AGGREGATION OF PREFERENCES *

DONALD J. BROWN

Consider a group of individuals who must collectively choose one of several mutually exclusive alternatives. Since, in general, there will be disagreement as to the desirability of the different alternatives, the group may find it easier first to choose a method or procedure for aggregating their preferences. At least there may be a consensus as to what properties an aggregation procedure ought to have before the group would consider it ethically and institutionally acceptable. For example, one such ethical consideration is the requirement that the group preference be Pareto-efficient, i.e., if every member of society prefers $a$ to $\beta$, then society should prefer $a$ to $\beta$. As an institutional requirement we might wish to restrict ourselves to methods of aggregation that can be implemented by voting rules.

We shall suppose that the tastes or preferences of the individuals in society can be represented in one of two ways. Each person has an ordinal utility function, $U$, or a preference relation, $P$, over the set of alternatives. Individuals whose tastes are described by a utility function will be called rational. If he has a preference relation $P$, $aP\beta$, is to be read “he prefers $a$ to $\beta$.” Operationally we shall interpret this to mean that if offered the choice between $a$ or $\beta$, he will choose $a$. We do not assume the converse. If someone chooses $a$ over $\beta$, then we can only infer that he does not prefer $\beta$ to $a$. This is not a pedantic distinction; it is essential if one is to recognize that an individual who makes a unique choice out of every subset of alternatives may well have an incomplete — even intransitive — preference relation over the set of alternatives.

Our problem is to aggregate the preferences of individuals into a social preference relation, where the aggregation procedure has certain desirable ethical and institutional properties. This is the

* The research described in this paper was carried out in part by grants from the National Science Foundation and the Ford Foundation. The author would also like to acknowledge the support of a Guggenheim Fellowship for the year 1973–74.

The author is pleased to acknowledge the stimulating and helpful criticisms of Peter Fishburn, Charles Plott, and John Ferejohn, Gerald Kramer, Susan Lepper, and Abraham Robinson have been kind enough to comment on several drafts of this paper. Preliminary sections of this paper were presented at the 1971 meeting of the Public Choice Society and the 1973–74 Berkeley–Stanford Seminar in Economic Theory and Econometrics.
problem that Arrow addresses in his classic monograph, *Social Choice and Individual Values*.

Arrow proposed a number of intuitively appealing conditions on individual preferences and the aggregation procedure — called by Arrow — a social welfare function. He first assumed that all members of the society were rational. Although this is a standard assumption in economic theory, it has recently been attacked by several economists, notably Georgescu-Roegen, Armstrong, and May. Their principal objection is that rational individuals must a fortiori have transitive indifference relations. They argue that intransitivities in indifference can arise out of a threshold in the perception of preference or if alternatives are perceived to be multi-dimensional. This assumption of individual rationality is primarily an institutional condition. By restricting our attention to groups of rational individuals, we enhance our chances that there will exist a social welfare function.

Arrow next required that the aggregation procedure produce an ordinal social utility function $U_s$ over the set of alternatives, a condition he called collective rationality. Given such a $U_s$, he then defined the socially optimal alternatives as those alternatives that maximize $U_s$ over the set of alternatives. He gave two justifications for collective rationality based on institutional requirements.

The first was that the social preference relation give necessary and sufficient conditions for an alternative to be socially optimal. His concern is social paralysis due to social indecision. This is certainly a legitimate concern, but the requirement that the social preference relation be complete so that it will provide sufficient conditions for optimality as well as necessary conditions is extremely stringent. As in the case of the individual, incomplete social preferences need not imply an inability on the part of society to choose or decide. An analogous social relation in economics, Pareto dominance, is used to define the Pareto-efficient points. These points are certainly not considered to be socially indifferent by economists, and they only recommend that the final distribution of goods and services be Pareto-efficient. That is, Pareto efficiency is a necessary but not sufficient condition for social optimality. A similar situation arises in game theory where the social relation is the blocking relation, which defines the core of the game. If one considers an exchange

economy as a cooperative game, then the core becomes the contract curve. As pointed out by Edgeworth, the social outcome ought to be somewhere on the contract curve, but our model of economic exchange does not have the institutional detail to predict what core allocation will be the actual equilibrium. Again, the social relation gives only a necessary condition for social optimality. Of course, the reason that these social preference relations provide only necessary conditions is their incompleteness, e.g., the Pareto-dominance relation says nothing about comparing two Pareto-efficient allocations. This incompleteness of the social relation is not allowed under the collective rationality assumption.

