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A Note on the Shape of the Pareto Optimal Surface*

1. The Problem

Given n individuals each of which has a complete preference ordering over k outcomes, we may represent the preference ordering of any individual on a utility scale. Thus any one of the k outcomes can be represented as a point in an n-dimensional Euclidean space.

As we only postulate an ordering over the outcomes many utility scales will reflect the preference structure. Any scale that can be derived from any other by an order preserving transformation will serve. Thus for example if: $a > b > c > d$ then a scale which assigns values 4, 3, 2, 1 or another scale which assigns values 4, .003, .0003, .0001 will serve equally well.

Given the freedom in selecting scales is it possible to select them in such a manner that the k points lie on a hyperplane?

2. Motivation

In the various theories of social choice such as voting, the economic theory of market exchange, bargaining, fair division procedures and various game theoretic solution theories, the solution suggested usually depends upon one of three sets of assumptions. They are (1) assumptions concerning the structure of the outcomes and the relationship between the outcome structure and the preference structure of the individual; (2) assumptions concerning a preference structure given *in abstracto* with no particular connection to or assumptions made about the structure of the outcomes; and (3) assumptions concerning the utility of the outcomes to each individual with a minimum of consideration given to the structure of the outcomes and the preference structure.

Examples of solutions pertaining to each of the three types of assumptions are given. The fair division procedure to decide how to divide a homogeneous cake between two individuals exemplifies (1). One individual cuts and the other chooses. The presumption is that the cake is divisible and homogeneous and (usually) that less cake is not preferred to more.

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The economic model of exchange in a market with prices provides another example of (1). The existence of a price system depends upon the shape of the preference contours over different bundles of commodities.

The Arrow [1] discussion of voting and preferences provides an example of (2). No particular properties of the outcomes are specified.

The Harsanyi [2] and other value solutions, [3] the bargaining set [4] and the core [5] are usually defined for situations in which utility functions for individuals can be specified up to a linear transformation. Thus, the solutions considered are composed of an imputation or set of imputations in the n-dimensional utility space. The details and physical aspects of the outcomes giving rise to the imputations in the solution need not be considered, beyond making the assumption that there are sufficient conditions to enable us to derive utility functions from preferences over outcomes.

How much structure is imposed on the shape of the Pareto optimal surface in the utility space as we make assumptions about the structure of the outcomes and about the structure of individual preferences over these outcomes?

In particular the interest in the question above comes when we search for intrinsic ways to compare utilities or to achieve transfers or side-payments. For example, if it were in general easy to obtain a flat Pareto optimal surface we might claim that a natural way of transferring and possibly even comparing welfare exists. If it is rarely possible to "flatten" the surface this appeal to an intrinsic measure loses its force.

This note examines part of this question for the case where no structure or description of the outcomes is given beyond observing that there are k outcomes.

3. A Planar Pareto Optimal Surface

Although our concern is with any number n of people and k outcomes we commence with the case for $n = 3$.

For 3 people and k outcomes with a complete preference ordering for each person, what is the smallest value of k such that there exists an ordering for which there is no way to assign nonnegative utilities that sum to one for each outcome that is consistent with the preference orderings.

Notation. Let u_j^i be the utility of the ith person for outcome j $i = 1, 2, 3$ and $j = 1, 2, \ldots, k$. Assume that the ordering for person 1 is $(1, 2, \ldots, k)$ where 1 is most preferred and k is least preferred. The orderings for persons 2 and 3 are $(m(1), m(2),..., m(k))$ and $(n(1), n(2),..., n(k))$ respectively. For a given ordering, utilities are sought such that:

(1)
$$
\begin{gathered}
u_1^1 > u_2^1 > \cdots > u_k^1 \\
u_{m(1)}^2 > u_{m(2)}^2 > \cdots > u_{m(k)}^2 \\
u_{n(1)}^3 > u_{n(2)}^3 > \cdots > u_{n(k)}^3 \\
u_1^1 + u_1^2 + u_1^3 = 1 \\
\vdots \\
u_k^1 + u_k^2 + u_k^3 = 1 \\
u_j^i \geqslant 0, \qquad i = 1, 2, 3 \qquad \text{and} \quad j = 1, 2,..., k.
\end{gathered}
$$

The problem is to find the minimum k for which there exists an ordering such that (1) has no solution.

