xi) = v\left(\bigcup_{j \in \Xi} j\right)$$

for all  $\Xi \subset \Pi$ . Now let  $T$  be a measurable subset of  $I$ , and let  $\bar{\Pi} = (\Pi_1, \Pi_2, \dots)$  be a decreasing sequence of measurable partitions whose first term  $\Pi_1$  is the partition  $\{T, I \setminus T\}$ . For each  $\nu$  let

$$T_\nu = \{j \in \Pi_\nu : j \subset T\};$$

$T$  is a coalition in the original infinite game  $v$ , and  $T_\nu$  is the corresponding coalition in the corresponding finite game  $v_{\Pi_\nu} = v_\nu$ , say. If the numbers  $\{^\lambda \xi(v_\nu)(T_\nu)\}$  approach a limit as  $\nu \rightarrow \infty$ , then this limit will be denoted  $[^\lambda \xi \bar{\Pi}(v)](T)$ . If the limit exists for all admissible  $\bar{\Pi}$  starting with  $\{T, I \setminus T\}$  and is independent of the choice of such  $\bar{\Pi}$ , then we will denote it by  $[^\lambda \xi(v)]$ . If that is the case for all measurable  $T$ , then we will call the set function  $^\lambda \xi(v)$  the *asymptotic  $\lambda$ -semivalue* of  $v$ .

Finally define, for  $\lambda \in \Lambda$ ,

$$\text{ASYMP}(\lambda) = \{v \in BV : \text{the asymptotic } \lambda\text{-semivalue of } v \text{ exists}\}.$$

<sup>2</sup> For a precise definition of this see [5, p. 22].

**3. The limit theorem.** We are prepared to state our result.

**THEOREM.**  $D \subset \text{ASYMP}(\lambda)$  for  $\lambda \in \Lambda$  and  $pNA \subset \text{ASYMP}(\lambda)$  for  $\lambda \in \Lambda'$ . Further, if  $v = f \circ \mu \in D$ , then (for  $\lambda \in \Lambda$ ) the asymptotic  $\lambda$ -semivalve of  $v$ ,  ${}^\lambda\xi(v)$ , has the following representation:

$$({}^\lambda\xi(v))(S) = \int_0^1 f_{\mu(S)}(t\mu(I)) d\lambda(t),$$

where  $f_{\mu(S)}$  is the directional derivative of  $f$  in the direction  $\mu(S)$ .

For the proof, we first establish a lemma. Fix an admissible sequence  $\bar{\Pi}$ . Denote the members of  $\Pi_\nu$  by  $\{A_1^\nu, \dots, A_m^\nu\}$ , and set  $\mu(A_i^\nu) = \alpha_i^\nu$ , where  $\mu$  is a nonatomic vector measure. Let  $\alpha_{\max}^\nu = \max_{1 \leq j \leq m^\nu} |\alpha_j^\nu|$ , where  $|\cdot|$  denotes the maximum norm.

**LEMMA.** Fix any  $t \in [0, 1]$ , and choose  $S \subset \Pi_\nu$  "at random" according to  $'P$ , i.e., with probability  $t^s(1-t)^{m^\nu-s}$ . Then  $\mu(S) \rightarrow t$  in probability as  $\nu \rightarrow \infty$ , i.e., for each  $\epsilon$  there is a  $\nu_0$  such that for  $\nu > \nu_0$ ,

$$\text{Prob}\{\|\mu(S) - t\mu(I)\| > \epsilon\} < \epsilon.$$

**PROOF.** First let  $\mu$  be one-dimensional and positive. For the moment, fix  $\nu$  and suppress the superscripts  $\nu$ . We have

$$\mu(S) = \sum_{i=1}^m \alpha_i X_i,$$

where the  $X_i$  are independent random variables taking the values 1 and 0 with probabilities  $t$  and  $1-t$ , respectively. Hence

$$E\mu(S) = t \sum_{i=1}^m \alpha_i = t\mu(I),$$

$$\text{Var } \mu(S) = \sum_{i=1}^m \alpha_i \text{Var } X_i = t(1-t) \sum_{i=1}^m \alpha_i^2 \leq \alpha_{\max} \sum_{i=1}^m \alpha_i = \alpha_{\max} \mu(I).$$

Chebycheff's inequality then yields

$$\text{Prob}\{|\mu(S) - t\mu(I)| > \epsilon\} < \alpha_{\max} \mu(I) / \epsilon^2;$$

since  $\alpha_{\max} \rightarrow 0$  as  $\nu \rightarrow \infty$  [5, p. 131], the lemma follows. The general case, when  $\mu$  may have any dimension, follows from the one-dimensional positive case. Q.E.D.

