A Dynamical Model of Political Equilibrium*

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I. Introduction

In a democratic society many social decisions are made by a political mechanism based on majority voting by the citizenry. Thus a fundamental task in analyzing the performance of a democratically structured public sector is to characterize the behavior of competitive voting processes, and in particular to see whether they lead to consistent social choices, or can be characterized by an equilibrium of some sort. These questions have been extensively studied, from several points of view. However, the major results in this voting theory literature (which we will discuss in more detail below) are either essentially negative, in the sense that they show that an equilibrium can exist only in very restrictive special cases, or else involve unsatisfactory ad hoc assumptions and formulations, or serious difficulties of interpretation (e.g., concerning the meaningfulness and existence of mixed strategies for certain agents in the process).

In this paper we take a different approach to the problem, which is more explicitly dynamical in character. We investigate the behavior of a political process which is driven by a competition for votes extending across an indefinite series of elections. In each period two political parties compete for votes by advocating particular policies or alternatives, and whichever party wins takes office and puts the policies it advocated into effect. In the next period a new election is held and another policy enacted. As this process is repeated over time, a sequence of successively enacted “winning” policies is generated. Our analysis focuses on the behavior of these sequences, or trajectories, of policies (which we assume can be represented as points in an appropriately defined policy or commodity space). We show that over time, these sequences converge on a particular subset of the feasible alternatives. This subset, typically a small proper subset of the Pareto set, thus seems to

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constitute a natural and useful type of equilibrium for this class of political processes. It can also be interpreted, in a more abstract social choice theory context, as the set of maximal elements of an essentially Arrowian social preference ordering.

To give a rough idea of the nature of the equilibrium, consider a collection of $n$ voters, each with a (complete, transitive) preference ordering defined on a set $A$ of alternatives. For simplicity we suppose all preferences are strict (no voter is indifferent between any pair of alternatives), and to avoid inessential detail we ignore continuity and related considerations, and simply assume by fiat that the various maxima and minima referred to below all exist. (These simplifications are relaxed later.) For any pair $x$, $y$ of alternatives, we define $n(x, y)$ as the number of voters who prefer $y$ to $x$; and by varying $y$ over the feasible set, we can also define the maximum vote against $x$, $v(x) = \max_{y \in A} n(x, y)$. This maximum-vote function gives a simple characterization of the usual type of voting equilibrium, a Condorcet or majority winner: a Condorcet winner is by definition an alternative which cannot be defeated in a pairwise majority vote, so clearly an alternative $x$ satisfies this condition if and only if $v(x) \leq n/2$. ($x^*$ in Fig. 1 is an example.)

\[\begin{align*}
\text{Votes} \\
\text{Alternatives}
\end{align*}\]

\[\begin{align*}
&n^* \\
&n/2 \\
&v(x) \\
&v'(x)
\end{align*}\]

**Fig. 1.** Maximum-vote functions.

As is well known, there are many voting situations in which no majority equilibrium of this kind exists, since pairwise majority voting leads to cycles and the "voters' paradox" phenomenon; this is virtually inevitable if the alternatives being voted on are points in a multidimensional commodity or
policy of some sort, for example. In such cases the maximum-vote function is always greater than \( n/2 \) in value (as with \( v'(\cdot) \) in Fig. 1), but its \textit{minima} are still of interest, at least heuristically. For example, we can interpret the gap between the minimum value of the maximum-vote function and \( n/2 \) as a measure of "how close" the society comes to having majority equilibrium. Similarly, we can think of the alternatives on which this minimum is achieved (such as \( x^* \) in Fig. 1) as the ones which "most resemble" Condorcet winners, and regard them as "pseudoequilibria" of some sort.

This set of minima of \( v(\cdot) \), or \textit{minmax} set (to use Simpson's [28] term), is always nonempty, and coincides with the set of majority winners whenever they exist. In this sense it generalizes the Condorcet winner concept. There are other ways of generalizing the Condorcet criterion as well, however, and it is not immediately apparent that this minmax approach is in any sense a more natural or useful extension. We shall show, however, that (under the assumptions developed in Section 3) the sequences of policies generated by the competitive electoral process described above, operating on a set of alternatives or social states which can be represented as points in an appropriately defined multidimensional issue space, necessarily converge on this minmax set. In this sense the minmax set emerges as a very natural and robust equilibrium for this type of process, and generalizes the well-known Hotelling-Downs results on electoral competition to a much broader class of situations.

The minmax notion is also of interest from a more normative, social choice theory point of view. In particular it can be shown that there exists a certain essentially Arrowian social welfare function or ordering, whose optimal or maximal elements are precisely the members of the minmax set. This ordering is based on a type of voting rule, though in contrast to the usual "fixed" voting rules, such as the Pareto or unanimity ordering \((n)\) or simple majority rule \((n/2)\), the minmax relation involves a "variable" rule, in which the number of votes required for one alternative to be socially preferred to another will vary from society to society. In particular, the minmax ordering for a particular society or profile is based on the sharpest or most decisive of the voting rules which are capable of aggregating that particular profile of individual preferences into a consistent (acyclic) social ordering. The idea that the appropriate voting rule should depend on the degree of social consensus or potential divisiveness of an issue is not a new one, of course, and indeed has some reflection in common constitutional and parliamentary practice. This "variable-majority" rule is more complex than the usual type of social choice method, but it yields a social ordering very much in the spirit of Arrow's general approach, and which satisfies appropriate weakenings of his axioms.
2. Voting on Multidimensional Choice Spaces

Before developing the dynamical model in detail, it will be useful to provide some background and motivation for our approach by briefly reviewing the major results on voting and electoral competition over multidimensional choice spaces.

When the underlying set of alternatives or polices being voted on is one dimensional—e.g., if society is deciding on the amount of a single public good to be provided, or on the level of a single policy instrument—then majority voting is typically well behaved, and as a consequence the Hotelling–Downs electoral equilibrium will generally exist. In the one-dimensional case, the convexity of voter preferences often implies satisfaction of Black’s “single-peakedness” condition, which in turn implies that the majority preference relation is transitive, and that there exists a Condorcet winner [3]. Even when the single-peakedness condition fails (e.g., because of nonconvexities in individual preferences, or the feasible set), there generally exist “local” equilibria, i.e., alternatives which cannot be defeated by neighboring alternatives [16]. Thus in one-dimensional choice problems, the behavior of political mechanisms based on majority rule can normally be described in terms of an equilibrium.

The situation is quite different, however, when there are two or more goods or quantities to be (simultaneously) voted on. In this multidimensional case, where the alternatives are points in $\mathbb{R}^k$, $k > 1$, and voter preferences satisfy the standard continuity and convexity assumptions, the usual conditions for transitivity of majority rule (e.g., [3, 24, 26]) all fail, in general [15]. Some rather different conditions for the existence of a majority winner in multidimensional voting situations have been proposed [6, 7, 21, 31], but they are quite restrictive. For example, the Plott conditions, which in the finite-voter case are both necessary (with minor qualifications) and sufficient, in effect require that for each voter who prefers a change (from the equilibrium) in one direction, there is one other voter with precisely the opposite preference. This very special type of symmetry in the distribution of voter preferences will be satisfied only accidentally, and is not preserved under arbitrarily small perturbations of voter preferences; hence the Plott conditions in effect show that it is an exceptional and unstable occurrence for a Condorcet winner to exist in multidimensional voting problems.