Assumption 1. If each person prefers outcome i to outcome j, then it is not possible to find utilities that satisfy (1). Therefore, we will assume that no outcome is preferred to another outcome by all persons. This condition implies that for every pair of outcomes i, j—at least one person prefers i to j and at least one person prefers j to i.

The question of whether utilities exist that satisfy condition (1) for a given ordering can be resolved by solving a linear programming problem.

(2)
$$
\begin{gathered}
\max t \\
s/t \qquad u_1^1 \geqslant u_2^1 + t \\
u_2^1 \geqslant u_3^1 + t \\
\vdots \\
u_{k-1}^1 \geqslant u_k^1 + t \\
u_{m(1)}^2 \geqslant u_{m(2)}^2 + t \\
\vdots \\
u_{m(k-1)}^2 \geqslant u_{m(k)}^2 + t \\
u_{n(1)}^3 \geqslant u_{n(2)}^3 + t \\
\vdots \\
u_{n(k-1)}^3 \geqslant u_{n(k)}^3 + t \\
u_1^1 + u_1^2 + u_1^3 = 1 \\
\vdots \\
u_k^1 + u_k^2 + u_k^3 = 1 \\
u_j^i \geqslant 0, \qquad i = 1, 2, 3 \qquad \text{and} \quad j = 1, 2,..., k.
\end{gathered}
$$

The optimal solution to this linear program is a solution to (1) if $t > 0$ and there is no solution to (1) if the optimal solution has $t = 0$. The linear program has a feasible solution ($u_i^j = 1/3$ for all i, j and $t = 0$), the optimal solution is bounded above by 1, thus (2) always has an optimal solution. Since the coefficients of (2) are all integer, a well-known result of linear programming gives:

LEMMA 1. *If there is a solution to* (1), *then there is a rational solution.*

LEMMA 2. *Assume that for every possible ordering with $k - 1$ outcomes satisfying assumption* 1 *there is a solution to* (1). *For any ordering with k outcomes such that one outcome is most preferred by at least one person and is least preferred by at least one person, there is a solution to* (1).

Proof. Without loss of generality, assume outcome 1 is most preferred by person 1 and least preferred by person 2. Remove outcome 1 from the orderings, then by the hypothesis there exists a solution to (1) for outcomes $2, 3,..., k$ denoted by $\bar{u}_j^i$. Set $\hat{u}_1^1 = 2$ and $\hat{u}_j^1 = \bar{u}_j^1, j = 2, 3,..., k$. Set $\hat{u}_1^2 = 0$ and $\hat{u}_j^2 = \bar{u}_j^2 + 1, j = 2, 3,..., k$. For the third person, one of these possible cases occurs:

(A) If outcome 1 is the most preferred set $\hat{u}_1^3 = 2$ and $\hat{u}_j^3 = \bar{u}_j^3$, $j = 2, 3,..., k$.

(B) If outcome 1 is least preferred set $\hat{u}_1^3 = 0$ and $\hat{u}_j^3 = \bar{u}_j^3 + 1$, $j = 2, 3,..., k$,

(C) If $1 = n(p)$, set $\hat{u}_1^3 = [\bar{u}^3_{n(p+1)} + \bar{u}^3_{n(p-1)}]/2$ and set $\hat{u}_j^3 = \bar{u}_j^3$, $j = 2, 3,..., k$.

The sum of utility for outcomes $2, 3,..., k$ are equal, denote the sum by t. Let s denote the sum for outcome 1. If $s = t$, dividing each $\hat{u}_j^i$ by t gives a rational solution to (1). If $t > s$, add $t - s$ to $\hat{u}_1^1$, dividing each $\hat{u}_j^i$ by t gives a rational solution to (1). If $t < s$ add $s - t$ to each $\hat{u}_j^2, j = 2, 3,..., k$, dividing by each $\hat{u}_j^i$ by s gives a rational solution to (1). Q.E.D.

LEMMA 3. *For any ordering with* 2 *outcomes that satisfies Assumption* 1, *there is a solution to* (1).

Proof. Allowing a renumbering of the outcomes and persons, there is only one case to consider.

$$u_1^1 > u_2^1$$
$$u_1^2 > u_2^2$$
$$u_2^3 > u_1^3$$

Then $u_1^1 = u_1^2 = 1/2$, $u_2^3 = 1$, $u_2^1 = u_2^2 = u_1^3 = 0$ satisfies (1). Q.E.D.

LEMMA 4. *For any ordering with 5 or fewer outcomes that satisfies Assumption* 1, *there is a solution to* (1).

