

Dynamic Models of Portfolio Behavior: Comment on Purvis

By GARY SMITH*

Douglas Purvis' discussion of an integrated approach to consumption and portfolio decisions is an attractive extension of the "pitfalls" framework advocated by William Brainard and James Tobin. The pitfalls model is concerned with the portfolio allocation of a level of wealth which is predetermined by beginning of period asset holdings and current period saving and capital gains. One of the innovative features of this model is the inclusion of all asset yields and lagged asset holdings as explanatory variables in the asset demand equations. Purvis supplements the Brainard-Tobin asset demands with a consumption-saving relationship that includes a similar list of explanatory variables and reinterprets this system as a model of integrated rather than sequential decision making. Despite his observation that, "when combined with a consumption-savings relationship such as (2), the Brainard-Tobin model will in principle give rise to exactly the same short- and long-run behavior as the integrated model" (p. 407), most of Purvis' discussion is concerned with alleged dissimilarities between the two approaches. This is apparently due to his implicit coupling of a simple consumption function and sequential decision making. In particular most of his comments on the Brainard-Tobin approach are actually concerned with whether or not lagged asset holdings should be included in a consumption function. This is rather unfair to Brainard and Tobin since there is no consumption function in the pitfalls model, and the two issues are really conceptually distinct. An integrated approach does not preclude, and a sequential approach does not require, a

simple consumption function. The spirit of Brainard and Tobin's work is in fact that the inherited composition of wealth is very important to consumption, but consumption decisions precede asset demand decisions. The substance of their sequential approach is not that the composition of wealth is unimportant to consumption but rather that there are some variables which influence consumption and yet do not separately affect asset demands; only the net amount of saving motivated by these influences is important. In this paper I have consequently tried to separate these two issues: the use of an integrated or sequential framework and the imposition of parametric assumptions.

One of the reasons for the merging of these two issues in Purvis' discussion is that he uses a deterministic scenario which makes the distinction between integrated and sequential decisions unimportant. In Purvis' integrated model, consumption and asset demands are constrained by lagged asset holdings plus income. In the relevant sequential interpretation of this model, consumption is first determined, setting the amount of saving and the level of end of period wealth. Asset demands are then decided upon, subject to the budget constraint that they sum to the predetermined end of period wealth. Thus the integrated asset demands include income as an explanatory variable while the sequential asset demands instead include end of period wealth. In a deterministic world there are no substantive differences between these approaches as long as income and wealth are related through a consumption-saving equation. This equivalence breaks down if the marginal propensity to save out of income is zero (since wealth is then no longer related to income) or if there is an unobserved disturbance term in the consumption

*Yale University. Note that equations numbered (1) through (11) are in Purvis' paper. My equations are numbered in the same sequence.

equation. With a stochastic error term, predictions will diverge and the use of observed wealth in the asset demand equations will introduce biases into the estimation procedure unless the decision making is truly recursive. If there is a simultaneous equations problem, then the use of the consumption explanatory variables as instruments for wealth will restore the equivalence of the sequential and integrated models.

In practice, the attractiveness of a sequential approach will depend upon the sector being studied. A hierarchical approach in which certain decisions are prior claims which constrain other decisions is often used with considerable success despite its weak theoretical underpinnings. Some examples are household saving preceding the allocation of wealth, deposits in financial institutions constraining their asset acquisition, corporate physical investment preceding financing, and corporate investment and long-term financing preceding short-term asset management.

Although it need not, a sequential approach is in practice often used to motivate the imposition of parametric restrictions. Some extreme examples are the assumptions that corporate investment depends only upon a comparison of the anticipated profit rate with some hurdle rate; that the supply of labor depends only upon the nominal or real wage rate; and that consumption depends only upon disposable income. This practice may underlie Purvis' implicit assumption that his complicated consumption function would not be used in a sequential model. I am instead stressing the points that a sequential approach does not require a simple consumption function and that a comparison of consumption functions does not provide a test of a sequential approach.

The actual Brainard-Tobin asset demands involve somewhat different parametric assumptions in that both income and wealth are included as explanatory variables. In Purvis' deterministic model, this is redundant since the saving relation makes wealth a linear function of income

and the other explanatory variables. In a stochastic world, wealth would pick up unexplained saving. In practice the substantive assumption actually embodied in the Brainard-Tobin sequential approach is that a number of explanatory variables in the consumption function do not separately appear in the asset equations but instead influence asset holdings only through wealth. Thus wealth appears in the asset demand equations because a number of other variables have been omitted from these equations, and not because lagged asset holdings have been omitted from the consumption function.