Condorcet, in his studies on elections, proposed that the candidate who receives a majority of votes against every other candidate should be elected to office. This suggestion has subsequently been termed the Condorcet criterion. Arrow's requirement of collective rationality is a generalization of the Condorcet criterion in that the social optimum would be socially preferred or indifferent to all the other alternatives under consideration. A necessary but not sufficient condition for an alternative $a$ to satisfy the Condorcet criterion is that no alternative $\beta$ receives a majority of votes against $a$. We define voting equilibria as those alternatives having this property. They are voting equilibria in the sense that if such an alternative is the status quo then it cannot be overturned under majority rule, i.e., the society will not vote to "move" from such a position. This, of course, does not preclude the society from moving, since voting is only an expression of the social preference and just requires society not to choose an alternative $\alpha$ as the optimum if there is another alternative $\beta$, which society prefers to $\alpha$. If we require our aggregation procedure to provide a social preference relation, $P$, then $\alpha$ is a social equilibrium if there does not exist a $\beta$ such that $\beta P \alpha$. In this sense the Pareto-efficient allocations are the social equilibria with respect to the Pareto-dominance relations; and the blocking relation has the core as its set of social equilibria. If the aggregation procedure satisfies the condition of collective rationality, then the set of social equilibria and social optima will be one and the same. In this very special case if an alternative is chosen, then it is socially preferred or indifferent to every other alternative.

Now suppose that the method of aggregation is some sort of voting rule — throughout our discussion we shall assume that people vote their true preferences, hence ignoring the very difficult and important question of strategic voting. In order for the voting rule to be well defined, we must describe not only the winning majorities
or winning coalitions but also the voting procedure. This last bit of institutional detail is often overlooked in the social choice literature, an outstanding exception being Black's seminal work, *The Theory of Committees and Elections*. Black discusses several types of voting procedures used by committees when voting on a set of motions. For our purposes we shall need to abstract one of them. We shall call it voting procedure (a).

In voting procedure (a) first an agenda is chosen, then alternatives are voted on sequentially and pairwise, the winner of a round being paired against the next alternative on the agenda. If there is a tie on a given round, then the alternative that is lower on the agenda is declared the winner of the round. The eventual winner of this process will be called a voting equilibrium of type (a).

This voting procedure, modulo minor modifications, is the one most commonly used by committees. This procedure has the disturbing property that there need not be any relationship between the social equilibrium defined by the social preference relation and the voting equilibria of type (a), the classic example being cyclical majorities. For example, in a society of three people having preferences

\[
\begin{array}{ccc}
(1) & (2) & (3) \\
 x & y & z \\
 y & z & x \\
 z & x & y
\end{array}
\]

if the social preference is defined by majority rule, then there are no social equilibria; but every alternative is a voting equilibria of type (a) for some agenda. We shall define a voting rule as path-dependent if for some configuration of preferences it leads to an alternative that is not a social equilibrium. Hence, majority rule together with voting procedure (a) define a path-dependent voting rule. This definition of path dependence is due to Plott.\(^2\) This may not agree with Arrow's intended definition of path dependence. Probably Arrow would define a voting rule as path-dependent if for some configuration of preferences it leads to an alternative that is not a social optimum. Of course, every voting rule that is path-independent in Arrow's sense is also path-independent under Plott's definition.

Arrow's second justification for collective rationality is that the social preference relation defined by \(U_s\) can be implemented by a path-independent voting rule. If we reexamine the case of cyclical

majorities, we shall see that the condition of collective rationality can be weakened a great deal and still have social preference relations that can be implemented by voting rules that are path-independent in Plott's sense.

In the instance of cyclical majorities the social preference relation defined by majority rule is both cyclic and intransitive. This so-called "paradox of voting" is often attributed to intransitivity, but in fact is caused by cyclicity of the social preference relation. Recall that a preference relation, $P$, is acyclic if there do not exist alternatives $a_1, a_2, \ldots, a_n$ such that $a_1Pa_2, a_2Pa_3, \ldots, a_{n-1}Pa_n, a_nPa_1$. A preference relation is transitive if for all triples of alternatives, $a, b, c$, if $aPb$ and $bPc$, then $aPc$. A preference relation that is both transitive and acyclic is called a partial order. It is easy to see that being a partial order is a sufficient condition for a social preference relation to be path-independent with respect to voting procedure (a). Moreover, it has been shown by Plott that this is also a necessary condition. Note that the outcome is completely determined by the agenda, i.e., every social equilibrium will be a voting equilibrium of type (a) for some agenda. In general, the social equilibria defined by a partially ordered social preference relation will not be socially optimal; hence, this voting rule is not path-independent in Arrow's sense. Since every social preference relation that is a partial order will have a nonempty set of social equilibria, Arrow's condition of collective rationality can be replaced by the requirement that the aggregation procedure provide a partial order as the social preference relation.

