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An Approximate Solution to Arrow's Problem

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We impose a lattice-theoretic structure on a class of social welfare functions, which properly include Arrowian social welfare functions, by ordering their families of decisive sets by set inclusion. We show that this lattice is compactly generated and characterize the completely meet irreducible elements of this lattice. Certain completely meet irreducible elements correspond to Arrowian social welfare functions. Those social welfare functions corresponding to the other completely meet irreducible families of decisive sets are approximate solutions to the problem of social choice as posed by Arrow.

1. INTRODUCTION

The problem of social choice, i.e., the aggregation of individual preferences, as posed by Arrow is the existence of an aggregation procedure —called by Arrow a social welfare function—which aggregates weak orders, representing individual preferences, into a social preference which is a weak order. In addition, Arrow requires that the social welfare function be Pareto efficient, exhibit positive responsiveness to individual and social values, and satisfy the condition of independence of irrelevant alternatives. More precisely,[1] let I be a (finite or infinite) set of individuals, and X, a (finite or infinite) set of alternatives. A weak order P on X is an asymmetric ($xPy \Rightarrow$ not yPx) and negatively transitive (not xPy and not $yPz \Rightarrow$ not xPz) binary relation on X. $\mathscr{W}$ denotes the set of weak orders on X. A quasitransitive order P on X is an asymmetric and transitive

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[1] The notation is that of Kirman and Sondermann [6].

(xPy and $yPz \Rightarrow xPz$) binary relation on X. $\mathscr{P}$ denotes the set of quasitransitive orders on X. An acyclic order P on X is a binary relation on X such that, for all n, x_1Px_2, x_2Px_3,..., $x_{n-1}Px \Rightarrow$ not x_nPx_1. $\mathscr{A}$ denotes the set of acyclic orders on X. If $|X| \geqslant 3$, then $\mathscr{A} \nsubseteq \mathscr{P} \nsubseteq \mathscr{W}$. Let $F_{\mathscr{R}}$ denote the set of all functions from I to $\mathscr{R}$ for $\mathscr{R} = \mathscr{W}$, $\mathscr{P}$, and $\mathscr{A}$. An $\mathscr{R}$-social welfare function (Arrow considered the case $\mathscr{R} = \mathscr{W}$) is a mapping σ which assigns to each $f \in F_{\mathscr{R}}$ a preference relation $\sigma(f)$ in $\mathscr{R}$. Given an $f \in F_{\mathscr{R}}$, $f(i)$ denotes the preference relation in $\mathscr{R}$ assigned by f to individual $i \in I$. If $a, b \in X$ and $J \subseteq I$, then $af(J)b \Leftrightarrow af(i)b$ for all $i \in J$. $f(i) = g(i)$ on $\{a, b\}$ means that $af(i)b \Leftrightarrow ag(i)b$ and that $bf(i)a \Leftrightarrow bg(i)a$. $f(J) = g(J)$ on $\{a, b\}$ means that $f(i) = g(i)$ on $\{a, b\}$ for all $i \in J$. $|A|$ is the cardinality of the set A. If J is a subset of I, $\bar{J}$ denotes the complement of J with respect to I. Arrow's conditions can now be stated as follows.

A.1 (Alternatives. $|X| \geqslant 3$.

A.2 ($\mathscr{R}$-Social welfare function). σ is a function from $F_{\mathscr{R}}$ into $\mathscr{R}$.

A.3 (Pareto). For all $a, b \in X$ and $f \in F_{\mathscr{R}}$, $af(I)b \Rightarrow a\sigma(f)b$.

A.4 (Positive responsiveness). For all $a, b \in X$ and $f, g \in F_{\mathscr{R}}$, $\{i \in I \mid af(i)b\} \subseteq \{i \in I \mid ag(i)b\}$ and $a\sigma(f)b \Rightarrow a\sigma(g)b$.

A.5 (Independence). For all $a, b \in X$ and $f, g \in F_{\mathscr{R}}$, $f = g$ on $\{a, b\} \Rightarrow \sigma(f) = \sigma(g)$ on $\{a, b\}$.

A.6 (Dictatorship). i_0 is a dictator if, for all $a, b \in X$ and $f \in F_{\mathscr{R}}$, $af(i_0)b \Rightarrow a\sigma(f)b$.