In order to separate in the present paper the selection of explanatory variables from the adoption of a sequential or integrated approach, I will first discuss the sequential vs. integrated argument using the Purvis model to more concretely illustrate the issues involved. I will then compare the Purvis model to the Brainard-Tobin asset demand equations and discuss the implications of the differences in explanatory variables.

I. The Relationship Between Integrated and Sequential Approaches

In order to analyze both deterministic and stochastic frameworks, I will add a stochastic disturbance term ϵ_0 to Purvis' consumption function (2) and stochastic terms ϵ_r to his asset flow demands (5). The presence of these terms introduces an additional adding up restriction to his equations (7a)-(7c):

$$(7d) \quad \epsilon_0 + \sum_r \epsilon_r = 0$$

In order to reinterpret his integrated model as a sequential process, the consumption function (2) can be substituted into the budget constraint (6) to yield a relationship between income X_0 and end of period wealth $X_M = \sum_{s=1}^m y_s$ as long as the marginal propensity to save out of income $(1 - b_0)$ is not zero,

$$(12) \quad X_0 = \left(X_M + \sum_{i=1}^{m-1} b_i X_i + \sum_{s=1}^n (e_s - 1) y_s (-1) + \epsilon_0 \right) / (1 - b_0)$$

The substitution of this relation into the asset demands (5) yields the sequential asset demands which depend upon wealth rather than income,

(13)

$$\begin{aligned} \Delta y_r &= \frac{\alpha_{r0}}{1 - b_0} X_M \\ &+ \sum_{i=1}^{m-1} \left(\alpha_{ri} + \frac{\alpha_{r0} b_i}{1 - b_0} \right) X_i \\ &- \sum_{s=1}^n \left(\gamma_{rs} + \frac{\alpha_{r0} (1 - e_s)}{1 - b_0} \right) y_s (-1) \\ &+ \left(e_r + \frac{\alpha_{r0}}{1 - b_0} \epsilon_0 \right) \\ &= \frac{\alpha_{r0}}{\sum_r \alpha_{r0}} X_M \\ &+ \sum_{i=1}^{m-1} \left(\alpha_{ri} + \frac{\left(\alpha_{r0} \sum_r \alpha_{ri} \right)}{\sum_r \alpha_{r0}} \right) X_i \\ &- \sum_{s=1}^n \left(\gamma_{rs} + \frac{\alpha_{r0} \left(1 - \sum_r \gamma_{rs} \right)}{\sum_r \alpha_{r0}} \right) y_s (-1) \\ &+ \left(\epsilon_r - \frac{\alpha_{r0}}{\sum_r \alpha_{r0}} \sum \epsilon_r \right) \\ &= \sum_{i=1}^M \bar{\alpha}_{ri} X_i - \sum_{s=1}^n \bar{\gamma}_{rs} y_s (-1) + \bar{\epsilon}_r \end{aligned}$$

The substitution of the definitions of the parameters contained in (13) into Purvis' adding up restrictions (7) for the integrated model yields the adding up restrictions for the sequential model,

$$(14a) \quad \sum_r \bar{\alpha}_{ri} = 0, \quad i = 1, \dots, m - 1$$

$$(14b) \quad \sum_r \bar{\alpha}_{rm} = 1$$

$$(14c) \quad \sum_r \bar{\gamma}_{rs} = 1, \quad s = 1, \dots, n$$

$$(14d) \quad \sum_r \bar{\epsilon}_r = 0$$

In a deterministic world where the $\epsilon_r = 0$, the $m + n$ explanatory variables in the sequential asset demands (13) are a linear transformation of the $m + n$ explanatory variables in the integrated demands (5), and the two approaches are fully equivalent. Notice that in contrast to Purvis, I have placed bars over the parameters of the sequential asset demands (13) to distinguish them from the integrated parameters in (5) because, although linearly related, they are not identical. The use of the same notation might give the misleading impression that the demand equations (5) and (13) and the corresponding consistency conditions (7) and (14) are competing alternatives rather than equivalent representations.