If our social preference relation is defined by a voting rule, i.e., $a$ is socially preferred to $b$ if the set of individuals who prefer $a$ to $b$ make up a winning coalition, then there is an obvious voting procedure, called voting procedure (c), whose set of voting equilibria coincides with the set of social equilibria. All alternatives are paired against one another and voted on. The voting equilibria are those alternatives that are not defeated by any other alternative. As is well-known, if the social preference relation is acyclic, then the set of social equilibria is nonempty. It is important to point out that an acyclic social preference relation may be path-dependent with respect to voting procedure (a). Consider the social preference relation $P_*$ over three alternatives $\{a, b, c\}$ where $aP_*b$, $bP_*c$, and the social preference relation is undefined with respect to $a$ and $c$. The agenda $\{(a, b), (b, c)\}$ has $c$ as a voting equilibrium of type (a), and $c$ is not a social equilibrium of $P_*$. If we replace the condition of collective rationality by requiring an acyclic social preference relation
from our aggregation process, then we must use voting procedure (c) to determine the set of social equilibria.

Arrow’s remaining institutional requirement is the independence of irrelevant alternatives. Simply stated, this condition is that the social preference relation between any pair of alternatives \( a \) and \( \beta \) depends only on the preferences of individuals between \( a \) and \( \beta \). The intent of this condition is to minimize the information society needs to aggregate individual preferences and to restrict the aggregation mechanism to voting rules.

The only ethical condition imposed by Arrow is that the method of aggregation be Pareto-efficient.

Arrow’s Possibility Theorem is that, if an aggregation procedure satisfies all of the stated conditions, then there exists an individual in society who is a dictator. An individual is a dictator if, whenever he prefers \( a \) to \( \beta \), society prefers \( a \) to \( \beta \) regardless of the preferences of the other individuals in society.

Much of the subsequent literature on social choice has been concerned with alternative formulations of the conditions suggested by Arrow as necessary properties of an aggregation procedure. For the purposes of this essay, the relevant work is presented in Hansson’s outstanding (unpublished) paper, “The Existence of Group Preferences.” The major concept in Hansson’s analysis of the problem of social choice, i.e., the aggregation of individual preferences, is the notion of a decisive set of individuals, a concept first introduced by Arrow.

Given an aggregation procedure, a set of individuals is decisive if whenever they prefer \( a \) to \( \beta \), then society prefers \( a \) to \( \beta \) regardless of the preferences of the rest of society, e.g., a dictator is a decisive set of one individual. In the case of a society of three people, if the social preference relation is defined by majority rule, then any coalition having more than one individual is decisive. A minimal decisive set is a decisive set that contains no proper decisive subset. Both the size of the minimal decisive coalitions and the number of decisive coalitions may be used as indices of domination in society. The two extremes of domination are dictatorship and the rule of unanimity. If our ethical criteria include a desire for individual liberty, i.e., lack of domination, then we want the family of decisive coalitions to be a family of “large” sets.

Let us denote the individuals in society by a nonempty finite set \( X \) and let \( F \) be a family of subsets (coalitions) of \( X \). What properties

should $F$ have in order to be a family of "large" subsets of $X$? Three reasonable conditions to impose are (i) $X \in F$; (ii) if $A \in F$ and $A \subset B$, then $B \in F$; (iii) $\phi$, the empty set, does not belong to $F$. A fourth not so obvious condition is that, if $A, B \in F$, then $A \cap B \in F$. A family of subsets of $X$ having these properties is called a filter over $X$. Having defined the families of "large" sets, i.e., filters, we have implicitly defined the families of "small" sets. Given a filter $F$, let $I_F$ be the family of complements of sets in $F$. $I_F$ has the following properties: (i) $\phi \in I_F$; (ii) if $A \in F$ and $B \subset A$, then $B \in I_F$; (iii) $X$ does not belong to $I_F$; (iv) if $A, B \in I_F$, then $A \cup B \in I_F$. Any family of subsets of $X$ having these properties is called an ideal over $I$.

