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# ATTAINABLE SETS OF QUASICONCAVE MARKETS, II: CONVEXIFIABLE SETS*†

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A set is convexifiable if there exists a strictly increasing, continuous rescaling of the coordinate axes which makes the set convex. Several classes of sets are investigated with regard to this property. It is shown that every convexifiable compactly generated set is the attainable set of a market in which the traders have quasiconcave utility functions.

**Introduction.** A topic of current interest is the characterization of games without side payments which arise from economic markets. In the case where the utility functions of the traders are concave, the characterization problem has been almost completely resolved [10]. However, the situation for markets in which the traders' utility functions need only be quasiconcave is extremely unclear. A natural first step is to attempt to characterize the "attainable sets" of such markets. In [13], sets which are comprehensively generated by finite sets were considered. In this paper, we approach the problem from another direction by treating sets generated by relatively smooth surfaces.

NOTATION. Consider a market consisting of a set of traders $N = \{1, 2, \dots, n\}$, and an $m$-dimensional commodity space $I^m = \{(y_1, \dots, y_m) : 0 \leqslant y_i \leqslant 1 \text{ for all } i\}$. For any collection $\{u_i\}_{i=1}^n$ of utility functions of the traders (real-valued functions on $I^m$), the *attainable set* of the market is

$$\mathcal{A}(u_1, \dots, u_n) = \Big\{x \in R^n : x \leqslant (u_1(y^1), \dots, u_n(y^n)), \text{ where each } y^i \in I^m \text{ and } \sum y^i = (1, \dots, 1)\Big\}.$$

This is the set of all utility outcomes which can be achieved by some distribution of the available commodities among the traders. A set $X$ in $R^n$ is the *comprehensive hull* of another set $Y$ if $X = \{x \in R^n : x \leqslant y \text{ for some } y \in Y\}$; in this case, we say that $X$ is (comprehensively) *generated by* $Y$. It is not difficult to show that if $u_1, \dots, u_n$ are upper-semicontinuous and lower-bounded, then $\mathcal{A}(u_1, \dots, u_n)$ is compactly generated (generated by a compact set). If a set is equal to the intersection of its comprehensive hull with $R^k_+ = \{x \in R^k : x \geqslant 0\}$, then we say that the set is comprehensive in $R^k_+$.

**Exponential convexifiability.** A great variety of compactly generated sets can be transformed into convex sets by a rescaling of the coordinate axes. Let $f$ be a real-valued function on $R^{n-1}$ which is monotone decreasing in each coordinate, and let $C$ be a compact, comprehensive subset of $R^{n-1}_+$. We define

$$A(f, C) = \{x \in R^n : x \leqslant (y_1, \dots, y_{n-1}, f(y)) \text{ for some } y \in C\}.$$

For any $k > 0$, let

$$A^k(f, C) = \{x \in R^n : x \leqslant (1 - e^{-ky_1}, \dots, 1 - e^{-ky_{n-1}}, 1 - e^{-kf(y)}) \text{ for some } y \in C\}.$$

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A set $A(f, C)$ is *exponentially convexifiable* if for all sufficiently large $k$, $A^k(f, C)$ is convex. For all $a \in R^{n-1}$ with $0 \leqslant a_i < 1$ for all $i$, let

$$g^k(a) = 1 - \exp\left[-kf\left(-\frac{1}{k}\ln(1-a_1), \ldots, -\frac{1}{k}\ln(1-a_{n-1})\right)\right].$$

Further let

$$D^k = \left\{a \in R_+^{n-1} : a_i = 1 - e^{-ky_i} \text{ for all } i, \text{ for some } y \in C\right\}.$$

Then $D^k$ is a compact, comprehensive subset of $R_+^{n-1}$, and $A^k(f, C) = A(g^k, D^k)$. Hence, in order for $A(f, C)$ to be exponentially convexifiable, it is necessary and sufficient that for all sufficiently large $k$, $D^k$ is convex and $g^k$ is concave.

