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EXISTENCE OF ELECTORAL EQUILIBRIUM*

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This note extends the theory of electoral competition by establishing the existence of candidate equilibria under quite general assumptions, in particular concerning the dimensionality of the underlying policy or issue space over which the candidates compete. In the classical Hotelling-Downs case this space is one-dimensional, and under the usual assumptions on voter and candidate preferences, there exists a pure strategy equilibrium in which both candidates adopt policies at the median of the voter policy-preference distribution. It is well known, however, that when the policy space is of greater dimensionality, pure strategy equilibria generally will not exist. Thus analysis of

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electoral competition in the general and substantively important multidimensional case must be based either on explicit hypotheses about disequilibrium behavior (as in Kramer [1977], for example), or else on a more general equilibrium concept, such as that involving the use of mixed strategies.

We focus here on the second of these possibilities, of characterizing candidate behavior in terms of an equilibrium in the domain of mixed strategies. Several studies of electoral competition which make use of the notion of mixed-strategy equilibria have appeared in the literature (for example, Shubik [1970], Ordeshook [1971], McKelvey and Ordeshook [1976]), but the question of whether such equilibria exist in the general multidimensional case is still unresolved. Since, in this case, the candidates' strategy sets are infinite, the usual matrix-game version of the minmax theorem is not applicable. Moreover, the candidates' vote shares are not continuous functions of the strategies they choose, so the standard continuous-game extension of the minmax theorem (for example, Theorem 1.5.1. of Karlin [1959]), which relies on continuity of the payoff functions, is also inapplicable. There are many examples of discontinuous games which fail to possess equilibria, so the existence issue is a potentially serious one.

The discontinuities in the electoral game are of two kinds. The first arises from the "lumpiness" of individual votes, since each voter must switch his vote from one candidate to the other in a discontinuous fashion as candidate strategies vary over the policy space. This is essentially a "small-sample" phenomenon, and is not important for a large mass electorate where the contribution of any single ballot would be so small as to be unnoticeable. The second type of discontinuity, however, is more fundamental. If one of the candidate's strategies is smoothly varied to approach and then pass through the policy chosen by his opponent, the set of voters who previously favored the first candidate will suddenly and simultaneously switch sides as the strategies cross. At the same time the voters previously favoring the second candidate will also switch, in the opposite direction; but unless these two coalitions happen to be of precisely the same size, the net effect will be a sudden jump, or discontinuity, in the vote shares of the two candidates. Since this discontinuity results from the behavior of entire coalitions rather than isolated individuals, it does not diminish as the electorate becomes large; it is, in fact, intrinsic to the nature of the situation.

In the argument which follows we use a generalized version of the minmax theorem, due to Mertens and Zamir, to show that minmax mixed strategies nevertheless do exist, in large electorates where discontinuities of the first kind do not arise. The first section introduces notation and defines the general multidimensional electoral game. The basic existence theorem is formulated and discussed in Section 2, and proved in Section 3.

1. DEFINITIONS AND ASSUMPTIONS

Two candidates or parties compete for votes by choosing policies x, y,... which can be represented as points in R^k. There is a set I of voters, and each voter $i \in I$ has a (complete, transitive) preference ordering $\underset{i}{\gtrsim}$ of the policies, the derived relations $\underset{i}{\succ}$, $\underset{i}{\sim}$ being defined as usual. The preference ordering of each voter i is assumed to be representable in the usual fashion by a continuous (ordinal) utility function $u_i : R^k \to R$, which is strictly quasi-concave, and satiated at a (unique) point s_i, $s_i = \arg\max_{R^k} u_i(\cdot)$. (The assignment of preferences is also assumed to satisfy measurability conditions given below.)

The composition of the electorate is characterized by a probability measure μ, defined on a σ-algebra $\mathcal{I}$ of I which includes all sets of the form $\{i \in I : x \underset{i}{\gtrsim} y\}$ for some $x, y \in R^k$. $\mu(C)$ is the size of the coalition $C \subset I$, as a fraction of the entire electorate.