Given a filter $F$, we would like every subset of $X$ to be either a "large" set or a "small" set, i.e., for all $A \subset X$, either $A \in F$ or $A \in I_F$.

The next example shows that not every $F$ has this property. Let $X = \{1, 2, 3, 4\}$ and $F = \{(1), (1, 2), (1, 2, 3), (1, 2, 3, 4)\}$, then $F$ is a filter over $X$ and $I_F = \{(3, 4), (4), (3), (\phi)\}$. In this instance, $\{2, 4\}$ is neither "large" nor "small." Filters $F$, having the property that every subset of $X$ belongs to $F$ or $I_F$, are called maximal filters or ultrafilters. If $F$ is a maximal filter, then $I_F$ is called a maximal ideal. Every ultrafilter over $X$ consists of a single point of $X$ and the family of subsets of $X$, which contain that point. For example, $F = \{(1), (1, 2), (1, 3), (1, 4), (1, 2, 3), (1, 3, 4), (1, 2, 3, 4)\}$ is an ultrafilter.

Every family of subsets of individuals $F$ defines an aggregation procedure $P_F$, where given individual preferences, $aP_\beta$ if the set of individuals who prefer $a$ to $\beta$ belongs to $F$. That is, every family of subsets of individuals $F$ generates an aggregation procedure $P_F$, where $F$ is the family of decisive sets or winning coalitions for $P_F$.

Those readers familiar with cooperative game theory will recognize this construction as a special case of the definition of a game in coalitional form.

We now want to look at some ways of aggregating individual preferences to see whether their decisive sets form a filter. No linear voting rule such as simple majority rule, two-thirds majority, etc., except unanimity has this property.

This observation makes even more surprising Hansson’s result that the decisive sets of any method of aggregation satisfying Arrow’s conditions form an ultrafilter over the set of individuals. Hansson also showed the converse that every ultrafilter $F$ defines a method of aggregation $P_F$ satisfying all of Arrow’s conditions. (These re-
AGGREGATION OF PREFERENCES

results were obtained independently by Kirman and Sondermann.)

Finally, Hansson showed that, if we require individual rationality,

independence of irrelevant alternatives, Pareto efficiency, and the

social preference relation be a partial order, then the decisive sets

form a filter. (A special case of this result was first announced by

Gibbard.) Hansson noted the converse result that, if $F$ is a filter,

then the social preference defined by $P_F$ is a partial order, and $P_F$

satisfies the remainder of Arrow's conditions. An example of this

last result was first given by Sen, where he observed that the Pareto-

dominance relation is a partially ordered social preference relation

satisfying the remainder of Arrow's conditions. In fact, we have

observed that one need not require individual rationality to obtain

these two results of Hansson; it suffices to assume that the individual

preference relations are partial orders. If $F$ is a filter over a finite

set $X$, then the intersection of all the sets that belong to $F$ is a set

that belongs to $F$, which we shall denote as $\cap F$; moreover, $F$ is just

the family of subsets of $X$, which contain $\cap F$. Therefore, the aggrega-

tion procedure $P_F$ generated by a filter $F$, i.e., given individual

preferences, $a P_F b$ iff the set of individuals who prefer $a$ to $b$ belong

to $F$ is an oligarchy. That is, society prefers $a$ to $b$ iff all of the

individuals in $\cap F$ prefer $a$ to $b$, and if at least one person in $\cap F$

prefers $a$ to $b$, then society does not prefer $b$ to $a$. It is easy to check

that $F$ is an ultrafilter iff $\cap F$ consists of a single point. This fact

combined with Hansson's result that the decisive sets of any social

certainty function satisfying Arrow's conditions form an ultrafilter

over the set of individuals proves Arrow's Possibility Theorem.

Reexamining the four conditions that define a filter, the first

three are certainly desirable properties to impose on the family of

decisive sets of a given method of aggregation. Hence, as Hansson

points out, it is the fourth condition requiring the intersection of
decisive sets to be a decisive set that must be weakened, if we
wish to rid ourselves of oligarchy.