Several authors, notably Tullock, have argued that this instability is a “small-sample” problem, and that majority equilibria will be more likely when the number of voters is large; examples and results supporting this thesis have been exhibited by Tullock [30] Davis and Hinich [6], Arrow [2], and others. Most of this work on infinite electorates (and on multidimensional voting problems in general) has assumed voter preferences to be of special form, in which each voter $i$ is satiated at a unique point $s^i$, and his
utility for any point $x$ decreases monotonically with the distance (by some metric, such as the Euclidean) from $x$ to $s^*$. (We shall refer to this as a Type I utility function henceforth.) The distribution of preferences is given by a density function $f$, $f(z)$ being the density (relative frequency, in the finite electorate case) of voters satiated at the point $z \in R^d$. The Tullock and Davis–Hinich examples are, in fact, rather special ones, since each uses a density function which is radially symmetric about some point $\theta$ (in the sense that $f(z) = f(\theta - (z - \theta))$, for any $z$). Moreover, McKelvey, Ordeshook, and Ungar [19] have subsequently shown that a condition only slightly weaker than radial symmetry is necessary as well as sufficient for the existence of a Condorcet winner. Their condition is a natural analog of the Plott conditions for the finite-voter case, and is just as strong. Thus in general, majority voting is not transitive and does not yield a majority winner in multidimensional voting problems, irrespective of the size of the electorate.

Conceivably majority rule might still be "well behaved" in some weaker sense. Buchanan [4], for example, has suggested that even when voting cycles do arise in such problems, the cycles will be confined to the Pareto optimal set; while Tullock [30], has gone further and argued (at least for Type I preferences) that the cycling will tend to move toward a central area in the interior of the Pareto set, and remain there. There have been related proposals in the social choice literature to define a social preference relation by the transitive closure of the majority preference relation (so that any two alternatives belonging to the same cycle are socially indifferent) [10, 14, 25], presumably based on the hope that the "top" cycle will be confined to a relatively small subset of the alternatives, akin to Tullock's "central region."

However, a result by McKelvey [19] shows that in multidimensional choice problems, there will typically be majority rule cycles which extend over the entire feasible region, and hence that all feasible alternatives are socially indifferent under the transitive closure relation. In particular, he has shown that if all voters have Type I preferences and no majority winner exists, then for any two alternatives $x, y$, a sequence of points $(x, x', x'', ..., y)$ can be found, which begins with $x$ and ends with $y$, such that each point is preferred by a majority in the preceding point. An illustration of such a trajectory is given in Fig. 2. There are seven voters, whose most-preferred points are labeled $s^1$ to $s^7$, respectively. Each voter has a Type I preference ordering and hence, given the choice between any two points $x$ and $y$, will prefer the point which is closest to his satiation point. The Pareto-optimal set is the shaded area. Beginning from an arbitrary initial point $x^1$, each subsequent point is preferred by a majority to the previous one. The point $x^2$ is closer to $s^1, s^2, s^3$, and $s^4$ than is $x^1$, so the majority composed of voters 1, 2, 3, 4 prefers $x^2$ to $x^1$; similarly $[3, 4, 5, 6]$ prefers $x^3$ to $x^2$, and so forth. The trajectory begins at a centrally located point $x^1$, but soon moves outside the Pareto set itself, and clearly could be extended to move further and further.
away from it, or to eventually reach an arbitrarily chosen point anywhere in
the feasible region. Though the McKelvey result (and the example) invokes
the assumption of Type I preferences, it seems clear that the conclusion is a
general one, and that except under very special conditions or restrictions on
voters' preferences, in multidimensional choice problems majority rule can
quite literally wander from anywhere to anywhere in the space.

Most of the work described above has been primarily concerned with
majority rule in the abstract, and not with any institutional mechanism for
implementing it, but these results are also relevant to the analysis of electoral
competition, the mechanism with which we shall be concerned. In the classical
work in this area [8, 13] two political parties are assumed to compute for
votes by advocating particular policies, or alternatives, and to be interested
only in winning the election, not in policies as such, and thus willing to adopt
whichever policies maximize their prospects in the election. In every period
an election is held, with each voter voting for the party whose policy he
prefers, and whichever party receives a majority is elected, and enacts the
alternative it advocated. Downs and Hotelling both assumed the underlying
space of policies to be one-dimensional, and (implicitly) voter preferences to
be single peaked, and argued that under these conditions the parties would
tend to converge, in their platform choices, on an equilibrium policy at the
median of the distribution of voters’ most-preferred points. From a more rigorous game-theoretic point of view, this competition for votes can be viewed as a two-player, zero-sum game between the parties, in which the feasible policies are the pure strategies available to each player, and the parties’ vote shares (or perhaps their respective probabilities of winning) are the relevant payoffs. It is not difficult to show that a pure strategy equilibrium for this electoral game will exist if, and only if, the electorate’s preferences are such that some feasible policy is a majority winner. In the one-dimensional case this is ensured by the single-peakedness of voter preferences, implicitly assumed by Downs and Hotelling. When the underlying policy space is multidimensional, however, the single-peakedness condition will not hold, and in general no Condorcet winner and hence no pure strategy equilibrium for the parties will exist.

Shubik [27] and McKelvey and Ordeshook [18] have explored the possibility and nature of a mixed strategy equilibrium for the parties. It has been shown that the set of policies which are played with positive probability (or density) in such an equilibrium constitutes a subset of the Pareto optimal policies. No sharp characterization of this subset (or of the equilibrium probability mixture itself) has yet been given, however. In fact, it is an open question whether such a mixed strategy equilibrium exists at all: the usual versions of the minimax theorem do not apply, since the pure strategy sets are infinite and (as is easily shown) the payoff function is neither continuous nor concave in each party’s strategies. There are also interpretive difficulties with this mixed-strategy approach. It is tempting to think of mixed strategies as uncertain or “ambiguous” policy commitments by the parties. If they were so interpreted, however, voters would be in the position of voting on lotteries over policies. The relevant pure strategies for each party would then be the set of possible lotteries, and within these expanded strategy sets no pure strategy equilibrium will exist, in general (Ordeshook [20]). Thus one must interpret mixed strategies as requiring the parties to simultaneously and randomly choose unambiguous or “pure” policies in advance of the electoral campaign. Under these conditions, however, either party could benefit by postponing its choice until its opponent commits itself to a policy, and then choosing (nonstochastically) a policy which ensures its victory. Hence the parties have strong incentives to abandon their equilibrium mixed strategies, so it is not clear that a mixed strategy equilibrium (even if one does exist) is operationally or descriptively meaningful in the context of electoral competition.