This equivalence breaks down in a stochastic world with unobserved disturbances, $\epsilon_r \neq 0$, since there will no longer be an exact linear relationship between explanatory variables for the integrated and sequential models. Even if one knew the true values of the parameters (or any common set for that matter), predictions would diverge since a unitary increase in the consumption disturbance term will not affect the integrated asset demand forecasts (5) but will lower X_m by a unit (see (12)) and hence lower the sequential asset predictions (13) by $\alpha_{r0}/(1 - b_0)$. The actual change in asset holdings will depend upon the correlation between ϵ_0 and ϵ_r (which is typically negative from (7d)). For any common set of parameters, the difference between the mean squared forecast errors for the sequential and integrated models is

$$\frac{\alpha_{r0}}{1 - b_0} E(\epsilon_0^2) + 2 \frac{\alpha_{r0}}{1 - b_0} E(\epsilon_r \epsilon_0)$$

The integrated model will consequently be more accurate if and only if

$$(15) \quad 2 \frac{E(\epsilon_r \epsilon_0)}{E(\epsilon_0^2)} > - \frac{\alpha_{r0}}{1 - b_0}$$

In practice the parameters will probably be estimated, and one must then confront the fact that wealth is likely to be correlated with the disturbance terms in the sequential asset demand equations, since

$$E(X_m \bar{\epsilon}_r) = -E(\epsilon_0 \epsilon_r) - \frac{\alpha_{r0}}{1 - b_0} E(\epsilon_0^2)$$

is not equal to zero unless

$$- \frac{E(\epsilon_0 \epsilon_r)}{E(\epsilon_0^2)} = \frac{\alpha_{r0}}{1 - b_0}$$

The left-hand side of this condition¹ describes the change in Δy_r associated with a unit increase in X_m that is due to a fall in ϵ_0 . The right-hand side is the increase in Δy_r associated with a unit increase in X_0 due to an increase in X_0 . Thus wealth will be correlated with the disturbance terms in the asset demand equations, unless the model is actually sequential in the specific sense that the effect of a change in wealth on asset demands is independent of whether the change in wealth is the result of a change in income or in the consumption disturbance term.² The earlier prediction criteria (15) can now be interpreted as stating that (for a common set of parameter estimates) the sequential model will have smaller forecast errors if it is more accurate to assume that a change in the consumption disturbance term has the same effect as a change in income on asset demands than to assume that it has no effect.

An obvious response to the simultaneity

¹An alternative interpretation is that the system must be recursive in that the error term ϵ_0 in the consumption function is uncorrelated with the error terms $\epsilon_r + \alpha_{r0}\epsilon_0/(1 - b_0)$ in the sequential asset demands (13).

²Purvis misstates the simultaneity issue as, "There will be econometric problems involved in using wealth as an explanatory variable and hence treating it as predetermined if it is in fact systematically related to the other explanatory variables" (p. 407). Simultaneity does not hinge upon the correlation between wealth and the other explanatory variables, but rather upon the correlation between wealth and the disturbance term.

problem is to use X_i ($i = 0, \dots, m - 1$) and $y_s(-1)$ ($s = 1, \dots, n$) as instruments for wealth X_m . The sequential asset demand equations will then include $m + n$ independent linear combinations of the $m + n$ integrated explanatory variables, rendering the two sets of parameter estimates fully equivalent. The only difference will be the superficial one that one set of equations will be in the integrated form (2) and (5) with its parameters subject to the adding up restrictions (7) while the other set will be in the sequential form (2) and (13) with its parameters subject to (14). Either set of parameter estimates could of course be directly derived from the other set. The forecasts will also coincide if X_m is replaced by its instrumental variables estimate. If the observed values of X_m are instead used, then the forecasts will diverge with the criteria (15) again determining the more accurate approach. It is also worth mentioning that wealth is not the only explanatory variable which may not be predetermined. Asset prices or yields, the prices of consumption goods and, more relevantly, income could all be plausibly considered sources of simultaneity problems.

II. The Pitfalls Model

The Brainard-Tobin asset demands discussed by Purvis (from (9) and (10)) modify this analysis somewhat since both income and wealth are included as explanatory variables,

$$(16) \quad \Delta y_r = \sum_{i=0}^m \bar{\alpha}_{rm} X_i - \sum_{s=1}^n \bar{\gamma}_{rs} y_s(-1) + \bar{\epsilon}_r$$