The celebrated Arrow Possibility Theorem is that any $\mathscr{W}$-social welfare function, for finite I, satisfying A.1–A.5 is dictatorial.[2]

In [2] we presented a critique of Arrow's problem, i.e., Arrow's formulation of the problem of social choice. This critique led us to consider $\mathscr{A}$-social welfare functions satisfying conditions A.1–A.5. Of course, there are many such social welfare functions which are nondictatorial. In this paper, we examine these social welfare functions and ask which of them "most nearly" satisfy Arrow's original conditions. This is a difficult question to pose formally because in one case the domain and range of the social welfare function is $\mathscr{W}$, and in the other case the domain and range of the social welfare function is $\mathscr{A}$. Our method of analysis is based on the observation that those social welfare functions which satisfy the condition of independence of irrelevant alternatives are, for the most part, characterized by their family of decisive sets; hence two social welfare functions are "similar" if they have the same family of decisive sets. Recall

[2] It may be noted that A.4 is not needed in the Arrow context to obtain a dictator.

that a set of individuals J is decisive for an $\mathscr{R}$-social welfare function σ if, for all $a, b \in X$ and all $f \in F_{\mathscr{R}}$, $af(J)\,b \Rightarrow a\sigma(f)\,b$. If Λ and Ξ are the families of decisive sets for the $\mathscr{R}$-social welfare functions σ and μ, respectively, and Λ and Ξ completely determine σ and μ, i.e., $\forall a, b \in X$ and $f \in F_{\mathscr{R}}$, $a\sigma(f)\,b$ iff $\{i \in I \mid af(i)\,b\} \in \Lambda$—similarly for μ—then $\Lambda \subseteq \Xi$ implies that, for every $f \in F_{\mathscr{R}}$, $\forall a, b \in X$, $a\sigma(f)\,b \Rightarrow a\mu(f)\,b$. That is, μ is more socially decisive than σ. An alternative formulation of the problem of social choice is the existence of nondictatorial social welfare functions which are socially more decisive than the Pareto rule. These observations suggest that we consider the class of families of decisive sets of $\mathscr{R}$-social welfare functions, satisfying conditions A.1–A.5, under the natural ordering relation of set inclusion. Those families of decisive sets which are maximal under this ordering merit special attention, because they characterize those $\mathscr{R}$-social welfare functions, satisfying conditions A.1–A.5, which are the most socially decisive. We now examine the internal structure imposed on the family of decisive sets of an $\mathscr{R}$-social welfare function if this function is required to satisfy conditions A.1–A.5. To simplify the analysis, we replace A.1 by the stronger assumption A.1′ that X is infinite. The case for finite X is treated in [3].

The condition of Pareto efficiency, A.3, implies that the that the coalition of the whole is a decisive set. Positive responsiveness, A.4, implies that if A is a decisive set and $A \subseteq B$, then B is a decisive set. In [2], we showed that, if σ is an $\mathscr{Q}$-social welfare function satisfying independence, A.5, then the intersection of any finite family of decisive sets is nonempty. Therefore, the family of decisive sets for any $\mathscr{Q}$-social welfare function σ satisfying A.1′ and A.3–A.5 forms a prefilter over the set of individuals. If I is a nonempty set, then a prefilter F over I is a family of subsets of I such that

(i) $I \in F$,

(ii) if $A \in F$ and $A \subseteq B$, then $B \in F$,

(ii) if G is a finite subfamily of F, then $\cap G \neq \varnothing$.

If we impose the stronger condition that σ be a $\mathscr{P}$-social welfare function, in addition to conditions A.1 and A.3–A.5 holding, then Hansson [5] has shown that the family of decisive sets of σ form a filter. F is a filter over I if F is a prefilter and $\forall A, B \in F$, $A \cap B \in F$. Finally, Hansson [5] and, independently, Kirman and Sondermann [6] have shown that Arrow's original requirement that σ be a $\mathscr{W}$-social welfare function, and conditions A.1 and A.3–A.5 hold, implies that σ's decisive sets form an ultrafilter. F is an ultrafilter over I, if F is a filter and, for all A, either A or $\bar{A}$ belongs to F. Every ultrafilter on a finite set I consists of a single point i_0 and the family of subsets of I which contain i_0. Consequently, for finite I, if the decisive sets of a social welfare function σ form an ultrafilter, then σ is dictatorial.