A rather different approach proceeds from the premise that some citizens will (stochastically) abstain from voting under certain conditions, for example, if there is little difference between the parties’ policies. Particular versions of this type of assumption yield a pure strategy equilibrium for the parties [11, 23]. This type of equilibrium does not seem a very robust or compelling
one, however. Its existence is sensitive to the specification used, and the particular formulations needed to ensure existence seem rather ad hoc, and not particularly consonant with the available empirical evidence on voter abstention [29]. In any event, such an equilibrium would fail to exist whenever compulsory voting laws or the salience of the election resulted in high voter turnout, and seems of little normative interest.

3. Sequential Electoral Competition

Most of the work described above has been concerned with extending the Hotelling-Downs analysis of electoral competition in an essentially static framework, and with characterizing an equilibrium in the context of a single electoral contest. Here we take a different approach, and embed the problem in a dynamic framework. We assume the two parties compete repeatedly, over an infinite series of elections. In each period one of the parties is elected, enacts the policy it advocated, and in the next election must defend this same policy. The “out” party may adopt any policy it wishes, to maximize its prospects in the next election. Since in general there will be no “majority winner” policy, the incumbent’s policy can always be defeated, so the “out” party will always win, and the two parties will alternate in office. A sequence of successively enacted “winning” policies is thus generated by this process.

Clearly one property of these sequences, or trajectories, is that each point is preferred by a majority of voters to the point which precedes it. This implies little, since in the absence of a Condorcet winner, such a majority trajectory need not converge on any single policy, and indeed, in view of the McKelvey [17] result, may continue to wander about the entire feasible set indefinitely. But many of the majority trajectories will not arise under competitive conditions, except under rather implausible assumptions about the objectives of the parties. In particular, to argue that any such trajectory might be generated by such a competitive process, it must be assumed that a party may choose any point \( y \) which defeats the incumbent’s policy \( x \), irrespective of the margin by which \( y \) defeats \( x \), and thus that it has zero marginal utility for votes, once a bare majority is achieved: it satisfies Riker’s “size principle” with a vengeance [22]. This assumption does not seem a very plausible one for electoral competition. Uncertainty about the election outcome itself, or in a parliamentary system, about future defections or deaths among the majority party, provides a strong incentive for parties to strive for larger-than-minimal majorities, as risk insurance. Moreover, a large winning margin is generally valued in itself as a “mandate” for the victor, and in many systems brings tangible benefits such as increased patronage, and the election of legislators from marginal districts whose indebtedness to the party leadership ensures a more cooperative legislature. Recent United States Presidential
elections, particularly the 1964 and 1972 contests, certainly provide little empirical evidence of any tendency for leading candidates to "satisfice" and settle for minimal margins.

Thus we shall adopt the stronger premise of the original Hotelling–Downs analysis of two-party competition, and assume that the parties attempt to maximize their vote shares.\footnote{A more natural assumption would be that each party attempts to maximize its plurality, or margin over the opposing party, since under the usual definition of majority rule, a party whose maximum vote share is less than its opponent's would still lose the election. Under our assumptions, however, this distinction is inessential, and vote- and plurality-maximization are equivalent (this is shown in Proposition 2 below).} Under this assumption, the trajectories generated by electoral competition are such that each policy is vote-maximizing against the preceding policy. It will be shown that these vote-maximizing trajectories are relatively well behaved, in contrast to the majority trajectories, and display strong convergence properties.

With this by way of motivation, we now introduce some formal notation and definitions:

There are \(n\) voters, denoted by \(N = \{1, 2, \ldots, n\}\). The alternatives are points in the Euclidean \(k\)-space \(R^k\), where \(k < n\). Each voter \(i\) has a preference ordering \(\succeq_i\) over the points in \(R^k\), representable by a Type I utility function, satiated at a unique point \(s_i\). Thus \(x \succeq_i y\) iff \(\|x - s_i\| \leq \|y - s_i\|\), where \(\|z\| = \left(\sum_{i=1}^{k} z_i^2\right)^{1/2}\). In what follows all points in \(R^k\) are assumed to be feasible, though all results would also hold without qualification if the set of feasible points \(A\) were a compact, convex body in \(R^k\), with each voter satiated in the interior of \(A\).

For any two points \(x, y\), the vote for \(y\) against \(x\), \(n(x, y)\), is the number of voters for whom \(y\) is (strictly) preferred to \(x\), i.e., \(n(x, y) = \{i \in N : y \succ_i x\}\). For any \(x\), we denote by \(v(x)\) the \textit{maximum vote against} \(x\), i.e., \(v(x) = \max_y n(x, y)\). If \(y\) is a point for which \(n(x, y) = v(x)\), then \(y\) is vote-maximizing against \(x\). If \(v(x) \leq n/2\) for any \(x\), then (and only then) \(x\) is a majority winner, since no other point can defeat it in a pairwise majority vote.\footnote{A slightly different definition of majority voting and of a majority winner is normally used in the voting theory literature, whereby \(x\) defeats \(y\) iff more voters prefer \(x\) to \(y\) than prefer \(y\) to \(x\). We can call this the \textit{relative} majority definition, given by: \(x \succ_P y\) iff \(n(x, y) > n(y, x)\). We have been implicitly using the game-theoretic, \textit{absolute} majority definition, in which \(x \succ_A y\) iff \(n(x, y) > n/2\); evidently \(x \succ_P y\) implies \(x \succ_A y\), though the converse need not hold if some voters are indifferent. Clearly a relative-majority winner is also an absolute-majority winner; moreover, under our assumptions, the converse of this is also true. More generally, even without the Type I assumption, we have:}

**Proposition 1.** If voter preferences are strictly convex, every absolute-majority winner is also a relative-majority winner (and conversely).

\textit{Proof.} Suppose to the contrary that \(y \succ_P x\), where \(x\) is an absolute-majority winner. Clearly \(y \succ x\), so from strict convexity, if \(z = \alpha x + (1 - \alpha)y\), \(\alpha \in (0, 1)\), then \(y \succeq_i x\) implies \(z \succ_i x\) for all \(i \in N\), whence \(n(x, z) > n(x, y) + I(x, y)\), where \(I(x, y) = \{|i \in N:\)}
though, no majority winner exists, so \( t(x) > \frac{n}{2} \) at all \( x \). The minimum value of \( t(x) \) is the \textit{minmax number} \([28]\), which we denote by \( n^* \): thus \( n^* = \min_x t(x) \). (Since the range of \( n'() \) is finite, the maxima and minima referred to in these definitions both exist.) Finally the set of points \( x \) for which \( t(x) = n^* \) is the \textit{minmax set}, denoted by \( M(n^*) \).

As noted above, our analysis focuses on the sequences or trajectories of alternatives which are generated under repeated electoral competition, where by a \textit{trajectory} we mean a sequence \((x^t)\), \( x^t \in \mathbb{R}^k \), all \( t = 1, 2, \ldots \). A \textit{vote-maximizing} trajectory\(^8\) is one such that for all \( t \), \( x^{t+1} \) is vote maximizing against \( x^t \).