Each candidate chooses a particular policy in R^k. For any $x, y \in R^k$, the vote for x over y is $v(x, y) = \mu\{i \in I : x \underset{i}{\succ} y\}$. (Thus, implicitly, indifferent voters do not vote at all, though they could alternatively be assumed to divide evenly without affecting any of the conclusions below.) If candidates 1 and 2 adopt as strategies x and y respectively, the vote share of 1 is $v(x, y)$, and his plurality, or margin over the other candidate is $v(x, y) - v(y, x)$; similarly for candidate 2. Each candidate is interested in winning more votes than his opponent, and is assumed to maximize some function of his plurality. More precisely, we assume the payoff functions can be obtained from a function $f : [-1, 1] \to R$, common to both candidates, such that the payoff for j is $f(v(x, y) - v(y, x))$ when j plays x and the other candidate plays y. $f(\cdot)$ is assumed to be continuous, strictly increasing, and symmetric around zero in the sense that $1/2[f(x) + f(-x)] = f(0) = 0$, for any $x \in [-1, 1]$. If the candidates maximize their expected pluralities, $f(\cdot)$ is the identity function. On the other hand if they pursue the more conservative objective of maximizing their respective probabilities of winning, this is equivalent to maximizing the expectation of a function which takes on value 1 for positive pluralities, and zero otherwise. Though such a function will have a discontinuity at zero, clearly we can approximate it as closely as desired by a continuous function in the class defined above; and in any event the probability-of-winning objective is probably best thought of as a convenient (for some purposes) simplification of a more general, continuous objective function.

The essential condition needed to establish existence is that the electorate be infinitely large, with each individual voter of negligible weight relative to the electorate as a whole, and that voter preferences be distributed over the policy space in continuous fashion. We formalize these premises as follows: say that two preference orderings $\succsim$, $\succsim'$ are of the same <u>type</u> if there exists $z \in R^k$ such that for all $x, y \in R^k$, $x \succsim y \Leftrightarrow x + z \succsim' y + z$; a type is thus an equivalence class of orderings, and two orderings of the same type differ only in the locations of their respective satiation points. Every ordering thus has a unique representation $(s, t) \in R^k \times \mathcal{T}$ where $s \in R^k$ is a satiation point and $\mathcal{T} \ni t$ is the set of types

Let the mapping $\mathcal{P} : I \to R^k \times \mathcal{T}$ which assigns preferences to voters be measurable with respect to $\mathcal{I}$ and $\mathcal{B} * \mathcal{T}$, the σ-algebra generated by the class $\mathcal{B}$ of Borel sets of R^k and some σ-algebra $\mathcal{T}$ of $\mathcal{T}$. From the vote distribution μ we can then derive a preference distribution ν on $R^k \times \mathcal{T}$, where for any measurable $T \in \mathcal{T}$, $\nu(T) = \mu\{i \in I : \mathcal{P}(i) \in T\}$. The marginal distribution across types is the measure ν_t on $\mathcal{T}$ given by $\nu_t(T) = \nu(R^k \times T)$, for any $T \in \mathcal{T}$. Finally, let μ_L denote Lebesgue measure on $\mathcal{B}$.

Our assumption on the preference distribution is, now, that the measure ν is absolutely continuous with respect to the product measure $\mu_L \times \nu_t$. (This is essentially equivalent to requiring that the conditional distribution of satiation points for any given type is representable by a finite density function on R^k, if these conditional probabilities are defined; the assumption above is a more general version of this condition.)