Guilbaud, in a provocative and penetrating analysis of the
problem of social choice, suggested that we require only the inter-

section of any pair of decisive sets be nonempty. An example is a

4. A. P. Kirman and D. Sondermann, "Arrow's Theorem, Many Agents


6. Iff is an abbreviation for "if and only if."


Program of Aggregation," in Readings in Mathematical Social Science, Lazars-

society of five people, where two persons are given two votes each, everyone else is given a single vote, and the voting rule is simple majority. Can Guillaume's voting rule cycle? It can only produce cycles of at least order three. Well, require that the intersection of any three decisive sets be nonempty. We can still have cycles of order four. Clearly, if we wish to prevent the occurrence of cycles of any length, then we must require the intersection of all the decisive sets to be nonempty, i.e., there exists at least one individual who belongs to every decisive set.

Let \( X \) be a finite nonempty set. We shall call a family of subsets of \( X \) a prefilter \( F \) over \( X \) if it has the following properties: (i) \( X \in F \); (ii) if \( A \in F \) and \( A \subseteq B \), then \( B \in F \); (iii) \( \cap F \), the intersection of all the sets belonging to \( F \), \( \neq \emptyset \). Note that every filter is a prefilter, but there exist prefilters that are not filters. If \( X = \{1, 2, 3\} \), then \( \{\{1, 2\}, \{1, 3\}, \{1, 2, 3\}\} \) is a prefilter that is not a filter.

Given a prefilter \( F \), we define the aggregation procedure \( P_F \), where given individual preferences, \( a P_F b \) iff the set of people who prefer \( a \) to \( b \) belong to the prefilter \( F \). The social preference relation defined by the last example is that society prefers \( a \) to \( b \) iff individual (1) prefers \( a \) to \( b \) and at least one other person prefers \( a \) to \( b \).

In general, the social preference relations generated by prefilters will only be acyclic — not partially ordered. In particular, if all the individual preference relations are acyclic, then the social preference relation generated by \( P_F \) is acyclic for any prefilter \( F \) over the set of individuals. This follows from the observation that for preorders the social preference relation will be a subrelation of any individual belonging to \( \cap F \), hence it can have no cycles. \( P_F \) will also be Pareto-efficient and satisfy the condition of independence of irrelevant alternatives. For an infinite number of alternatives the converse is also true, i.e., if a mechanism aggregates every family of acyclic individual preference relations into an acyclic social preference relation and the aggregation procedure is Pareto-efficient and satisfies the condition of independence of irrelevant alternatives, then the decisive sets of that method of aggregation form a prefilter.

A proper prefilter over \( X \) is a prefilter that is not a filter. If the prefilter \( F \) is a family of decisive sets for some aggregation procedure, say \( P_F \), then \( F \) is proper iff there exist two decisive sets whose intersection is not a decisive set. Given any proper prefilter \( F \), we define the collegium of \( F \) as \( \cap F \). We shall call collegial polities the aggregation procedure \( P_F \), generated by proper preorders. If for the moment we think of a collegial polity defined over members of
a committee and the social alternatives as motions before the committee, then a necessary but insufficient condition for a motion \( a \) to defeat a motion \( \beta \) is that the members of the collegium vote in the affirmative for \( a \), not just concur by abstaining from voting. If this condition is met, then the vote of the collegium must be ratified by a coalition of committee members, none of whom belongs to the collegium, such that the larger coalition comprised of the collegium and the given coalition belongs to the prefilter. In principle, the coalition outside the collegium could be drawn from society at large. Institutional this form of collegial polity does not seem to exist, although there may be cases where unanimous approval of a commission is required for a measure that must then be ratified by a specified majority on a referendum. Institutional cases where collegial polities do exist are those involving the executive and legislative branches of the government. In any country where the executive branch has an absolute veto over the bills passed by the legislature, we have a collegial polity, i.e., the executive belongs to every decisive set. Examples are the parliamentary governments of Spain and Ghana. Those governments, like the United States, where the executive veto can be overridden are not collegial polities. The cabinet government affords us with two examples of collegial polities, where it is important that at least part of the polity is elected. By a cabinet government we mean a parliamentary form of government where the cabinet is chosen either by the parliament or head of state. In the cabinet we have the executive or prime minister. One collegial polity is defined by the voting rule that a motion passes if it receives the support of the prime minister, some fixed fraction of the cabinet — say, a third — and a majority of parliament. Another collegial polity is defined by the rule that a law passes if it is supported by the prime minister and a majority of parliament. In the second case the members of the cabinet are merely administrators and informal advisors. Crick has argued that the British parliamentary system has changed from the strong cabinet government of our first example to a parliament dominated by the prime minister as in our second example. The British form of government provides a novel solution to the problem of social indeterminacy arising out of using a collegial polity. On issues basic to the government’s program, either the prime minister is supported by a majority of parliament, or new elections are called. That is, in the event of an irresolvable conflict between the execu-