\( x \sim y \) is the number of voters indifferent between \( x \) and \( y \). By hypothesis, \( y \text{P}_{\text{Max}} \), i.e., \( n(x, y) > n(y, x) = n - n(x, y) - I(x, y) \). But this implies \( n < 2n(x, y) + I(x, y) \leq 2n(x, z) \) and thus \( z \text{P}_{\text{Max}} \), a contradiction which proves the result.

Hence \( t(x) \leq \frac{n}{2} \) is necessary and sufficient for \( x \) to be a majority winner under either definition of majority rule.

\(^8\) As noted earlier, it might be more natural to suppose that each party is interested in maximizing its plurality, or margin over its opponent, rather than its vote share, as assumed above. Under our assumptions, however, plurality- and vote-maximization are equivalent, and lead to precisely the same set of trajectories. To show this, first define the \textit{plurality} for \( y \) over \( x \), as \( n(x, y) - n(y, x) \). (This implicitly assumes indifferent voters do not vote; we could alternatively assume they divide evenly, without affecting the conclusion below.) The \textit{maximum} plurality against \( x \), \( p(x) \), is then

\[ p(x) = \max_y \{ n(x, y) - n(y, x) \} \]

Then we have:

**Proposition 2.** If voter preferences are \textit{strictly convex} then \( p(x) = 2t(x) - n \) for any \( x \) which is not a Condorcet winner, and a point which maximizes votes against \( x \) also maximizes the plurality over \( x \), and conversely.

\textit{Proof.} We first show that if any voter is indifferent between \( x \) and some point \( y \), then \( y \) cannot maximize either objective function. Since \( x \) is not a Condorcet winner, \( t(x) > \frac{n}{2} \), whence \( p(x) > 0 \). Hence the point \( y = x \) cannot maximize either function, since that would imply either \( t(x) = 0 \) or \( p(x) = 0 \), a contradiction. The remaining possibility is \( y \neq x \). Let \( z = \alpha x + (1 - \alpha) y \) for some \( 0 < \alpha < 1 \). From the strict convexity of voter preferences, it follows that \( y \succeq z \succeq x \) or, equivalently \( x \succeq z \succeq y \), implying \( n(x, z) \geq n(x, y) \) and \( n(z, x) \leq n(y, x) \), respectively. If any voter were indifferent between \( x \) and \( y \) the first of these inequalities would be strict, and \( y \) would not maximize either votes \((n(x, z))\) or plurality \((n(x, y) - n(y, x))\) against \( x \).

Hence if \( y^* \) maximizes \( n(x, y) \), no voters are indifferent between it and \( x \), so \( n(x, y^*) + n(y^*, x) = n \). The resulting plurality for \( y^* \) over \( x \) must then satisfy

\[ p(x) > n(x, y^*) - n(y^*, x) = 2n(x, y^*) - n = 2t(x) - n. \]

Similarly, if \( y^{**} \) maximizes the plurality against \( x \), then we must also have

\[ p(x) = n(x, y^{**}) - n(y^{**}, x) = 2n(x, y^{**}) - n < 2t(x) - n. \]

This implies the weak inequalities above are actually equalities, and hence that \( y^* \) and \( y^{**} \) simultaneously maximize both functions.

Q.E.D.
With these definitions, we can now state the main result, to be proven in the next section. Define the distance \( d(x, A) \) from a point \( x \in \mathbb{R}^k \) to a set \( A \subseteq \mathbb{R}^k \) in the obvious fashion: \( d(x, A) = \inf_{y \in A} \| x - y \| \). Then we have:

**Theorem 1.** Let \( (x^t) \) be a vote-maximizing trajectory. For any \( t \), if \( x^t \) does not belong to the minimax set \( M(n^*) \), then

\[
d(x^{t+1}, M(n^*)) < d(x^t, M(x^*)).
\]

The proposition thus implies that on any vote-maximizing trajectory the distance to the minimax set must be monotonically decreasing (until the trajectory reaches the set). In this sense the minimax set is an equilibrium which every vote-maximizing trajectory will tend to approach.

4. **Proof of the Main Result**

To prove Theorem 1 (actually a slightly stronger version) we first establish three preliminary results. The first two are elementary:

For any set of voters \( C \subseteq \mathcal{N} \), let \( \mathcal{P}(C) \) be the set of points which are (weakly)

Pareto optimal with respect to \( C \), i.e., \( \mathcal{P}(C) = \{ x : \text{for no } y \text{ is } y \succeq x, \text{ all } i \in C \} \). Then we have:

**Lemma 1.** \( \mathcal{P}(C) \) is the convex hull of the satiation points \( \{ s^i : i \in C \} \).

**Proof.** Omitted. (A straightforward consequence of the assumption of Type 1 preferences.)

Next, let \( M(m) \) be the set of points which cannot be defeated by more than \( m \) votes, i.e., \( M(m) = \{ x : \epsilon(x) \leq m \} \). Evidently, the set \( M(m) \) consists of the entire feasible region \( \mathbb{R}^k \) if \( m \geq n \), is empty if \( m < n^* \), and is precisely the minimax set if \( m = n^* \). For intermediate values of \( m \), the \( M(m) \) sets are clearly nested, in the sense that \( M(m') \supseteq M(m) \) if \( m' \geq m \), and moreover can be explicitly characterized by the proposition:

**Lemma 2.** If \( m < n \), then \( M(m) = \bigcap_{C \subseteq \mathcal{N}, |C| = m} \mathcal{P}(C) \).

**Proof.** Omitted. (Follows immediately from the definition, and does not depend on the Type 1 assumption. In particular Lemma 2 would still be true if the Type 1 assumption were replaced by the weaker assumption that voter preferences are quasi-concave.)

To obtain the \( M(m) \) sets (which will be important for characterizing the behavior of vote-maximizing trajectories), it thus suffices to examine the Pareto sets of each of the \( m + 1 \)-membered coalitions, \( M(m) \) being given by the intersection of these sets. A simple example (the same as that of
Fig. 2) with \( k = 2, \ n = 7 \) is given in Fig. 3. \( M(6) \) is simply the Pareto optimal set \( \mathcal{P}(N) \). \( M(5) \) is given by the intersection of the \( \mathcal{P}(C) \) of all the 6-membered coalitions, and \( M(4) \) by that of the 5-membered coalitions. Since the Pareto sets of the 4-coalitions have no common intersection, \( M(3) \) is empty, and the minmax number \( n^* \) is 4. (Note that the minmax set \( M(4) \) lies well within the interior of the Pareto set \( \mathcal{P}(N) \), and is small relative to \( \mathcal{P}(N) \).

![Diagram](image)

Fig. 3. Construction of \( M(m) \) sets.

We now use these two lemmas to prove a fundamental result:

**Lemma 3.** Let \( x, y \) be two feasible points for which \( n(x, y) > n^* \), and \( n \) an integer such that \( n(x, y) > m \geq n^* \). Then \( d(y, M(m)) < d(x, M(m)) \).