THEOREM 1. *Let $f$ be twice continuously differentiable, with negative first-order partial derivatives, and let $C$ be exponentially convexifiable. Then $A(f, C)$ is exponentially convexifiable.*

PROOF. We restrict our attention to $k$ so large that the corresponding set $D^k$ is convex. It will suffice to show that for sufficiently large $k$, $g^k$ is concave on $D^k$. Since $g^k$ is an analytic function in $f$, it follows that $g^k$ is twice continuously differentiable. Therefore, the desired conclusion will hold if, for sufficiently large $k$, the Hessian of $g^k$ is negative definite throughout $D^k$.

Consider any fixed $x \in C$, and let $a^k = (1 - e^{-kx_1}, \ldots, 1 - e^{-kx_{n-1}}) \in D^k$. The Hessian under consideration is

$$H_{g^k}(a^k) = \left[\frac{\partial^2 g^k}{\partial a_i\, \partial a_j}(a^k)\right].$$

Hence,

$$\left(H_{g^k}(a^k)\right)_{ij} = \begin{cases} \dfrac{e^{-kf}}{(1-a_i)^2}\left[\dfrac{f_{ii}}{k} - f_i^2 + f_i\right] & \text{if } i = j, \\[2ex] \dfrac{e^{-kf}}{(1-a_i)(1-a_j)}\left[\dfrac{f_{ij}}{k} - f_i f_j\right] & \text{if } i \neq j, \end{cases}$$

where the subscripts indicate partial derivatives, and $f$ and its derivatives are evaluated at $x$. After common positive factors are deleted from the rows and columns of this matrix, we are left with the matrix $H^{(k)}$, where

$$(H^{(k)})_{ij} = \begin{cases} \dfrac{f_{ii}}{k} - f_i^2 + f_i & \text{if } i = j, \\[2ex] \dfrac{f_{ij}}{k} - f_i f_j & \text{if } i \neq j. \end{cases}$$

Let $\overline{H} = \lim_{k\to\infty} H^{(k)}$. If $\overline{H}$ is negative definite, then all matrices in a neighborhood of $\overline{H}$, including all $H^{(k)}$ for sufficiently large $k$, are also negative definite. Consider $\overline{H}_m$, the order-$m$ principal minor of $\overline{H}$ obtained by deleting all but the first $m$ rows and columns. Then

$$\det\left(\overline{H}_m\right) = \det\begin{bmatrix} (1-f_1)f_1 & -f_1 f_2 & \cdots & -f_1 f_m \\ -f_2 f_1 & (1-f_2)f_2 & \cdots & -f_2 f_m \\ \vdots & \vdots & \ddots & \vdots \\ -f_m f_1 & -f_m f_2 & \cdots & (1-f_m)f_m \end{bmatrix}$$

$$= f_1 f_2 \cdots f_m\left(1 - \sum f_i\right).$$

(For each $i$, factor $f_i$ from column $i$. Subtract the last column from all others. Expand along the first row by cofactors. The result follows by induction.) Since all of the first-order partial derivatives of $f$ are negative, the sign of the determinant of $\overline{H}_m$ is $(-1)^m$. Therefore, $\overline{H}$ is negative definite. It follows that, for all sufficiently large $k$, the Hessian of $g^k$ is negative definite at $a^k$.

Since the first and second partial derivatives of $f$ are continuous, we can associate with each $x \in C$ an integer $K(x)$ and a neighborhood $N(x)$ such that, for all $k \geqslant K(x)$, the Hessian of $g^k$ is negative definite throughout the subset of $D^k$ corresponding to $N(x)$. From the compactness of $C$, it follows that a finite collection $\{N(x^1), \ldots, N(x^m)\}$ of these neighborhoods covers $C$. For any $k \geqslant \max(K(x^1), \ldots, K(x^m))$, we conclude that the Hessian of $g^k$ is negative definite throughout $D^k$. ∎

**Dented sets.** Consider any compactly generated set $S$ in $R^n$. Let $g$ be a monotone increasing, continuous real-valued function on $R$, and define

$$S_g = \{x \in R^n : x \leqslant (g(y_1), \ldots, g(y_n)) \text{ for some } y \in S\}.$$

We say that $S$ is *g-convexifiable* if $S_g$ is convex. Notice that the function $g$ serves to rescale the coordinate axes.