tive and the legislature, a new collegial polity is chosen. An interesting historical example is the Roman Republic. The Roman Republic used the power of veto along with separation of powers as their primary check on the potentially abusive political power of the majority. We mention one such check, which is discussed in Abbott. The Roman Senate was attended by ten tribunes and three hundred senators. Each tribune had an absolute veto and a motion required a majority vote of the senators. Since the affirmative votes of tribunes did not count for passage, we have a collegial polity where every decisive set consists of the collegium of tribunes and at least a majority of senators.

Collegial politics are distinguished from oligarchies by the asymmetry between the power to enact and the power to veto. The necessary ratification of the collegium’s preferences acts as a significant check against oligarchy if the size of the ratifying coalition is large. The absolute vetoes of the collegium act as a check against Madison’s “tyrannical majority.” In the normative literature of mathematical political theory, Buchanan and Tullock have argued that among the linear voting rules unanimity “minimizes” the asymmetry between the power to enact and the power to veto. Their argument has been challenged by Baumol, who asserts that majority rule “minimizes” the difference between the two powers. Both of these analyses are heuristic, but they do suggest that some quantitative measure of an individual’s power in a polity might provide a means of ethically discriminating among the plethora of collegial politics. Shapley and Shubik have proposed the a priori probability that an individual is pivotal to the success of a winning coalition be used as an index of his power. Again we imagine the set \( X = \{1, 2, \ldots, n\} \) to be the set of individuals in a committee and consider the set of all roll call votes, with respect to alternatives \( a \) and \( b \), obtained by permuting the order of individuals in the committee. Given a voting rule for the committee, for each individual we count the number of permutations in which he is pivotal in a defeating \( b \). By pivotal we mean that he along with the members of the committee preceding him in the roll call form a minimal

AGGREGATION OF PREFERENCES

decisive set. To simplify matters, we shall assume that this number is independent of the alternatives \( a \) and \( b \). The ratio of the number of roll calls in which he is pivotal to the total number of roll calls is his (Shapley-Shubik) power index.

The generic example of a social preference relation generated by a prefilter is given next. Suppose that we have a society of \( k \) people. Choose \( m \) people as a collegium and pick a number \( n \) such that \( m + n \leq k \). Let the social preference be that society prefers \( a \) to \( b \) iff everyone in the collegium prefers \( a \) to \( b \) and at least \( n \) other members of society prefer \( a \) to \( b \). The interesting cases are when both \( m \) and \( n \) are positive. Suppose that we have four individuals, \( X = \{1, 2, 3, 4\} \) and consider the prefilters,

\[ F_1 = \{\{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3\}, \{1,3,4\}, \{1,2,4\}, \{1,2,3,4\}\} \]
\[ F_2 = \{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{1,2,3,4\}\} \]
\[ F_3 = \{\{1,2,3\}, \{1,2,4\}, \{1,2,3,4\}\}. \]

\( a \mathrel{\mathop{P_{F_i}}^b} b \) if individual (1) prefers \( a \) to \( b \) and at least one other person in the collegium prefers \( a \) to \( b \). \( a \mathrel{\mathop{P_{F_0}}^b} b \) if individual (1) prefers \( a \) to \( b \) and at least two other persons in the collegium prefer \( a \) to \( b \). \( a \mathrel{\mathop{P_{F_2}}^b} b \) if both individuals (1) and (2) prefer \( a \) to \( b \) and at least one other person in the collegium prefers \( a \) to \( b \). The following matrix summarizes the three cases, where the row index is over the pre- filters, the column index over the individuals, and the elements of the matrix are the Shapley-Shubik indices:

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 9/12 & 1/12 & 1/12 & 1/12 \\
2 & 6/12 & 2/12 & 2/12 & 2/12 \\
3 & 5/12 & 5/12 & 1/12 & 1/12 \\
\end{array}
\]