**Proof.** The general idea of the proof is illustrated in Fig. 4. The set \( \{ z : \| z - y \| < \| z - x \| \} \) of points closer to \( y \) than to \( x \) constitutes an open halfspace \( H \), bounded by the set of perpendicular bisectors of the line segment \( [x, y] \). A voter \( j \) prefers \( y \) to \( x \) if and only if his satisfaction point \( s^j \) is closer to \( y \) than to \( x \), so the set \( C^* \) of voters who vote for \( y \) is the set whose satisfaction points lie in \( H \). From Lemma 1, their Pareto set \( \mathcal{P}(C^*) \) is the convex hull of these satisfaction points, which is therefore contained in \( H \) also, i.e., \( \mathcal{P}(C^*) \subseteq H \). Since \( |C^*| = n(x, y) > m \), it follows from Lemma 2 that \( M(m) = \bigcap_{C^* \subseteq C, \|C\| = m} \mathcal{P}(C) \subseteq \mathcal{P}(C^*) \), so \( M(m) \) is contained in \( H \) also. Let \( q \) be the point in \( M(m) \) closest to \( x \) (\( M(m) \) is nonempty since \( m \geq n^* \) and
clearly compact from Lemma 1, so such a point exists), i.e., such that $d(x, M(m)) = \| x - q \|$. Then since $q \in M(m) \subset H$, $q$ must be closer to $y$ than to $x$, i.e., $d(x, M(m)) = \| x - q \| > \| y - q \| \geq d(y, M(m))$, which proves the lemma. Q.E.D.

The main results follow immediately from Lemma 3. In particular, we can now establish the following strengthened version of Theorem 1 (which corresponds to the case $m = n^*$ of the proposition below):

**Theorem 1'**. Let $(x')$ be a vote-maximizing trajectory and $M(m) \neq \emptyset$. Then for any $t$, $x' \notin M(m)$ implies $d(x'^{t+1}, M(m)) < d(x', M(m))$.

**Proof.** Since $x'^{t+1}$ is vote maximizing against $x'$, $v(x', x'^{t+1}) - v(x')$. $x' \notin M(m)$ implies $v(x') > m$, and $M(m) \neq \emptyset$ implies $m \geq n^*$, so the conditions of Lemma 3 hold and the result follows. Q.E.D.
5. Convergence and Stability

Theorem 1' implies that every vote-maximizing trajectory tends to move inside the nested $M(\cdot)$ sets, and hence to approach the innermost minmax set $M(\pi^*)$. This is a somewhat surprising result in one sense, since the vote-maximization process itself is quite indeterminate, and the number of possible trajectories is enormous. Some of this indeterminacy is illustrated in Fig. 5. Any point in the interior of the region $A$ is preferred by all seven voters to $a$, so any such point could be chosen as the next point in a vote-maximizing trajectory through $a$. At the point $b$, $v(b) = 6$, and there are two six-membered coalitions $\{1, 2, 3, 4, 5, 6\}$ and $\{2, 3, 4, 5, 6, 7\}$ whose Pareto sets do not contain $b$, and the set of vote-maximizing points is $B_1 \cup B_2$. At point $c$, the set of vote-maximizing points is even more complex, as shown on the figure. At any point $x^t$ the number of votes $\nu(x^t)$ by which $x^{t+1}$ must defeat $x^t$ is unique, but the set of individuals who cast these votes, and the direction and distance from $x^t$ to $x^{t+1}$, are quite indeterminate. But despite this indeterminacy, every vote-maximizing trajectory necessarily converges on the minmax set in the sense of Theorem 1.

Figure 5 also illustrates some possible anomalies in the behavior of such trajectories, however. In particular, many of the points which are vote
maximizing against \( c \) lie outside the minmax set, even though \( c \) itself is a minmax point. Thus the equilibrium is not a stable one, in the sense that there is no guarantee that a trajectory which enters the minmax set will necessarily remain inside thereafter. Theorem 1 does ensure that a trajectory which jumps outside must immediately return toward the minmax set, but reentries and subsequent departures may recur repeatedly. The apparent seriousness of this instability in the example is essentially an artifact of the small number of voters there; however, with large \( n \), vote-maximizing trajectories which enter the minmax set will typically remain "close" to it (though not necessarily inside). (This is readily shown by examples, though we do not attempt a formal statement and proof of the assertion here.)

Figure 5 also illustrates how a vote-maximizing trajectory may fail to reach the minmax set at all. From the points \( a \) and \( \bar{a} \), we can construct the sequence \((a^i)\), where \( a^i = \bar{a} + (1/i)(a - \bar{a}) \). Clearly \((a^i)\) converges on \( \bar{a} \), and moreover from the geometry of the situation, \( a^{t+1} \) is vote maximizing against \( a^t \) for all \( t \).

Hence \((a^i)\) is a vote-maximizing trajectory which fails to reach the minmax set, or even the Parkto set \( \mathcal{P}(N) \), though the distance to \( M(n^N) \) does decrease in the manner of Theorem 1. \((a^i)\) is in fact a Cauchy sequence, i.e., for all \( \delta > 0 \) there exists \( T > 0 \) such that \( \| a^s - a^t \| < \delta \) for all \( s, t > T \). This type of nonconvergence arises from the fact that at some point both agents get "tired" and begin taking ever-shorter steps; eventually each party will be differentiating itself only infinitesimally from its opponent. There is no particular reason or incentive for parties to behave in this manner; indeed, it seems more plausible to suppose the opposite, that each would attempt to maintain a certain minimal differentiation and distinctiveness in its policies. Hence this type of nonconvergence, though analytically possible, seems unlikely under competitive electoral conditions.

A more serious type of nonconvergence would be a limit cycle, in which the successive steps taken by the parties are all substantial in magnitude, but the trajectory itself approaches some sort of orbit and never reaches the minmax set. Only a slight strengthening of our assumptions is required to preclude this possibility, however. We shall say that a set \( A \subset \mathbb{R}^k \) is a body if it has an interior (relative to \( \mathbb{R}^k \)), and that a trajectory \((x^t)\) enters a set \( A \) if \( x^t \in A \) for some \( t > 0 \). We now show

**Theorem 2.** If \( M(m) \) is a body and \((x^t)\) is a vote-maximizing trajectory which is not a Cauchy sequence, then \((x^t)\) enters \( M(m) \).