By considering the family of all prefilters, $\mathscr{F}$, over the set of individuals, I, we can examine all of the $\mathscr{R}$-social welfare functions satisfying conditions A.1′ and A.3–A.5. If $\Xi, \Lambda \in \mathscr{F}$, then $\Xi \subseteq \Lambda$ if every decisive set belonging to Ξ also belongs to Λ. Under this ordering $\langle \mathscr{F}, \subseteq \rangle$ is not only a partially ordered set, but is also a compactly generated lattice. This is an important property of $\langle \mathscr{F}, \subseteq \rangle$ because it allows us to distinguish a particularly simple class of prefilters, called completely meet irreducible prefilters. Every prefilter F can be expressed as the meet (intersection) of all the completely meet irreducible prefilters which properly contain it. Those prefilters which are maximal with respect to the ordering $\subseteq$ are completely meet irreducible. In fact, they are ultrafilters, and consequently their social welfare functions are dictatorial. The existence of completely meet irreducible prefilters which are not ultrafilters is guaranteed by the existence of proper prefilters—i.e., prefilters which are not filters—since, if every completely meet irreducible prefilter is an ultrafilter, then every prefilter is the intersection of ultrafilters. But, as is well known, the intersection of a family of filters is a filter.

We propose as approximate solutions to Arrow's problem those social welfare functions whose decisive sets form a completely meet irreducible prefilter which is not an ultrafilter. These are the most socially decisive nondictatorial social welfare functions.

If $I = \{1, 2, 3, 4\}$ and $F_1 = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}$, then we shall see that F_1 is a completely meet irreducible prefilter; it is clear that F_1 is not a filter since $\cap F_1 = \{1\} \notin F_1$. The social preference defined by F_1 is that society prefers alternative a to b if individual 1 prefers a to b and at least one other individual prefers a to b. Prefilters resolve the Arrow paradox by allowing a single individual—a potential dictator—to have a veto but requiring him to have his affirmative vote confirmed by other members in society. For additional examples and discussion of the normative and descriptive properties of social welfare functions defined by prefilters, see [2]. Since every completely meet irreducible prefilter which is a filter must be an ultrafilter, we see that oligarchies, i.e., social welfare functions whose decisive sets form a filter, are not approximate solutions to Arrow's problem.

In order for our results to be comparable to those of Hansson and of Kirman and Sondermann, we have considered the lattice of prefilters over an arbitrary set I. These authors have shown that for infinite I there exists nondictatorial $\mathscr{W}$-social welfare functions satisfying conditions A.1 and A.3–A.5. The decisive sets of such social welfare functions form a free ultrafilter. F is a free ultrafilter if F is an ultrafilter and $\cap F = \varnothing$. Consequently, in this idealized case of an infinite number of individuals, Arrow's problem has a solution. It is therefore important to examine the relationship between completely meet irreducible prefilters and ultrafilters in this instance. We wish to emphasize that the applications we have in mind are for finite I as in the example of the social welfare function generated by F_1 and the examples in [2].

To state our major results, Theorems 1 and 2, we need the notion of an ideal of subsets of I. Filters are one way of formalizing the notion of a family of "large sets." For example, if I is infinite, then the family of cofinite sets, i.e., sets whose complements are finite, form a filter; if I is the real line, then the family of Lebesgue measurable subsets which have measure 1 form a filter; if I is the real line, the family of second-category subsets form a filter. If F is a filter, then the ideal generated by F, denoted K_F, is the family of complements of sets in F. In general, an ideal K is a family of subsets such that

(i) $\varnothing$, the empty set, is in K,

(ii) if $A \in K$ and $B \subseteq A$, then $B \in K$,

(iii) if $A, B \in K$, then $A \cup B \in K$.

An ideal is a family of "small sets": the finite subsets of an infinite set, the sets of Lebesgue measure 0 on the real line, the sets of first category on the real line. Particularly simple ideals are the so-called principal ideals. An ideal K is principal if K is the family of subsets of a given set. If $\mathscr{G}$ and $\mathscr{H}$ are families of subsets of I, then $\mathscr{G}/\mathscr{H}$ is the set-theoretic difference of $\mathscr{G}$ and $\mathscr{H}$. $\mathscr{G}/\mathscr{H}$ is read $\mathscr{G}$ modulo $\mathscr{H}$.

THEOREM 1. *A prefilter F is meet irreducible iff F is an ultrafilter or there exists an ultrafilter U and a nonmaximal ideal K such that $F = U/K$.*

THEOREM 2. *A prefilter F is completely meet irreducible iff F is an ultrafilter or there exists an ultrafilter U and a nonmaximal principal ideal K such that $F = U/K$.*

If I, the set of individuals, is finite, then F is meet irreducible iff F is completely meet irreducible; also, if I is finite, then every ideal is principal. Consequently, for finite I, Theorems 1 and 2 say the same thing. In any event, a prefilter is completely meet irreducible iff it is an ultrafilter modulo a "small" family of "small sets." It is in this sense that completely meet irreducible prefilters approximate ultrafilters and thus afford an approximate solution to Arrow's problem.