2. EXISTENCE OF EQUILIBRIUM

To recall some standard terminology, a zero-sum game is a triple (C, D, g), where C and D are sets and $g : C \times D \to R$ is a real-valued function on $C \times D$. The game has a value, k, if $\sup_c \inf_d g(c, d) = \inf_d \sup_c g(c, d) = k$, c and d ranging over C and D, respectively. $\hat{c}$ is optimal for 1 if $g(\hat{c}, d) \geq k$, all $d \in D$, and $\hat{d}$ is optimal for 2 if $g(c, \hat{d}) \leq k$, all $c \in C$; a pair $(\hat{c}, \hat{d})$ of optimal strategies is an equilibrium. If C is a set of mixed strategies, i.e., of probability measures on (a σ-algebra of) some underlying set $\underline{C}$ of pure strategies, then player 1 has ϵ-optimal strategies with finite support if for every $\epsilon > 0$ there exists a strategy $c' \in C$ with finite support in $\underline{C}$ such that $g(c', d) \geq k - \epsilon$, all $d \in D$; similarly for player 2. Strategies with finite support are actually playable, since they involve randomization over only a finite subset of pure strategies. ϵ-optimality thus ensures that a player can (nearly) achieve his value in the game by means of actually-playable strategies.

To now formulate electoral competition as a game, denote by A the set of pure strategies a available to candidate 1, and by $\widetilde{A}$ his mixed strategies α. A is assumed to be a compact convex subset of R^k, and $\widetilde{A}$ is the space of probability measures on the Borel sets of A. $b \in B$ and $\beta \in \widetilde{B}$ are similarly defined for 2. Clearly, $f[v(a, b) - v(b, a)] + f[v(b, a) - v(a, b)] = 0$ for all a and b, so the game is zero-sum, and these functions are evidently measurable on the Borel sets of A x B, so the extended payoff function

$$\widetilde{f}(\alpha, \beta) = \int f[v(a, b) - v(b, a)]d(\alpha \times \beta)$$

is well-defined for all α and β. Say a candidate is dominant if he has a pure strategy which guarantees him a nonnegative plurality; thus 1 is dominant if for some $a' \in A$, $v(a', b) - v(, a') \geq 0$ for all $b \in B$, and similarly for 2. We now have:

THEOREM. The mixed-strategy electoral game $(\tilde{A}, \tilde{B}, \tilde{K})$ has a value. If both candidates are dominant, both have optimal pure strategies; otherwise, each non-dominant candidate has an optimal mixed strategy, while a dominant candidate has an ϵ-optimal strategy with finite support.

In the usual kind of unconstrained electoral contest, we would set A and B equal to some large set D within which most voters are satiated, and from symmetry the value will be zero. If, exceptionally, there exists a Condorcet winner $d \in D$ (i.e., $v(d, d') \geq 1/2$, all $d' \neq d$), both candidates are dominant and (d, d) is a pure strategy equilibrium. In the more typical case of no Condorcet winning policy, neither candidate will be dominant, and both will have optimal, as well as ϵ-optimal, mixed strategies.

The formulation of the theorem also allows the more general situation in which the candidates' strategy sets differ, whether because of ideological constraints, assymetries in the role of incumbent and challenger, or whatever. Differences of this kind make the game assymetrical, and may bias the value in favor of one candidate or the other, or even make one of them dominant. ϵ-optimal strategies exist for both candidates, and the non-dominant candidate will have an optimal mixed strategy. The dominant candidate, however, may not: he may find it always possible to improve slightly upon any particular strategy, so that none of his strategies is strictly optimal. As an example of this, suppose the second candidate is obliged to play some fixed strategy b^* (perhaps because he is incumbent and must defend his record in office), while candidate 1 can play some larger set $A \ni b^*$. Unless b is a Condorcet winner in A, some strategy $a \neq b^*$ will guarantee a strictly positive plurality for 1, making him dominant. But if voters are densely distributed on some neighborhood of b^*, there will exist some convex combination $\lambda a + (1 - \lambda)b^*$, $\lambda \in (0, 1)$, which yields an even higher plurality against b^* (cf. Remark 2, Section 3). Hence none of 1's pure strategies is optimal, and this reasoning extends readily to his mixed strategies.