We shall give an example of a set which cannot be convexified by any continuously differentiable $g$. Let $f$ be decreasing on $[0, 1]$. Assume that $f$ is continuously differentiable at all points except $w$, where $0 < w < 1$ and $f'_-(w) < f'_+(w)$. (The set $A(f, [0, 1])$ has a "dent" at the point $(w, f(w))$.) Let $g$ be any continuously differentiable function with positive derivative. The boundary of $A(f, [0, 1])$ contains all points of the form $(x, f(x))$, and the boundary of $A_g(f, [0, 1])$ contains all points of the form $(g(x), g(f(x)))$. Let $a = g(w)$. Then the boundary of $A_g(f, [0, 1])$ in a neighborhood of $(g(w), g(f(w)))$ consists of all points of the form $(a + h, gfg^{-1}(a + h))$ for small values of $h$. The differentiability assumptions allow us to write

$$gfg^{-1}(a+h) = gfg^{-1}(a) + h \cdot g'(f(w)) \cdot f'_{\pm}(w) \cdot \frac{1}{g'(w)} + o(h), \qquad (*)$$

where the subscript of $f'$ corresponds to the sign of $h$. Assume that $A_g(f, [0, 1])$ is convex. Then it must be the case that, for all small $h > 0$,

$$\tfrac{1}{2}\big(a + h, gfg^{-1}(a + h)\big) + \tfrac{1}{2}\big(a - h, gfg^{-1}(a - h)\big)$$
$$= \big(a, \tfrac{1}{2}\big[gfg^{-1}(a + h) + gfg^{-1}(a - h)\big]\big) \in A_g(f, [0, 1]).$$

Since $(a, gfg^{-1}(a))$ is on the boundary of $A_g(f, [0, 1])$, it follows that

$$\tfrac{1}{2}\big[gfg^{-1}(a + h) + gfg^{-1}(a - h)\big] \leqslant gfg^{-1}(a).$$

From $(*)$, we can then conclude that

$$h \cdot g'(f(w)) \cdot (f'_+(w) - f'_-(w)) \cdot \frac{1}{g'(w)} \leqslant o(h).$$

This can hold for all small $h > 0$ only if $f'_+(w) \leqslant f'_-(w)$, in contradiction to our original assumption. Therefore, $A_g(f, [0, 1])$ cannot be convex.

If we remove the requirement that $g$ be continuously differentiable, many sets with dents can then be convexified. For example, let $f$ be defined by

$$f(x) = \begin{cases} 1 - 2x & \text{if } 0 \leqslant x \leqslant \frac{1}{3}, \\ \frac{1}{2} - \frac{1}{2}x & \text{if } \frac{1}{3} \leqslant x \leqslant 1. \end{cases}$$

Define a rescaling function $g$ by

$$g(x) = \begin{cases} x & \text{if } x \leqslant \frac{1}{3}, \\ \frac{1}{3} + k(x - \frac{1}{3}) & \text{if } x \geqslant \frac{1}{3}. \end{cases}$$

For every $k > 0$, $g$ is strictly increasing and continuous. If, furthermore, $k \leqslant \frac{1}{4}$, it is easily shown that $A_g(f, [0, 1])$ is convex.