Proof. Each person has a most preferred and least preferred outcome. Since there are 5 or fewer outcomes, among these 6 numbers must be at least one outcome appearing twice. Assume this is Outcome 1. Outcome 1 must be most preferred by some person and least preferred by another person in order to satisfy Assumption 1 (if Outcome 1 was most preferred by two persons and the third person preferred Outcome 1 to Outcome j then Assumption 1 would be violated; similarly if Outcome 1 was least preferred by two people). Now for $k = 3$, Lemmas 2 and 3 imply the result for $k = 3$; this result implies the result for $k = 4$ and this latter result implies the result for $k = 5$. Q.E.D.

LEMMA 5. *For* 6 *outcomes, there is an ordering satisfying Assumption* 1 *for which there is no solution to* (1).

Proof. Assume the following is a solution to (1)

$$u_1^1 > u_2^1 > u_3^1 > u_4^1 > u_5^1 > u_6^1$$
$$u_3^2 > u_6^2 > u_5^2 > u_2^2 > u_1^2 > u_4^2$$
$$u_5^3 > u_4^3 > u_1^3 > u_6^3 > u_3^3 > u_2^3$$

Sum the 9 strict inequalities that involve an outcome with an odd number being preferred to an outcome with an even number, this yields:

$$u_1^1 + u_3^2 + u_5^3 + u_3^1 + u_5^2 + u_1^3 + u_5^1 + u_1^2 + u_3^3 >$$
$$u_2^1 + u_6^2 + u_4^3 + u_4^1 + u_2^2 + u_6^3 + u_6^1 + u_4^2 + u_2^3.$$

This can be simplified to $3 > 3$ which is a contradiction. Thus, there is no solution to (1) for this ordering. Q.E.D.

Lemmas 4 and 5 completely resolve the case $n = 3$. It is also possible to answer the question for problems with a different number of persons where problem (1) is extended in an obvious manner.

LEMMA 6. *For two persons, any positive number of outcomes and any ordering that satisfies Assumption* 1, *there is a solution to* (1).

Proof. The only ordering satisfying assumption 1 is:

$$1 > 2 \qquad > \cdots > k$$
$$k > k - 1 > \cdots > 1.$$

A solution satisfying (1) is $u_j^1 = (k - j + 1)/(k + 1)$ and $u_j^2 = j/(k + 1)$, $j = 1, 2,..., k$. Q.E.D.

LEMMA 7. *For $n \geqslant 2$ persons,* 2 *outcomes and any ordering satisfying Assumption* 1, *there is a solution to* (1).

Proof. The first k persons $1 \leqslant k < n$ have ordering $1 > 2$ and the remainder have $2 > 1$. A solution to (1) is $u_1^j = 1/k$ and $u_2^j = 0$, $j = 1, 2,..., k$, $u_2^j = 1/n - k$ and $u_1^j = 0, j = k + 1,..., n$. Q.E.D.

LEMMA 8. *For $n \geqslant 2$ persons,* 3 *outcomes and any ordering satisfying Assumption* 1, *there is a solution to* (1).

Proof. Person 1 has ordering $1 > 2 > 3$. If there exists a person with outcome 1 the least preferred outcome, then an obvious generalization of the proof of Lemma 1 together with Lemma 7 gives the result. If no person has outcome 1 as the least preferred outcome, then in order for assumption 1 to be met there must be a person with $2 > 1 > 3$ and another person with $3 > 1 > 2$. The above argument is repeated with outcome 3 replacing outcome 1. Q.E.D.

LEMMA 9. *For $n \geqslant 4$ persons and* 4 *outcomes, there is an ordering satisfying Assumption* 1 *for which there is no solution to* (1).

Proof. The first 4 persons have orderings

$$1 > 2 > 3 > 4$$
$$1 > 4 > 3 > 2$$
$$3 > 2 > 1 > 4$$
$$3 > 4 > 1 > 2$$

which satisfy Assumption 1 and each of the remaining $n - 4$ persons has one of the above orderings. Summing the $2n$ inequalities with an odd numbered outcome preferred to an even numbered good contradicts the existence of a solution to (1). Q.E.D.