**PROOF OF THEOREM.**<sup>3</sup> First consider the case where  $\lambda$  concentrates on the point  $(t, \dots, t)$ . Let  $H^\nu = (h_1^\nu, \dots, h_{x^\nu}^\nu)$  be such that  $A_i^\nu \in T_\nu$  iff  $i \in H^\nu$ . Then

$$\begin{aligned} [{}^t\xi(v_\nu)](T_\nu) &= \sum_{i \in H^\nu} \sum_{S \subset \Pi_\nu} {}^tP(S \setminus i) [v(S) - v(S \setminus i)] \\ &= \sum_{i \in H^\nu} \sum_{S \subset \Pi_\nu} {}^tP(S \setminus i) [f(\mu(S)) - f(\mu(S \setminus i))] \\ &= \sum_{i \in H^\nu} \sum_{S \subset \Pi_\nu} {}^tP(S \setminus i) (\nabla f)(x_S) \cdot \mu(A_i^\nu). \end{aligned}$$

The last equality comes from the mean value theorem.  $\nabla$  stands for gradient,  $\cdot$  for dot product, and  $x_S$  is a point on the line joining  $\mu(S)$  and  $\mu(S \setminus i)$ . By Lyapunov's

<sup>3</sup> We sometimes use  $i$  to stand for the player  $A_i^\nu$  in  $\Pi_\nu$ . No confusion results.

theorem. the range of  $\mu$ , which is the domain of  $f$ , is compact: since  $f$  is  $C^1$  on its domain, its gradient  $\nabla f$  must be bounded. By the lemma,  $\sum_{S \subset \Pi} P(S \setminus i)(\nabla f)(x_S)$  tends to  $(\nabla f)(t\mu(I))$ . Clearly  $\sum_{i \in H^*} \mu(A_i^*)$  tends to  $\mu(T)$ . Thus we have proved that  $\lim[\xi(v_\nu)(T_\nu)]$  exists and is equal to  $f_{\mu(T)}(t\mu(I))$ .

Now take any  $\lambda \in \Lambda$ . Note that:

- (a)  $\int \xi(v_\nu)(T_\nu) d\lambda(t)$ ;
- (b)  $\lim[\xi(v_\nu)(T_\nu)] = f_{\mu(T)}(t\mu(I))$  for  $t \in [0, 1]$ .

The theorem then follows for  $D$  from Lebesgue's dominated convergence theorem, since  $[\xi(v_\nu)(T_\nu)] \leq M e \cdot \mu(T)$ , where  $M$  is a bound on the coordinates of  $\nabla f$ , and  $e = (1, \dots, 1)$ .

To complete the proof it will suffice to show that  $\text{ASYMP}(\lambda)$  is a closed subset of  $BV$  whenever  $\lambda \in \Lambda'$ . This can be done along exactly the same lines as in Proposition 18.4. We indicate the argument. First observe that<sup>4</sup> for any  $v \in G_N$ ,  $\|\xi(v)\| \leq M(\lambda)\|v\|$ . Using this it can be established, as in Proposition 18.1, that for all  $v \in \text{ASYMP}(\lambda)$ ,  $\xi(v) \in BV$ , and  $\|\xi(v)\| \leq M(\lambda)\|v\|$ . This fact, in turn, permits us to mimic the argument of Proposition 18.4 to prove that  $\text{ASYMP}(\lambda)$  is closed. Q.E.D.

#### 4. Concluding remarks.

(1) Take  $\lambda = \mu$  (the Lebesgue measure) in the theorem. We get Kannai's result in [6], with a simpler proof.

(2) The asymptotic  $\lambda$ -semivalues for  $\lambda \in \Lambda'$ , restricted to  $pNA$ , can be characterized by certain axioms (linearity, symmetry, positivity, and projection). See [1] for this development. Here we note that it can easily be shown, as in Chapter 3, that the asymptotic  $\lambda$ -semivalued satisfies these axioms on the domain  $\text{ASYMP}(\lambda)$ .

(3) For a probability measure  $\lambda \notin \Lambda$  it can be shown easily by counterexamples that  $pNA \not\subset \text{ASYMP}(\lambda)$ . Thus  $pNA \subset \text{ASYMP}(\lambda) \Leftrightarrow \lambda \in \Lambda'$ .

(4) It is worth pointing out that for the set  $M$  of nonatomic differentiable market games (with transferable utilities), i.e., for games of the type in Chapter 6, all the asymptotic semivalues coincide. This will be detailed elsewhere [7].

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<sup>4</sup>The variation norm  $\|v\|$  of a set function  $v : 2^N \rightarrow R$  is defined in the obvious way.

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