**A nonconvexifiable set.** We next give an example of a set in $R^3$ which cannot be convexified by any monotone increasing (essentially nonconstant), continuous function. Let $T_1$ be the triangle with vertices (0, 0, 1), (1, 1, 0), and (3, 0, 0), and let $T_2$ be the triangle with vertices (0, 0, 1), (1, 1, 0), and (0, 3, 0). Finally, let $T$ be the comprehensive hull of the union of $T_1$ and $T_2$ (see Figure 1: $T$ is "creased" along the line between (0, 0, 1) and (1, 1, 0)). Consider any function $g$ which is monotone increasing and continuous, with $g(0) < g(1)$. We shall show that $T_g$ is not convex.

Assume, to the contrary, that $T_g$ is convex. For any $0 \leqslant z \leqslant 1$ and $0 \leqslant \epsilon \leqslant 1$, the points $a = ((1 - z)(1 - \epsilon), (1 - z)(1 + 2\epsilon), z)$, $b = ((1 - z)(1 + 2\epsilon), (1 - z)(1 - \epsilon), z)$, and $c = (1 - z, 1 - z, z)$ are on the boundary of $T$. Let $a_g$, $b_g$, and $c_g$ be the images of $a$, $b$, and $c$ when all components are transformed by $g$; these three points are on the boundary of $T_g$. The first two components of $\frac{1}{2}(a_g + b_g)$ are equal, as are the first two components of $c_g$. These two points have the same third component. Furthermore, since $T_g$ is convex, $\frac{1}{2}(a_g + b_g)$ is in $T_g$. Therefore, from the comprehensiveness of $T_g$ we conclude that

$$\left(\tfrac{1}{2}(a_g + b_g)\right)_1 = \tfrac{1}{2}\left[ g((1 - z)(1 - \epsilon)) + g((1 - z)(1 + 2\epsilon))\right] \leqslant g(1 - z) = (c_g)_1.$$

Let $s = 1 - z$ and $d = s\epsilon$. Then for all $0 \leqslant d \leqslant s \leqslant 1$, we have

$$\tfrac{1}{2}\left[ g(s - d) + g(s + 2d)\right] \leqslant g(s),$$

which may be restated as

$$g(s + 2d) - g(s) \leqslant g(s) - g(s - d).$$

Replacing $s$ with $t + kd$ in the preceding inequality, we obtain

$$g(t + (k + 2)d) - g(t + kd) \leqslant g(t + kd) - g(t + (k - 1)d), \qquad (*)$$

which must hold whenever $0 \leqslant d \leqslant t \leqslant t + kd \leqslant 1$. For any integer $m$ for which $md \leqslant 1 - t$, we may add the inequalities $(*)$ for all integers $0 \leqslant k \leqslant m$, obtaining

$$g(t + (m + 2)d) + g(t + (m + 1)d) + g(t - d) \leqslant g(t + d) + g(t + d) + g(t). \qquad (**)$$

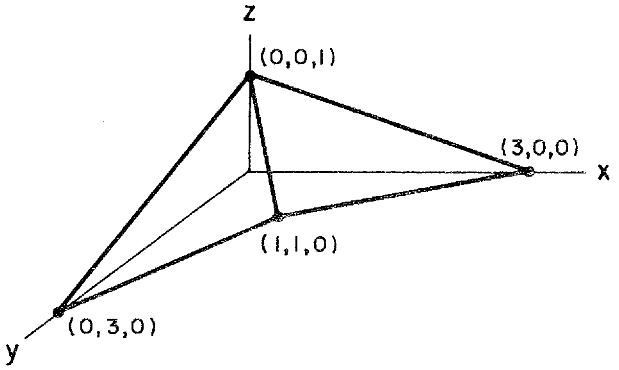

FIGURE 1. A nonconvexifiable set.