**Proof.** Let \( M(m) \) be a body and \( x^t, x^{t+1} \) be any two points such that \( x^t \in M(m) \) and \( x^{t+1} \) is vote maximizing with respect to \( x^t \). On Fig. 6, \( L \) is the line through \( x^t \) and \( x^{t+1} \), \( \delta_t = \| x^{t+1} - x^t \| \), and \( H = \{ z : \| z - x^{t+1} \| < \| z - x^t \| \} \) is the set of points closer to \( x^{t+1} \) than to \( x^t \), an open half-space bounded by the plane \( P \) formed by the perpendicular bisectors of the line...
segment \([x^t, x^{t+1}]\). From Lemma 3, \(M(m) \subset H\), as shown. By assumption \(M(m)\) is a body, and from Lemmas 1 and 2 is compact, so there exists a largest ball \(B\) contained in \(M(m)\), with radius \(\rho > 0\). Let the point \(c\) be its center, and define \(h = d(c, L)\), the distance from \(c\) to \(L\), and \(a\), the point in \(L\) such that \(a - c\) is \(h\); similarly let \(b\) be the distance from \(c\) to \(P\), and \(b \in P\) such that \(b - c\) is \(k\). Finally let \(d_i = \|x^t - c\|\) and \(d_{i+1} = \|x^{t+1} - c\|\). Clearly the line segment \([a, c]\) is parallel to \(P\), so \(d(a, P) = k\), and is perpendicular to \(L\), which implies \(d_i^2 = h^2 + \|a - x^t\|^2 = h^2 + (k + \frac{1}{2} \delta)^2\) and \(d_{i+1}^2 = h^2 + \|a - x^{t+1}\|^2 = h^2 + (k - \frac{1}{2} \delta)^2\). Hence \(d_i^2 - d_{i+1}^2 = 2h \delta\), \(k > 2h \delta\), since \(M(m) \subset H\) implies \(k > \rho\).

Consider now a trajectory \((x^t)\) which never enters \(M(m)\). The above
inequality must hold for all \( t > 0 \), which after rewriting and summing for \( i = 1, 2, \ldots, T \), implies

\[
2\rho \sum_{t=1}^{T} \delta_t < \sum_{t=1}^{T} (d_t^2 - d_{t+1}^2) = d_1^2 - d_{T+1}^2,
\]
or equivalently

\[
d_{T+1}^2 < d_1^2 - 2\rho \sum_{t=1}^{T} \| x^t - x^{t+1} \|.
\]

If \((x^t)\) were not a Cauchy sequence, the sum \( \sum_{t=1}^{T} \| x^t - x^{t+1} \| \) would be unbounded, so the right-hand side of the above inequality would eventually become negative, contradicting the assumption that \((x^t)\) never enters \( M(m) \). Hence \((x^t)\) must be a Cauchy sequence.

Q.E.D.

The condition that \( M(m) \) be a body is of course an extremely weak one (for \( m \geq n^* \)); it will hold in "almost every" society for which \( n \) is large relative to \( k \). (It is worth noting that one case in which the minimax set is not to be a body and a limit cycle is possible is when a Condorcet winner exists, the possibility which has received so much attention in the voting theory literature.)

6. SOCIAL CHOICE INTERPRETATION OF THE MINMAX EQUILIBRIUM

Our primary concern has been to characterize the behavior of a certain class of competitive electoral processes. However, the resulting equilibria seem also to be of some interest from a normative, social choice theory point of view.

To develop this interpretation, consider first the possibility of defining a social preference relation on the basis of the unanimity or weak Pareto criterion. The unanimity relation (given by \( xP_U y \) if \( n(y, x) = n \)) is necessarily transitive, from the transitivity of individual preferences, so this rule thus leads to consistent social choices. However, it has the disadvantage of (typically) being rather indecisive, since many pairs of alternatives cannot be compared by the Pareto criterion, and the set of undominated or Pareto optimal states is often quite large.

In contrast to this, the simple majority relation (given by \( xP_{M,y} y \) iff \( n(y, x) > n/2 \)) is quite decisive; indeed, if \( n \) is odd and all preferences are strict it is completely decisive, since either \( xP_{M,y} y \) or \( yP_{M,x} \) will hold for all \( x, y \). But the majority preference relation is generally inconsistent, in the sense that it cycles, and there is no "best" or "optimal" choice since every alternative is dominated by some other alternative.
To find a more satisfactory social decision method, which can reconcile the conflicting demands of consistency and decisiveness, it thus seems natural to turn to some of the intermediate voting rules which lie between unanimity and simple majority rule, such as the two-thirds or three-fourths rules. The family of such rules can be indexed by a parameter $\lambda$, $\lambda \in [\frac{1}{2}, 1)$, which represents the size of the majority required for one alternative to be socially preferred to another; thus we shall say $x$ is preferred to $y$ by the $\lambda$-majority rule, or $xP_\lambda y$, if and only if $n(y, x) > \lambda n$. Clearly these relations are nested, in the sense that if $\lambda' < \lambda$, then $xP_{\lambda'} y$ implies $xP_\lambda y$; thus $P_{\lambda'}$ is sharper or more decisive than $P_\lambda$ and in general the smaller the $\lambda$ the more decisive the relation $P_\lambda$.

To explore the consistency properties of these relations, we must first define the notion of "consistency" more carefully. The weakest useful definition would be to say a social preference relation is consistent if it has a nonempty set of undominated or optimal elements; more precisely, a relation $P \subseteq A \times A$ has a maximal element if there exists an $x \in A$ such that for no $y \in A$ is $yPx$. This maximality property does not preclude the possibility of cycles among the inoptimal states; if such cycles do exist, then there will be subsets of alternatives within which maximal elements fail to exist (i.e., every element in the set will be dominated by some other element also belonging to the set). A stronger consistency property, which eliminates these possibilities, is to require that the preference relation not cycle; more precisely, a relation $P \subseteq A \times A$ is acyclic if for no $r > 1$ do there exist alternatives $a_i \in A$, $i = 1, 2, \ldots, r$, such that $a_{i+1} Pa_i$ for all $i = 1, 2, \ldots, r - 1$, and $a_1 Pa_r$. If $A$ is finite, acyclicity implies the existence of a maximal element, though in general the converse need not hold. However, with respect to the specific $P_\lambda$ relations defined on the set of Type I societies, acyclicity and maximality are equivalent, and moreover are related to the minmax concept, in the following sense:

**Theorem 3.** The following statements are equivalent:

(a) $P_\lambda$ is acyclic,

(b) $P_\lambda$ has a maximal element,

(c) $\lambda \geq n^*/n$.

**Proof.** The equivalence of (b) and (c) follows immediately from the definition of the minmax number $n^*$. It remains to be shown that (a) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a).

(a) $\Rightarrow$ (c). Let $B$ be a ball with center $c \in M(n^*)$ and radius such that $M(n^*)$ lies in the interior of $B$. For any $x$, clearly $\{z : zP_\lambda x\} = \bigcup_{\{c \in N: |c| < 2\delta\}} \bigcap_{\{e \in \mathcal{E}\}} \{z : z \geq_i x\}$ is open, since it is composed of finite unions and intersections of open sets. $B$ is compact, so if $P_\lambda$ is acyclic, there exists a maximal element in $B$ [32, Theorem 7, p. 8]; i.e., an element $x^* \in B$ such that for no
\( y \in B \) is \( yP_nx^* \). By definition there exists \( y \) such that \( n(x^*, y) = \tau(x^*) \). If \( \tau(x^*) > n^* \) it follows from the argument of Theorem 1 that \( d(y, c) < d(x^*, c) \), and hence that \( y \in B \). If \( \tau(x^*) = n^* \), then \( x^* \in M(n^*) \) is an interior point of \( B \). Since Type \( I \) preference orderings are convex it must be true for all \( \alpha \in (0, 1) \) that \( n(x^*, y(\alpha)) \geq n(x^*, y) \), where \( y(\alpha) = \alpha y + (1 - \alpha) x^* \), and for sufficiently small \( \alpha \), \( y(\alpha) \in B \) also. Hence in either case \( B \) contains a point \( x \) such that \( n(x^*, z) = \tau(x^*) \geq n^* \), so if \( \lambda < n^*/n \), \( x^* \) would not be a maximal element of \( P_n \).