Summary. For $n \geqslant 2$ persons, the smallest number of outcomes k for which there exists an ordering satisfying Assumption 1 for which there is no solution to (1) is given by:

n	2	3	4	5	6 ⋯
k	∞	6	4	4	4 ⋯

We have assumed strict preferences; however, our results extend to the case where there is at least one good strictly preferred by at least one person. It is easy to see that the counterexamples in the proofs of Lemmas 5 and 9 remain counterexamples.

Further Problems

Given n individuals and k outcomes there are in total $(k!)^n$ preference states for the society of n individuals. What fraction of these could have arisen from trading economies with indivisible goods?

A trading economy with n individuals trading in m goods where for any good i there are b_i units, will have a preference structure for each individual that can be described by a modified lattice. An example to illustrate this is given by $n = 2$, $m = 2$, $b_1 = 2$, $b_2 = 2$.

Assumption 2. The preferences of an individual depend only upon the commodities he obtains, not on the holdings of others.

Assumption 3. An individual does not prefer less to more. Figure 1 shows the preference structure among the 9 outcomes that exist in this 2 person 2 goods 2 units of each good trading economy.

Assumption 4. Diminishing rate of substitution between goods.*

This is the assumption of strict convexity in "level sets" or indifference curves. In the example in Fig. 1 it would require that (1, 1) be preferred to (2, 0) or (0, 2).

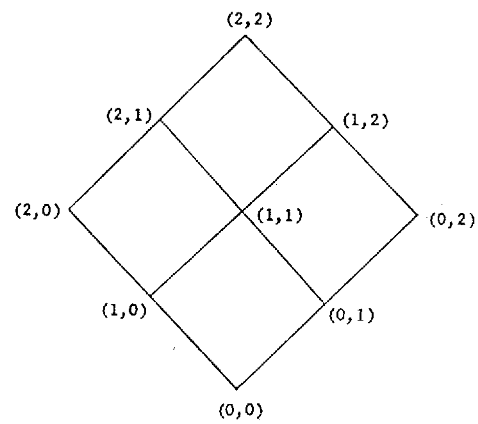

Figure 1

It is easy to see that only an extremely small fraction of preference states can arise from trading economies. For a two good economy with b units of each good and only integral amounts of goods possible, there are $(b + 1)^2$ possible states for each person, (0, 0), (1, 0),..., (b, b). Of the $[(b + 1)^2]!$ preference states, fewer than $2^{(b+1)(b+1)}$ satisfy Assumption 3.

* We could modify this to a nonincreasing rate of substitution.

For preference states that arise from trading economies (that is, satisfy Assumptions 2, 3, and 4) with n persons what is the smallest number of goods k so it is possible to obtain a Pareto Optimal set that does not satisfy (1)? By Lemma 6, for 2 people there is no k. For one good every preference ordering yields a Pareto Optimal set that satisfies (1). Two examples show that for n persons ($n \geqslant 3$) k is 2.

EXAMPLE 1. For 3 persons in a trading economy with 24 units each two goods, consider the following 6 outcomes where x, y is the amount of Good 1 and Good 2.

	Person 1	Person 2	Person 3
Outcome 1	18, 0	6, 0	0, 24
2	0, 15	4, 9	20, 0
3	9, 5	15, 4	0, 15
4	0, 11	0, 4	24, 9
5	8, 0	14, 0	2, 24
6	0, 6	0, 18	24, 0

If persons 1, 2, 3 determine their preferences for the outcomes using functions $x + y$, $x + y$, $2x + 3y$ respectively, then the resulting preferences are the same as in the proof of Lemma 5. Thus there is no solution to (1).

EXAMPLE 2. For 4 persons in a trading economy with 10 units each of two goods, consider the following 4 outcomes:

	Person 1	Person 2	Person 3	Person 4
Outcome 1	6, 0	0, 6	4, 0	0, 4
2	0, 5	4, 0	2, 5	4, 0
3	4, 0	0, 4	6, 0	0, 6
4	0, 4	5, 0	0, 4	5, 2

If persons 1, 2, 3, 4 determine their preferences using functions $6x + 5y$, $5x + 6y$, $2x + y$, $x + 2y$ respectively, then the resulting preferences are the same as in the proof of Lemma 9. For $n > 4$, use the same outcomes for persons 1, 2, 3 and give each of the others $1/(n - 3)$ times the outcomes for person 4.

The utility functions ($x + y$, $2x + 3y$, etc.) used to define preferences in Examples 1 and 2 do not have strictly convex level sets. However, slight modifications of the functions do yield utility functions with the same preference orderings and strictly convex level sets.

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