Fix $0 < t \leqslant 1$, and let $m \to \infty$ and $d \to 0$ in (**) in such a manner that $md \to (1-t)/2$. Since $g$ is continuous, we find in the limit that

$$2g\left(t + \frac{1-t}{2}\right) + g(t) \leqslant g\left(t + \frac{1-t}{2}\right) + 2g(t),$$

and therefore

$$g\left(\frac{t+1}{2}\right) \leqslant g(t).$$

But $(t+1)/2 \geqslant t$, and it follows from the monotonicity of $g$ that this last relationship must hold with equality for all $0 < t \leqslant 1$. Therefore, $g$ must be constant on $(0, 1]$. Again invoking the continuity of $g$, we finally have $g(0) = g(1)$, in contradiction with the original assumption. Hence, $T_g$ cannot be convex.

It should be noted that *any* compactly generated set $T$ in $R^n$, with the generating set lying (without loss of generality) in the unit $n$-cube, can be convexified by a function $g$ which is constant on $[0, 1]$. However, $T_g$ will simply be a "corner" (a set comprehensively generated by a single point); the convexifying operation will destroy all distinguishing properties of $T$.

**Flat sets.** In Theorem 1, two conditions were imposed on $f$ in order that $A(f, C)$ be convexifiable. It was required that $f$ be sufficiently differentiable, and also that the partial derivatives of $f$ never be zero. We will briefly discuss the second condition.

Assume that $f$ is strictly decreasing and continuously differentiable on $[0, 1]$, but that $f'(w) = 0$ for some $0 < w < 1$. Then, even if distinct rescaling functions $g_1$ and $g_2$, both differentiable with positive derivatives, are applied to the two axes (so that a typical point $(x, y)$ is mapped into $(g_1(x), g_2(y))$), it can be shown that $A(f, [0, 1])$ will not be convexified. However, if we allow these functions to have zero or infinite derivatives, then $A(f, [0, 1])$ may be convexifiable. For example, take $f(x) = \frac{1}{8} - (x - \frac{1}{2})^3$. Then $f'(\frac{1}{2}) = 0$. But if $g_1(x) = \frac{1}{8} + (x - \frac{1}{2})^3$ and $g_2(y) = y$, or if $g_1(x) = x$ and $g_2(y) = \frac{1}{2} + (y - \frac{1}{8})^{1/3}$, then the rescaled set $A_{(g_1, g_2)}(f, [0, 1])$ is convex.

As in the case of the differentiability conditions, matters become significantly more involved in three dimensions. Let $f$ be strictly decreasing and continuously differentiable on $C$ in $R^2$. Assume that $\partial f(x_0, y_0)/\partial x = 0$ at every point $(x_0, y_0)$ on some specific curve in $C$ (for example, see $A(f, C)$ in Figure 2). It appears that $A(f, C)$

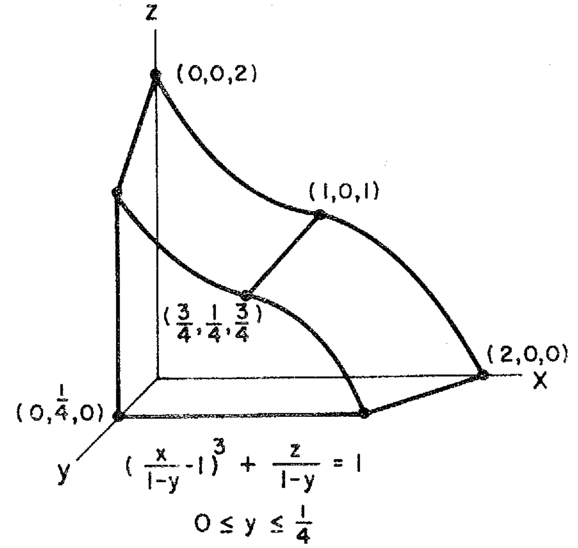

FIGURE 2. $\frac{\partial z}{\partial x} = 0$ for all points $(t, 1-t, t)$.

cannot usually be convexified by any collection $\{g_1, g_2, g_3\}$ of strictly increasing, continuous rescaling functions. (Otherwise, for instance, in the diagrammed example we would need to have $g_3'(1-y)/g_1'(1-y)$ infinite for every $0 \leqslant y \leqslant \frac{1}{4}$.)