(c) \( \Rightarrow \) (a). If \( P_n \) were not acyclic there would exist \( r \geq 2 \) points \( x^1, x^2, ..., x^r \) such that \( x^iP_nx^j \), all \( i = 1, 2, ..., r - 1 \), and moreover \( x^iP_nx^r \). Since \( \lambda \geq n^*/n \), Theorem 1 implies that \( d(x^{i-1}, M(n^*)) < d(x^i, M(n^*)) \), all \( i = 1, 2, ..., r - 1 \), and hence that \( d(x^r, M(n^*)) < d(x^i, M(n^*)) \). But if \( x^iP_nx^r \), Theorem 1 would also imply \( d(x^1, M(n^*)) < d(x^i, M(n^*)) \), a contradiction which shows that \( P_n \) must be acyclic.

Hence with Type \( I \) preferences acyclicity and maximality are equivalent, and the family of voting rules which are consistent in either sense is the set of \( P_n \) for which \( \lambda \geq n^*/n \).

Of these, the minmax voting rule \( P_n^* \), where \( \lambda^* = n^*/n \), has some attractive features as a social welfare function. Since \( \lambda^* \) is the smallest \( \lambda \) satisfying the above inequality, the corresponding \( P_n^* \) relation is the largest or most decisive of the consistent \( P_n \) relations, and yields the smallest or most sharply characterized (nonempty) set of social optima (which is precisely the minmax set \( M(n^*) \)).

The minmax relation can also be justified in a more abstract axiomatic social choice framework. It clearly satisfies Arrow's [1] Pareto and Non-Dictatorship axioms (and in fact also satisfies the much stronger Anonymity axiom, since the \( P_n^* \) relation does not depend on the labeling of voters). It also satisfies the natural weakenings of the Domain axiom (to the class of Type \( I \) societies), and of Arrow's implicit transitivity axiom (replacing transitivity by acyclicity). The one condition which apparently is violated in a serious way is the Independence of Irrelevant Alternatives condition: since \( P_n^* \) is defined in terms of the global parameter \( n^* \), it depends on alternatives (which minimize the maximum-vote function) that may be "irrelevant" (in the sense of the axiom) to a particular choice. Even here, however, there is a

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* Craven [5] and Forejohn and Grether [9] have explored the structure of the \( P_n \) relations over finite sets of discrete alternatives, and have shown (essentially) that if \( r \) is the number of alternatives, a necessary and sufficient condition for \( P_n \) to be acyclic for all societies is that \( \lambda \geq (r - 1)/r \). This is a conservative bound, since in many societies lower values will still yield acyclic \( P_n \), and in any event it does not apply to multidimensional choice spaces in which the feasible set is infinite. However, we believe the topological structure of the multidimensional case can be exploited to obtain a comparable bound, and in particular conjecture that such a bound will be given by \( k/(k + 1) \), where \( k \) is the dimension of the space of alternatives.
strong sense in which the minmax ordering satisfies the basic idea behind the Independence condition, if not its precise axiomatic statement. For it is clear from Arrow's justification and defense of the axiom (particularly the discussion of [1, pp. 109–111]) that the Independence axiom is intended to formalize a certain kind of informational decentralization condition. The underlying idea is that the overall or global social ordering of all social states (including all future and contingent consumptions, events, and so forth) should be decomposable into a series of smaller partial-equilibrium choices, each concerned with a subset of immediately available or “relevant” states or alternatives. These various partial-equilibrium choices can be decided separately, and the transitivity (or acyclicity) property ensures that the resulting decisions will be mutually consistent, and will increase the (global) social welfare function. The Independence condition ensures that each of these “local” choices is informationally feasible, in the sense that it depends only on citizen’s preferences for the immediately “relevant” alternatives which constitute the choice, and not on how they would order (if they could) various hypothetical “irrelevant” states or contingencies. Clearly a method of social choice which required complete information on every agent's preferences for all conceivable states would not be institutionally or operationally viable, and some informational condition of this kind is essential. Arrow’s Independence axiom is nevertheless a very strong version of such a requirement, however, and there is no intrinsic reason for not considering more flexible formulations of the idea, which permit the choice from a certain subset of states to depend, in part, on information concerning a certain delimited set of other states, such as the status quo, or alternatives which are “near” (in some topological sense) the ones under consideration.

Viewed in these terms, the minmax ordering can be realized by an informationally decentralized process of this kind, specifically the competitive electoral mechanism described in Sections 3 and 4. Clearly if v is vote maximizing against x, then yPv,x (unless x is already a maximal element of P), so vote maximization increases the P ordering, and (from Theorems 1' and 2) repeated vote maximization will eventually reach M(v*), the set of maximal or optimal elements of P. Moreover, the convexity of voter preferences implies that if y is vote maximizing against x, then so also is any point αy + (1 − α)x, α ∈ (0, 1]. Hence, for any x, there always exists a nearby vote-maximizing point; or conversely, we need only know voters' preferences in some small neighborhood of x to find such a point. Thus vote maximization can be done in an entirely “local” fashion, by searching in the immediate vicinity of the current status quo, and does not depend on voter preferences for more distant “irrelevant” alternatives: In this sense the P* social ordering is realizable (for Type I societies) by an informationally decentralizable mechanism which satisfies the spirit of Arrow’s Independence axiom.
7. Discussion and Interpretation

The results reported above provide a rather general characterization of the behavior of a Downsian two-party political system acting over a multidimensional issue or policy space, with a Type I electorate. Competitive vote maximization in such a situation leads the two parties to converge on the minimax set. Most previous work on electoral competition in this type of situation spaces has been based on the much more restrictive concept of a "majority winner," whose existence is so rare and unstable a phenomenon as to make it of little practical interest as a descriptive equilibrium. The equilibrium proposed here, the minimax set, always exists, so the convergence results apply quite generally. These results thus generalize the Downsian analysis of party competition to multiple-issue elections, and suggest that the minimax set is the natural generalization of the "median" (of the distribution of voters' most-preferred points), the point on which the parties' platform choices tend to converge in the unidimensional case.