3. PROOF OF THE RESULT

The proof makes use of a generalized form of the minmax theorem, which is a special case of an even more general result (Proposition 3 bis) proven by J. F. Mertens and S. Zamir in their forthcoming volume on game theory. Let A and B be compact subsets of R^k, and the bounded real-valued function $K : A \times B \to [c, d]$ be measurable on the Borel sets of A x B. Denote by $\tilde{A}$ (resp. $\tilde{B}$) the space of probability measures on the Borel sets of A (resp. B), and by $\tilde{K}$ the extended payoff function $\tilde{K}(\alpha, \beta) = \int K(a, b)d(\alpha \times \beta)$, all $(\alpha, \beta) \in \tilde{A} \times \tilde{B}$. For each $b \in B$ define the function $\bar{\varphi}_b(a) = \limsup_{a' \to a} K(a', b)$, and let $\bar{F}$ be the class of functions g on A for which $g(\cdot) \geq \limsup_{n \to \infty} \bar{\varphi}_{b^n}(\cdot)$ for some sequence $b^n \in B$. Similarly define $\underline{\varphi}_a(b) = \liminf_{b' \to b} K(a, b')$ and $\underline{F}$, the family of functions g on T for which $g \leq \liminf_n \underline{\varphi}_{a^n}$ for some sequence $a^n \in A$. Then we have:

PROPOSITION (Mertens - Zamir). If $K(\cdot, b) \in \bar{\bar{F}}$ for every $b \in B$, the game $(\tilde{A}, \tilde{B}, \tilde{K})$ has a value. Player 1 has an optimal strategy, and player 2 has an ε-optimal strategy with finite support. Similarly when reversing the roles of the players, if $K(a, \cdot) \in \underline{F}$ for all $a \in A$.

For a proof, see Mertens and Zamir [1978]. Clearly the payoff function of the electoral game is bounded; it remains to be shown that it satisfies the condition of the Proposition. We first note two properties of the vote-share function $v(\cdot, \cdot)$:

Remark 1. $v(x, y) + v(y, x) = 1$ if $x \neq y$, $= 0$ otherwise. For fixed y the function $g(x) = v(x, y)$ is continuous except at $x = y$; similarly for fixed x, the function $h(y) = v(x, y)$.

Proof. Partition I into $A_1 = \{i : x \succ_i y\}$, $A_2 = \{i : x \sim_i y\}$ and $A_3 = \{i : x \prec_i y\}$; clearly $v(x, y) + v(y, x) = \mu(A_1) + \mu(A_2) = 1 - \mu(A_3)$. If $x = y$ all voters are indifferent and $\mu(A_3) = 1$. To prove the rest of the first assertion we now show that $\mu(A_3) = 0$ if $x \neq y$. Define $B = \mathcal{P}(A_3)$ and let B^t denote the section $B^t = \{s : (s, t) \in B\}$. The strict quasi-concavity assumption implies that within any single type t the set of satiation points which make x and y indifferent is of Lebesgue measure zero, i.e., $\mu_L(B^t) = 0$, all t; hence $\mu_L \times \nu_t(B) = \int \mu_L(B^t)d\nu_t = \int 0 \cdot d\nu_t = 0$. Since ν is absolutely continuous with respect to $\mu_L \times \nu_t$, this implies $\mu(A_3) = \nu(B) = 0$.

To show the continuity of g, suppose $x \neq y$, let $x^n \to x$, and define $C^n = \{i \in I : x^n \succsim_i y\}$. Clearly $\mu(C^n) = \mu(C^n \cap A_1) + \mu(C^n \cap A_2) + \mu(C^n \cap A_3)$, and from the argument of the preceding paragraph, $\mu(C^n \cap A_3) = 0$ except possibly for some finite number of the C^n. From the continuity assumption on voter preferences,

$$\limsup_{n\to\infty} C^n \cap A_1 = \liminf_{n\to\infty} C^n \cap A_1 = A_1$$

(we henceforth write this as: $\lim_n C^n \cap A_1 = A_1$), and similarly $\lim_n C^n \cap A_2 = \phi$. Hence

$$\begin{aligned} \lim_n \mu(C^n) &= \mu(\lim_n C^n \cap A_1) + \mu(\lim_n C^n \cap A_2) \\ &\qquad + \lim \mu(C^n \cap A_3) \\ &= \mu(A_1) + \mu(\phi) + 0 \\ &= v(x, y), \end{aligned}$$

Q.E.D.