It is interesting to compare these indices against those determined by a dictatorial society, with individual (1) as dictator and the rule of unanimity. In the case of dictatorship, individual (1) of course has all the power, i.e., his index is one and everyone else’s index is zero. For the case of unanimity the power is shared equally, and everyone has the power index \( 3/12 \). Coincidentally, this is the same as the power indices for simple majority rule. If we make the same computations for veto power, then we get radically different results. For \( F_1 \) and \( F_2 \), individual (1) has all the veto power. For \( F_3 \) individuals (1) and (2) share the veto power equally between them. In our dictatorial society (1) has all the veto power. Under the rule of unanimity the veto power is shared equally. Again, coincidentally, this agrees with the distribution of veto
power for simple majority rule. These naive calculations support our intuition that the members of the collegium have a disproportionate amount of "power" in the polity. A not so obvious result is that individuals other than (1) are "more powerful" in the polity defined by \( F_2 \) than they are in the polity defined by \( F_1 \). This result runs counter to our intuition that we stand a better chance of having our preferences adopted in the polity defined by \( F_1 \) than in the polity defined by \( F_2 \), since in \( F_1 \) we need only the agreement of individual (1), whereas in \( F_2 \) we need both individual (1) and some other person to agree with our preferences. This suggests that the a priori probability of an individual supporting the proposal that society adopts may be a better index of personal power in a collegial polity.

The requirement that each member of the collegium must vote in the affirmative before a motion can carry seems a little strong. The significance of this requirement becomes clear in regard to the U.N. Security Council. Consider the voting rule used in the Security Council prior to August 31, 1965. The Security Council consisted of eleven members; each of the five permanent members had an absolute veto, suggesting that this body might also be a collegial polity. On substantive matters a motion required a total of seven affirmative votes and the concurring votes of the five permanent members. Abstentions or affirmative votes are counted as concurring votes. Suppose that each member of the Security Council is rational, then we shall show that the Security Council voting rule cannot cycle. Suppose there are \( n \) alternatives, denoted \( a_1, a_2, \ldots, a_n \), where \( a_1 \) defeats \( a_2 \), \( a_2 \) defeats \( a_3 \), \ldots, \( a_{n-1} \) defeats \( a_n \). In order for \( a_1 \) to defeat \( a_2 \), at least one permanent member, say the United States, voted affirmatively for \( a_1 \) over \( a_2 \). In all the remaining votes the United States either voted in the affirmative or abstained, i.e., was indifferent. Therefore, the United States, by assumption of rationality, prefers \( a_1 \) to \( a_2 \). Hence, \( a_n \) cannot defeat \( a_1 \), since the United States, being a permanent member, will impose its veto. If all of the permanent members are indifferent between alternatives \( a \) and \( \beta \), hence abstain from voting, then \( a \) is not preferred to \( \beta \) even if all the nonpermanent members prefer \( a \) to \( \beta \). To redress this imbalance of power, the nonpermanent membership of the Security Council was enlarged to ten members, as of September 1, 1965, and the voting rule was changed to the following: on substantive matters a motion requires a total of nine affirmative votes and the concurring votes of the five permanent members. This new rule, even assuming individual rationality, can cycle. This suggests the following rule: on
substantive matters a motion requires a total of ten affirmative votes and the concurring votes of the five permanent members. If the individuals are rational, then this rule cannot cycle. For suppose that $\alpha_1$ defeats $\alpha_2$, $\alpha_2$ defeats $\alpha_3$, $\ldots$, $\alpha_{n-1}$ defeats $\alpha_n$, then either at least one permanent member voted in the affirmative for say $\alpha_j$ over $\alpha_{j+1}$, in which case our proof for the original Security Council voting rule applies, or all the permanent members were indifferent and abstained in each vote and the nonpermanent members unanimously preferred $\alpha_i$ over $\alpha_{i+1}$ for $i = 1, 2, \ldots, n-1$. Since all members are assumed to be rational, the nonpermanent members will unanimously prefer $\alpha_1$ over $\alpha_n$, and the permanent members will be indifferent between $\alpha_1$ and $\alpha_n$. Therefore $\alpha_1$ will defeat $\alpha_n$. This rule accomplishes the goal of preventing a voting deadlock when there is a unanimous preference among the nonpermanent members and unanimous indifference among the permanent members, with the attendant virtue of being acyclic. Of course, none of these Security Council rules is acyclic if we only assume acyclic individual preferences. Consequently, the Security Council voting rules are not generated by a prefilter, i.e., they are not collegial politics. In fact, the families of decisive sets for these rules do not define prefilters; this does not contradict our theorems, since as we have pointed out, these rules can be made to cycle if we weaken the requirement of rational preferences to that of acyclic preferences.