2. The Lattice of Prefilters

In our discussion, we shall quote without proof some elementary properties of the lattice of filters of a Boolean algebra and some well-known facts concerning compactly generated lattices. The interested reader is referred to Bell and Slomson [1] for proofs of the assertions pertaining to filters, and to Crawley and Dilworth [4] for those proofs relating to the properties of compactly generated lattices. The following definitions and notation are taken from Crawley and Dilworth.

Let A be a nonempty set and $\leqslant$ a binary relation on A. $\leqslant$ is a partial order if, for all $x, y, z \in A$, (a) $x \leqslant x$, (b) if $x \leqslant y$ and $y \leqslant x$, then $x = y$, (c) if $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$. $\langle A, \leqslant\rangle$ is said to be a partially ordered set if $\leqslant$ is a partial order. If x and y are elements of a partially ordered set, then we write $x < y$ if $x \leqslant y$ and $x \neq y$. When $x < y$, we say that y is greater than x. If B is a subset of a partially ordered set, then there is at most one element $a \in B$ such that $x \leqslant a$ for all $x \in B$. Such an element a, if it exists, is called the greatest element of B. Similarly, if B contains an element b, such that $b \leqslant x$ for all $x \in B$, then b is called the least element of S. An element a in a partially ordered set A is said to be an upper bound of a subset $B \subseteq A$ if $x \leqslant a$ for every $x \in B$. Similarly, a is a lower bound of B if $a \leqslant x$ for every $x \in B$. Let A be a partially ordered set and B a subset of A. We say that an element $a \in P$ is a join (or least upper bound) of B if a is an upper bound of B, and $a \leqslant x$ for every upper bound x of B. Similarly, b is a meet (or greatest lower bound) of B if b is a lower bound of B, and $y \leqslant b$ for every lower bound y of B.

If the join of the subset B exists in the partially ordered set A, then we denote this join by either the formulas $\bigvee B$ or $\bigvee_{b \in B} b$. When B is a two-element set, say $B = \{x, y\}$, we write $\bigvee \{x, y\}$ or $x \vee y$. The meet of B, when it exists, is denoted by $\bigwedge B$ or $\bigwedge_{b \in B} b$ and $\bigwedge \{x, y\} = x \wedge y$.

A partially ordered set in which every pair of elements has a join and a meet is called a lattice. A partially ordered set in which every subset has a join and a meet is called a complete lattice. An element x in a complete lattice is called compact if, whenever $x \leqslant \bigvee B$, there exists a finite subset $C \subseteq B$ with $x \leqslant \bigvee C$. We define a lattice L to be compactly generated if L is complete and each of its elements is the join of compact elements. We say that an element y in a lattice L is meet irreducible if, for all $w, z \in L$, $y = w \wedge z$ implies $y = w$ or $y = z$. We say that an element y in a complete lattice L is completely meet irreducible if, for every subset B of L, $y = \bigwedge B$ implies that $y \in B$. If L has a finite number of elements, then the notions of meet irreducible and complete meet irreducible are the same. Compactly generated lattices have an abundance of completely meet irreducible elements as is shown by the following lemma.

LEMMA 1 (*Crawley and Dilworth*). *If L is a compactly generated lattice, then every element of L is the meet of all the completely meet irreducible elements which are greater than it.*

Let $\mathscr{L}$ be the family of prefilters over a nonempty set X, X may be finite or infinite, and let $^*\mathscr{L} = \mathscr{L} \cup B(X)$, where $B(X)$ is the family of subsets of X.

LEMMA 2. *$^*\mathscr{L}$ ordered under set inclusion is a compactly generated lattice.*

Proof. Omitted.

If G and H are families of subsets of X, then G/H will denote those subsets of X which belong to G and not to H.

LEMMA 3. *If U is a filter and I is an ideal, then U/I is a prefilter.*

Proof. Omitted.

The intersection of all the filters which contain a family of subsets G will be called the filter generated by G. Similarly, the intersection of all the ideals which contain a family of subsets K will be called the ideal generated by K. If H is a family of subsets having the FIP,[3] then there is a smallest prefilter containing H. If H does not have the FIP, then we shall define $B(X)$ as the smallest prefilter containing H. In either case we shall refer to the smallest prefilter containing H as the prefilter generated by H.