The size of the minimax set (or more generally, the shape of the \( r(\cdot) \) function) in some rough sense reflects the degree of consensus within the society. In a purely distributional question, in which a limited amount of a single private commodity must be distributed among the citizenry, each citizen will want to maximize his own share, at the expense of everyone else. In this case the minimax set will coincide with the entire Pareto optimal set, \( \mathcal{P}(N) \), and the voting process will be relatively indeterminate. Such redistributational questions are the most divisive ones a society must face, and it is not surprising that voting is unable to resolve them. In a more general public-goods type problem, in which the interests of the citizens are not so diametrically opposed, we will have a situation more like that of Fig. 3, where the minimax set lies well within the interior of \( \mathcal{P}(N) \), and the voting process will be more determinate. Moreover, if the number of voters increases in such a way that their satisfaction points become spread more smoothly across the space, the minimax set will shrink, and the process will tend to become more determinate in large societies.\(^4\) These qualitative characteristics of minimax sets seem eminently reasonable to us, and reinforce our feeling that the minimax equilibrium concept is a compelling one which deserves to be taken seriously.

The basic motivational premise of our model is that parties are primarily interested in maximizing votes over the current election period. The vote maximization assumption is a plausible abstraction, at least as defensible (for a competitive two-party system) as the assumption of profit maximization by firms. Nevertheless political parties (like firms) may sometimes have other objectives as well, and in a more general treatment, we might suppose the parties to be maximizing an objective function involving both vote and

\(^4\) These assertions are precisely formulated, and proved, in a forthcoming paper.
policy (or other) goals, and to be willing to sacrifice some extra votes in favor of these other objectives in elections in which a comfortable margin of victory is already assured. Some generalization of our results in this direction is possible. In particular, if the vote–policy trade-offs are such that no “out” party would ever settle for fewer than $n^*$ votes, then the sequence of successively enacted policies would constitute a subset of the $n^*$-trajectories (i.e., trajectories for which $m(x^t, x^{t+1}) \geq n^*$). It follows from Lemma 3 that every such $n^*$-trajectory still converges on the minmax set, so the model generalizes immediately to this extent.

A somewhat different issue concerns the myopia of the model. In each period the “out” party is assumed to maximize its prospects in the coming election, without regard to the possible consequences in subsequent periods. This is a strong assumption, though not necessarily an unrealistic one, in an electoral context. Many observers have noted the relatively short horizons of elected officials, and the fact that their preoccupations rarely extend beyond the next election (in this respect they differ from private firms and nonelected officials). Indeed, this short-sightedness is often cited as a major shortcoming of the decision-making process in a democracy. It is therefore of some interest, in this regard, to find that a completely myopic democratic process nevertheless does succeed, over time, in attaining what is in many respects an eminently reasonable social optimum, the minmax set.

The myopia assumption could be relaxed in various ways. In a completely general treatment the competitive process might be modeled as a fully dynamic game, in which each party strives to maximize a discounted stream of all future election returns. While we do not think the gain in descriptive realism in such a treatment is necessarily great, it is nevertheless an interesting, and open, question as to whether the minmax set would still emerge as an equilibrium in this more general framework. A less general though not implausible intertemporal extension in this direction can be developed along the following lines. In the long run both parties will tend to rotate in office, since each election can always be won by the “out” party. Hence the most important intertemporal effect of a party’s policy choice in period $t$ will be the extent to which it makes the party vulnerable to large losses in the following election $t + 1$, when it will be incumbent and obliged to defend this same policy; since it is free to adopt a new policy (and win the election) in the period after that, $t + 2$, it is not unreasonable to suppose that the party ignores the possible minor consequences its current choice may have in period $t + 2$ and subsequently, and considers only the effects in periods $t$ and $t + 1$ in choosing its current platform. If it chooses a policy $y$ against the incumbent’s policy $x^t$, the “out” party guarantees itself $m(x, y)$ votes in the current election, but also exposes itself to the possibility of receiving as few as $n - r(y)$ votes in the following election; thus let us assume that instead of simply maximizing its current vote share, the party maximizes a more general two-
period objective function of the form \( f(n(x', y), v(y)) \), which is strictly increasing in its first argument, and strictly decreasing in the second. If each of the parties maximize (possibly different) objective functions of this form in successive elections, a class of trajectories is generated which in general will differ from the pure vote-maximizing trajectories. Nevertheless, almost all of these two-period trajectories also converge on the minmax set. This is implied by the following result.

**Proposition 3.** For almost every point \( x \), a point \( y \) which maximizes the two-period objective function \( f(n(x, \cdot), v(\cdot)) \) against \( x \) is either a minmax point, or else \( v(x) > n^* \) and \( y \) satisfies \( d(y, M(v(x) - 1)) < d(x, M(v(x) - 1)) \).

**Proof.** Suppose \( x \) is an interior point of \( M(v(x)) \). The convexity of voter preferences and the fact that \( x \) is interior imply the existence of a point \( z \) which is vote maximizing against \( x \) and such that \( v(z) = v(x) \). If \( y \) maximizes \( f(n(x, \cdot), v(\cdot)) \) against \( x \), then since \( f \) is strictly increasing in \( n(x, \cdot) \) and decreasing in \( v(\cdot) \), and \( n(x, y) \leq v(x) = n(x, z) \), \( y \) must satisfy either

(i) \( n(x, y) < n(x, z) \) and \( v(y) < v(z) \) or
(ii) \( n(x, y) = n(x, z) \) and \( v(y) \leq v(z) \).

Since \( v(y) \leq v(z) = v(x) \) in either case, if \( x \) is a minmax point, \( y \) is also. Conversely, if \( x \) is not a minmax point then \( M(v(x) - 1) \) is nonempty and \( d(x, M(v(x) - 1)) > 0 \). Then if (i) holds \( v(y) < v(z) = v(x) \) implies \( y \in M(v(x) - 1) \), i.e., \( d(y, M(v(x) - 1)) = 0 \); while if (ii) holds \( y \) is vote maximizing against \( x \) and the conclusion follows from Theorem 1'.

The reasoning above applies to all \( x \) except those lying on the boundary of some \( M(m) \), i.e., for almost all \( x \).

Q.E.D.

This result implies that any "two-period" trajectory will approach the "next" set it has not yet reached, \( M(v(x') - 1) \), and thus (subject to the qualifications of Theorem 2) tend to move inside the nested \( M(\cdot) \) sets. Hence the minmax set again emerges as the relevant equilibrium in this two-period version of the model.

All of the results proven here (except Lemma 2) make essential use of the assumption that all voters have Type I preferences. Most previous work in this area has been based on this or essentially equivalent assumptions. The Type I societies are an important class, within which all of the intrinsic difficulties of voting intransivities emerge. Nevertheless the Type I assumption is a restrictive one, and the extent to which it can be relaxed is clearly an important issue. Some obvious immediate generalizations can be obtained by replacing the Euclidean metric by other metrics. Clearly the minmax and \( M(m) \) sets exist and are characterized by Lemma 2 under much weaker assumptions, for example, that voter preferences are representable by smooth, strictly quasi-concave utility functions. However, the extent to which the
essential qualitative properties expressed in Theorems 1 through 3, concerning the acyclicity of the $P_i$ relations and the convergence of the vote-maximizing trajectories on the minmax set, can be shown to hold under such weaker preference assumptions is an important and still unresolved question.

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