**Quasiconcave markets.** We now apply the ideas of the preceding sections to the investigation of attainable sets of markets in which the traders have quasiconcave utility functions.

THEOREM 2. *Let $C$ be a compactly generated set in $R^n$, and let $\{g_1, \dots, g_n\}$ be a collection of strictly increasing, continuous functions such that $D = \{(g_1(x_1), \dots, g_n(x_n)) : (x_1, \dots, x_n) \in C\}$ is convex. Then there is a collection $\{u_1, \dots, u_n\}$ of monotone increasing, quasiconcave utility functions such that $C = \mathcal{A}(u_1, \dots, u_n)$.*

PROOF. Let $G$ be a compact set which generates $C$. Without loss of generality, assume $C$ to have been so affinely scaled that $G$ lies in the unit $n$-cube, and assume that $g_i(0) = 0$ and $g_i(1) = 1$ for all $i$. Let $G_g$ be the image of $G$ under the mapping induced by $\{g_1, \dots, g_n\}$. Then $G_g$ lies in the unit $n$-cube, and compactly generates $D$.

We define a collection $\{v_1, \dots, v_n\}$ of utility functions on the $(n-1)$-dimensional commodity space $I^{n-1}$. For $1 \leqslant i \leqslant n-1$ and $y \in I^{n-1}$, let $v_i(y) = g_i^{-1}(y_i)$. Each such $v_i$ is continuously increasing in variable $i$ and is constant in the remaining $n-2$ variables. Hence, each such $v_i$ is monotone increasing and quasiconcave.

Write $e = (1, \dots, 1) \in I^{n-1}$, and for $y \in I^{n-1}$ let

$$v_n(y) = \begin{cases} \max\{z : (v_1(e-y), \dots, v_{n-1}(e-y), z) \in C\}, & \text{if this set is nonempty,} \\ 0 & \text{otherwise.} \end{cases}$$

It follows from the comprehensiveness of $C$ and the monotonicity of $v_1, \dots, v_{n-1}$ that $v_n$ is monotone increasing. Consider any level set $L(t) = \{y \in I^{n-1} : v_n(y) \geqslant t\}$ of $v_n$. Then

$$\begin{aligned} L(t) &= \{y \in I^{n-1} : (v_1(e-y), \dots, v_{n-1}(e-y), t) \in C\} \\ &= \{y \in I^{n-1} : (g_1^{-1}(1-y_1), \dots, g_{n-1}^{-1}(1-y_{n-1}), t) \in C\} \\ &= \{y \in I^{n-1} : (1-y_1, \dots, 1-y_{n-1}, g_n(t)) \in D\}. \end{aligned}$$

Since $D$, and therefore every cross-section of $D$, is convex, we can conclude that every level set of $v_n$ is convex. Hence, $v_n$ is quasiconcave.

It remains to determine $\mathcal{A}(v_1, \dots, v_n)$. Let $B$ be the comprehensive hull of the single point $(1, \dots, 1, 0) = (g_1^{-1}(1), \dots, g_{n-1}^{-1}(1), 0) \in R^n$. We will show that $\mathcal{A}(v_1, \dots, v_n) = C \cup B$.

Given any $y \in I^{n-1}$, we associate with $y$ a collection $\{y^1, \dots, y^n\}$ of allocations in $I^{n-1}$. For $1 \leqslant i \leqslant n-1$,

$$(y^i)_j = \begin{cases} y_i & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

and $y^n = e - y$. Consider any $x \in C$. Let $y \in I^{n-1}$ be defined by $y_i = g_i(x_i)$ for all $i$, and let $\{y^1, \dots, y^n\}$ be the associated collection of allocations. Then $v_i(y^i) = g_i^{-1}(g_i(x_i)) = x_i$ for all $1 \leqslant i \leqslant n-1$. Also, $v_n(y^n) = \max\{z : (x_1, \dots, x_{n-1}, z) \in C\}$. Therefore, $x \in \mathcal{A}(v_1, \dots, v_n)$. The point $(1, \dots, 1, 0)$ is also in $\mathcal{A}(v_1, \dots, v_n)$, as is seen by considering the collection of allocations associated with $e$. Thus, $\mathcal{A}(v_1, \dots, v_n) \supset C \cup B$. The reverse inclusion follows directly from the definition of $v_n$.