The next proposition characterizes the discontinuity that arises as $x \to y$:

Remark 2. Suppose $x \neq y$, and define $q(\lambda) = y + \lambda(x - y)$ and $g(\lambda) = v(q(\lambda), y)$, $\lambda \in (-\infty, \infty)$. Then g(.) is a monotonically increasing (resp. decreasing) function of λ for all $\lambda < 0$ (resp. > 0), and

$$\lim_{\lambda \to 0^+} g(\lambda) = 1 - \lim_{\lambda \to 0^-} g(\lambda).$$

Similarly for $v(x, \cdot)$.

Proof. Let $\lambda^n \overset{+}{\to} 0$ be a monotone sequence. Define $x^n = q(\lambda^n)$ and $z^n = q(-\lambda^n)$. For each n, partition I into

$$\begin{aligned}
C_1^n &= \{i : x^n \succ_i y \quad \text{and} \quad y \succ_i z^n\} \\
C_2^n &= \{i : z^n \succ_i y \quad \text{and} \quad y \succ_i x^n\} \\
C_3^n &= \{i : y \succ_i x^n \quad \text{and} \quad y \succ_i z^n\} \\
C_4^n &= \{i : x^n \succ_i y \quad \text{and} \quad z^n \succ_i y\} \\
C_5^n &= \{i : x^n \sim_i y \quad \text{or} \quad z^n \sim_i y\}.
\end{aligned}$$

Since this is an exhaustive classification we can write

$$\begin{aligned}
v(x^n, y) + v(z^n, y) &= \mu(C_1^n \cup C_4^n) + \mu(C_2^n \cup C_4^n) \\
&= \mu(C_1^n) + \mu(C_4^n) + \mu(C_2^n) + \mu(C_4^n) \\
&= 1 + \mu(C_4^n) - \mu(C_3^n) - \mu(C_5^n).
\end{aligned}$$

Quasi-concavity implies $C_4^n = \phi$ and that C_1^n and C_2^n are monotone increasing, which proves monotonicity of $g(\cdot)$, and also that the limits the payoff must both be ≥ 0. But there is also a strategy b' for 2 which guarantees these quantities are ≤ 0. Hence the value of the game is zero, and (a', b') is a pure strategy equilibrium.

The remaining possibility is that at least one of the candidates, say 1, is not dominant. For each $b \in B$ define

$$\bar{h}_b(a) = \limsup_{a' \to a} [v(a', b) - v(b, a')],$$

and note that

$$\bar{h}_b(a) = v(a, b) - v(b, a) \quad \text{if} \quad a \neq b$$

from the continuity part of Remark 1. If $b \in A \cap B$, let b^* be a strategy in B for which $v(b^*, b) > 1/2$; such a strategy must exist, for otherwise 1 would be dominant, since $b \in A$. Define the sequence $b^n = b + 1/n(b^* - b)$, and note that $v(b, b^n) < 1/2 < v(b^n, b)$, from Remark 2. Hence, if $a = b$,

$$\limsup_n \bar{h}_{b^n}(a) = \limsup_n \bar{h}_{b^n}(b)$$

$$= \limsup_n [v(b, b^n) - v(b^n, b)] \leq 0 = v(b, b) - v(b, b).$$

Otherwise, if $a \neq b$, b^n is distinct from a for all but possibly one n, so

$$\limsup_n \bar{h}_{b^n}(a) = \limsup_n [v(a, b^n) - v(b^n, a)]$$

$$= v(a, b) - v(b, a)$$

from Remark 1 (continuity) and the fact that $a \neq b$.

For $b \in B - A$, take the sequence $b^n = b$, and the last conclusion holds for all a. Hence, for any $b \in B$, $v(\cdot, b) - v(b, \cdot) \in \bar{F}$. The continuity and monotonicity of f also ensure that the same conclusion holds for the more general payoff function $f[v(\cdot, b) - v(b, \cdot)]$, so the Theorem is proved.

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