The adding up restrictions are now³

³Purvis argues that the Brainard-Tobin adding up restrictions "reflect ... the treatment of wealth as exogenous, ..." and that, "Condition ... [(17c)] implies that any reshuffling of initial assets $y_s(-1)$, holding $X_m(-1)$ constant does not necessarily influence current wealth X_m " (p. 406). The model and the restrictions in fact contain no information about the determination of wealth. Conditions (17a), (17c), and (17d) do reflect a mental *ceteris paribus* experiment in which wealth is held constant while another ex-

$$(17a) \quad \sum_r \bar{\alpha}_{ri} = 0, \quad i = 0, \dots, m-1$$

$$(17b) \quad \sum_r \bar{\alpha}_{rm} = 1$$

$$(17c) \quad \sum_r \bar{\gamma}_{rs} = 1, \quad s = 1, \dots, n$$

$$(17d) \quad \sum_r \bar{\epsilon}_r = 0$$

I have again labelled the parameters with bars, both for notational simplicity and because the sequential version (13) of Purvis' model can be interpreted as the special case of the Brainard-Tobin model (16) in which $\bar{\alpha}_{r0} = 0$. This implies that since Purvis' integrated model, (2) and (5), is of comparable generality to the sequential version, (2) and (13), he is incorrect in arguing that the Brainard-Tobin model "is in some sense a special case of the integrated model" (p. 408), and that with an integrated approach the adding up restrictions "are now relaxed somewhat (compare (7a) and (7b) with ... [(17a)] and ... [(17b)])" (p. 409). The latter argument is also directly refuted by the observation that (7) is a linear transformation of the sequential adding up restrictions (14) which are the special case of (17) where $\bar{\alpha}_{r0} = 0$.⁴

planatory variable changes, but such experiments can be performed regardless of whether or not wealth is exogenous. For example, condition (17a) states that holding wealth as well as the other explanatory variables constant, an increase in X_i involves only a reshuffling of asset holdings. Purvis' condition (7b) on the other hand states that holding the explanatory variables other than wealth constant, an increase in X_i will increase net asset holdings by the amount that saving increases. If one were to perform comparable mental experiments, one would obtain comparable answers. For example, the asset demand effects of a change in X_i in the Brainard-Tobin model, taking into account the effect of X_i on wealth, also sum to $-b_i$.

⁴A tangential issue raised by Purvis is that the pitfalls framework is best suited for a static environment since the target asset holdings and adjustment behavior do not reflect the anticipated course of asset demands and resources. In a growth situation, the model is consequently always struggling to catch up to a moving target. It is unfortunately difficult to construct a simple and yet sophisticated description of this dynamic multivariate problem, although attempts have been made by Manuel Barbosa and

An alternative interpretation is provided by using the implication of the consumption function (2) that

$$(18) \quad X_m = \sum_1^n y_s(-1) + X_0 - C = (1 - b_0)X_0 - \sum_1^{m-1} b_i X_i + \sum_1^n (1 - e_s)y_s(-1) - \epsilon_0$$

to eliminate X_m from (16):

$$(19) \quad \Delta y_r = (\bar{\alpha}_{r0} + \bar{\alpha}_{rm}(1 - b_0))X_0 + \sum_1^{m-1} (\bar{\alpha}_{ri} - \bar{\alpha}_{rm}b_i)X_i - \sum_1^n (\bar{\gamma}_{rs} - \bar{\alpha}_{rm}(1 - e_s))y_s(-1) + (\bar{\epsilon}_r - \bar{\alpha}_{rm}\epsilon_0)$$

This is identical to Purvis' integrated asset demands (5).

In a deterministic world the Brainard-Tobin model (16) and the Purvis model (19) are thus equivalent if the consumption function (2) is appropriate. This is because the consumption function (2) implies that X_m is redundant since it is linearly related to the remaining explanatory variables in the asset demand equations. As Purvis notes the adding up restrictions (17) are then sufficient but not necessary. This is analytically equivalent to the Ladenson-Clinton problem discussed by the author, though one could draw a distinction between linear dependencies arising from identities and those due to behavior and/or events. With the former, one can only use fanciful terms to describe behavior in the Wonderland situation where an explanatory variable is increased while it is also being held constant. In the latter case, however, it is entirely reasonable (and even interesting) to speculate on how agents might behave if for legal, policy, or behavioral reasons variables which had been tied together were to be set free. When engaging

Benjamin Friedman. Since Purvis' asset demands are less general than those of Brainard and Tobin, his contribution here is limited to his advocacy of the consumption function (2).