LEMMA 4. *Let F be a prefilter, let U be the filter generated by F, and let I be the ideal generated by U/F. If U/F is closed under finite unions, then $F = U/I$.*

Proof. Omitted.

LEMMA 5. *Let F be a prefilter and let U be the filter generated by F. F is meet irreducible iff U is an ultrafilter and U/F is closed under finite unions.*

Proof. Let F be meet irreducible. If $A, B \in U/F$ and $A \cup B \in F$, then A and B are incomparable. If F_A is the prefilter generated by $F \cup \{A\}$ and F_B is the prefilter generated by $F \cup \{B\}$, then $F_A \neq F_B$ and $F = F_A \cap F_B$. This contradicts the meet irreducibility of F, and therefore U/F is closed under finite unions. If U is not an ultrafilter, then there exists a set A and its complement $\bar{A}$ such that $A, \bar{A} \notin U$. If F_A is the prefilter generated by $F \cup \{A\}$ and $F_{\bar{A}}$ is the prefilter generated by $F \cup \{\bar{A}\}$, then $F_A \neq F_{\bar{A}}$ and $F = F_A \cap F_{\bar{A}}$, which contradicts the meet irreducibility of F.

[3] FIP means the finite intersection property, i.e., every finite family has nonempty intersection.

If F is meet reducible and U is an ultrafilter, then there exist prefilters F_1 and F_2 such that $F_1/F \neq \varnothing$, $F_2/F \neq \varnothing$, and $F = F_1 \cap F_2$. If $A \in F_1/F$ and $B \in F_2/F$, then A and B are incomparable and $A \cup B \in F_1 \cap F_2$. Let U_1 be the filter generated by F_1, and let U_2 be the filter generated by F_2. $F \subset F_1$ and $F \subset F_2$ imply that $U \subseteq U_1$ and $U \subseteq U_2$. U is an ultrafilter and therefore $U_1 = U = U_2$. Consequently, $A \in U/F$, $B \in U/F$, $A \cup B \in F$, which proves that U/F is not closed under finite unions.

THEOREM 1. *A prefilter F is meet irreducible iff F is an ultrafilter or there exists an ultrafilter U and a nonmaximal ideal I such that $F = U/I$.*

Proof. Every ultrafilter is a meet irreducible prefilter. Let $F = U/I$ for some ultrafilter U and nonmaximal ideal I, and $A \in U \cap I$. Since I is not maximal, there exist nonempty sets B_1 and B_2 such that $B_1 \cap B_2 = \varnothing$, $B_1 \cup B_2 = X$, and $B_1, B_2 \notin I$. If $A_1 = A \cup B_1$ and $A_2 = A \cup B_2$, then $A_1, A_2 \in U/I$ and $A_1 \cap A_2 = A$. If F^+ is the filter generated by U/I, then $F^+ = U$. Hence, by Lemma 5, F is meet irreducible.

If F is meet irreducible and U is the filter generated by F, then U is an ultrafilter and U/F is closed under finite unions. If I is the ideal generated by U/F, then by Lemma 4 $F = U/I$. If I is a maximal ideal, then $\bar{I}$, the complement of I, is an ultrafilter. Hence $F = U/I = U \cap \bar{I}$, which contradicts the assumption that F is a meet irreducible prefilter.

THEOREM 2. *A prefilter F is completely meet irreducible iff F is an ultrafilter or there exists an ultrafilter U and a nonmaximal principal ideal I such that $F = U/I$.*

Proof. Let F be meet irreducible and assume that there does not exist an ultrafilter U and a nonmaximal principal ideal I such that $F = U/I$. By Theorem 1 there exist some ultrafilter U and a nonmaximal ideal I such that $F = U/I$. For each $A \in I$, let I_A be the principal ideal generated by A. Then $I = \bigcup_{A \in I} I_A$ and $\bar{I} = \bigcap_{A \in I} \bar{I}_A$. Hence $F = \bigcap_{A \in I} F_A$, where $F_A = U/I_A$ and F_A is a prefilter by Lemma 3. By assumption $F \neq F_A$ for all $A \in I$. Therefore F is completely meet reducible.

Let $F = U/I_B$ for some ultrafilter U and nonmaximal principal ideal I_B, where I_B is the principal ideal generated by the subset B. If $B \notin U/I_B$, then $U/I_B = U$. If $B \in U$ and F^+ is a prefilter properly containing F, then the only sets which are consistent with those in F are members of U. If $E \in F^+/F$, then $E \in I_B$ and $B \in F^+$. Therefore the intersection of all the prefilters which properly contain F also contains B. That is, F is completely meet irreducible.

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