Rather than distinguishing trader $n$ in the preceding construction, we can select any other single trader for similar treatment. In such a manner, $n$ distinct markets can be defined. Each market will be defined on a different $(n-1)$-dimensional commodity space, and will have as its attainable set the union of $C$ with a set generated by a point with a zero in the coordinate corresponding to the distinguished trader. The "intersection" of these markets, in the sense of Billera and Bixby [2], is a market with utility functions $u_1, \ldots, u_n$ on a space of $n(n-1)$ commodities. The attainable set of an intersection of markets is the intersection of the separate attainable sets, and therefore $\mathscr{Q}(u_1, \ldots, u_n) = C$. Furthermore, this operation of intersection involves only taking the minimum of collections of utility functions, and hence is an operation which preserves monotonicity and quasiconcavity. ∎

**Remarks.** The purpose of this paper has been to present the notion of convexifiability and to indicate its application to the study of attainable sets. A number of our results are open to refinement.

Let $C$ be any compactly generated set in $R^n$, and let $G$ be a collection of monotone increasing, continuous functions on $R$. Under what conditions on $C$ can we conclude that there exists a subcollection $\{g_1, \ldots, g_n\} \subset G$ such that

$$C_{(g_1, \ldots, g_n)} = \{(g_1(x_1), \ldots, g_n(x_n)) : (x_1, \ldots, x_n) \in C\}$$

is convex? Conditions related to "creases," and to the "flatness" of $C$, seem crucial.

Of major interest is the characterization of all sets which are the attainable sets of markets in which all of the traders have (upper-semicontinuous, lower-bounded) quasiconcave utility functions. Two classes of such sets are presented in this paper and in [13]. In approaching the characterization problem from the opposite direction, it would be useful to know of classes of compactly generated sets which never occur as the attainable sets of quasiconcave markets. A natural starting place is with the example of a three-dimensional nonconvexifiable set discussed earlier. Two other papers are relevant in this regard. In an unpublished manuscript, Mantel derives a result essentially equivalent to Theorem 1 (the exponential scaling transformation is there called the "deFinetti transformation"), and discusses the nonconvexifiability of the set illustrated in Figure 1. A recent paper of Kannai and Mantel [9] presents a mild generalization of this example, and also sheds light on the characterization of attainable sets by giving a result equivalent to the following: if the set in Figure 1 is truncated, by intersecting it with the halfspace $\{x \in R^3 : x_3 \leqslant 1 - \epsilon\}$ (for any $0 < \epsilon < 1$), then the resulting set is not convexifiable, but *is* the attainable set of a market with continuous, strictly increasing, quasiconcave utility functions. This result serves to localize the problem: if the set in Figure 1 is not the attainable set of a quasiconcave market, it is because of the nature of the set near the point $(0, 0, 1)$.

A separate question deals with the "complexity" of the representation of attainable sets, as measured by the dimension of the commodity space. Kalai [7] has shown that every convex compactly generated set can arise from a market with concave utility functions over a space of at most $(n-1)^2 - (n-2)$ commodities. In another paper [6], we indicate how the construction given in this paper can be modified to yield quasiconcave representations in at most $n(n-1)/2$ commodities, and we show how a similar approach yields representations of a wide class of convex compactly generated sets by concave utility functions of at most $n(n-1)/2$ commodities. This bound has been conjectured to be the best possible.

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