in such an exercise, one should assign parameter values which would yield consistent behavior if the variables were to move independently. The full set of adding up restrictions (17) would then be necessary as well as sufficient. In practice, the linear dependency in (16) will be broken if the parameters of the consumption function (2) change or if there is a stochastic error term in that equation. In a stochastic world, even with the common set of parameters described in (19), predictions will diverge⁵ since an increase in the consumption disturbance term ϵ_0 will not affect the Purvis forecasts (19), but will lower wealth and hence the Brainard-Tobin asset forecasts (16) by $\bar{\alpha}_{rm}$. The actual mean change in Δy_r of $-\bar{\alpha}_{rm} + cov(\epsilon_0\epsilon_r)/var(\epsilon_0)$ is generally assumed negative.⁶ In their predictions (and estimation) Brainard-Tobin implicitly assume that $cov(\epsilon_0\bar{\epsilon}_r) = 0$ and assign a value $\bar{\alpha}_{rm}$ to the change in Δy_r . Purvis implicitly assumes that $cov(\epsilon_0\bar{\epsilon}_r)/var \epsilon_0 = \alpha_{rm}$ and predicts no change in Δy_r . Purvis will have a smaller mean squared error if and only if

$$\frac{cov(\epsilon_0\epsilon_r)}{var(\epsilon_0)} > \frac{\bar{\alpha}_{rm}}{2}$$

Thus, there is here a substantive difference in that Purvis neglects, while Brainard-Tobin do not, the effects on asset demands of influences which are omitted from the consumption function.

This brings up the more general point that Brainard and Tobin include wealth in the asset demands to catch the effects not only of ϵ_0 , but also of any explanatory variables in the consumption function which are not otherwise included in the asset demand equations. The Brainard-Tobin asset demands are exceptionally rich and detailed, but they typically include only income, saving, capital gains, asset yields, and lagged asset stocks as explanatory variables, and neglect such obvious influences on saving as the composition of income, commodity prices, lagged stocks of

durable commodities, and accustomed consumption or production levels. Instead they have adopted a sequential approach in which asset demands depend upon available saving, but not upon all of the factors which determine the level of saving. Thus the appearance of wealth in the asset demands does not reflect the simplifying assumption that asset stocks are unimportant to consumption but rather the simplifying assumption that many consumption influences are not separately important to portfolio decisions.

Although the Brainard-Tobin model reflects a simplification of asset demands rather than consumption, the presence of lagged asset stocks in the consumption function is interesting and deserves discussion.⁷ Purvis specifically raises the issue of the monetarist transmission mechanism, although his mental experiment in which households find themselves with more money and fewer bonds reflects a benevolent mugger analogy rather than an open market operation in which households are persuaded to make a voluntary exchange.

The specific issue is the relative size of the coefficients e_s on the lagged asset stocks in the consumption function (2). If the coefficient for money is larger than that for bonds then a *ceteris paribus* swap of money for bonds will increase consumption. The coefficient e_s measures the reduction in saving occurring when the actual holdings of an asset increase relative to desired holdings. This coefficient $e_s = \sum_r \gamma_{rs}$ represents not only the own speed of asset adjustment (γ_{ss}), but also the induced changes in holdings of other assets (γ_{rs} , $r \neq s$). If these cross effects are negative then $e_s > \gamma_{ss}$, since saving must finance not only the acquisition of the own asset but of other assets as well. In the case of money one would expect the own effect to be large and the cross effects to be small but more likely negative than positive; other assets might be sold to obtain money but there is little reason to acquire other assets. With more

⁵If X_m itself is predicted from (2), then the asset demand forecasts will of course coincide.

⁶In Purvis' framework, this assumption is that $cov(\epsilon_0\epsilon_r) < 0$.

⁷Lawrence Klein; Don Patinkin; James Tobin; Harold Watts and Tobin; Arnold Zellner, David Huang, and L. C. Chau discuss the influence of the composition of asset holdings on consumption.

illiquid items, the own speed of adjustment is probably smaller but there may be significant positive cross effects; a hesitancy to move quickly into illiquid high transactions-cost items may motivate the temporary acquisition of liquid assets.

My own priors are that the e_s for liquid items are probably larger than for illiquid ones due to the extra transactions costs of using liquid assets as a temporary buffer for illiquid transactions. Within the liquid and illiquid and high and low transactions costs categories, the differences in e_s are probably negligible. Brainard and Tobin argued that model building often requires generality in order to avoid an inadvertantly implausible or even inconsistent specification. However, one should also seek to avoid the uselessness that follows from completely general models. Some subtle effects are surely so minor that they can be safely neglected.

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