Risk Aversion and Wealth Effects on Portfolios with Many Assets ¹, ²

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I. INTRODUCTION

This paper is concerned with describing the qualitative behavior of alternative characterizations of portfolios with many assets or securities; in particular, we wish to know what can be said about changes in portfolios as wealth changes.

At least three alternative characterizations seem available:

1. A portfolio can be characterized by the percentage of its value held in the form of a safe asset or money. Those like Tobin [5] and Arrow [1] who have attempted to derive properties of the demand curve for money from portfolio analysis have implicitly employed this characterization.

2. A portfolio can be characterized by certain statistical properties, e.g., its mean, variance, range, etc.

3. A portfolio can be characterized in terms of certainty equivalents: at what certain rate of return would the individual investor be indifferent between his (risky) portfolio and the portfolio consisting of just a safe asset with that rate of return?

In Section 2 we present theorems describing the effect of changes in wealth on portfolios in terms of all three characterizations for the special case of two assets, one of which is perfectly safe. In Sections 3 and 4 we consider whether these theorems can be extended to situations where there are more than two assets. In Section 3, we show that if there are as many securities as states of nature, although the theorems relating to the second and third characterizations still obtain, that relating to the first does not. This suggests that without stringent conditions it does not appear possible to derive a simple theory of the demand for money from portfolio analysis. In Section 4, we show that only the theorem relating to certainty equivalents can be extended to the general case where there are more states of nature than securities.

Throughout the paper we assume that individual investors choose their portfolios to maximize the expected utility of terminal wealth. ³ The individual has an initial wealth of $W_0$, which he can allocate among a number (two or more) of assets. The $i$th asset

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³ There is no question that these assumptions—as well as, for example, the implicit assumption that there are no transaction costs, or our later assumption that there are no other restrictions on short sales—severely limits the usefulness of this model. However, our primary purpose is to show that even with such drastic simplifications the possibilities for characterizing wealth effects are quite restricted.
yields a return per dollar invested in it of $\rho_\theta$ in state $\theta$. Money is the particular asset which yields the same return in every state of nature, which for simplicity we shall denote by $\rho_M$.  \footnote{In the sequel, whenever there exists a safe asset we shall let it be the first asset, so $\rho_\theta = \rho_M$ all $\theta$, and we shall let $a_0$ denote the proportion of wealth allocated to the safe asset.} If the individual allocates a fraction $a_i$ of his initial wealth to the $i$th asset, his terminal wealth in state $\theta$ will be

$$W_\theta = \Sigma a_i \rho_\theta W_0$$  (1)

where

$$\Sigma a_i = 1.$$  (2)

Thus $a_i$ are chosen to maximize

$$EU[W_\theta] = EU[\Sigma a_i \rho_\theta W_0] = EU[\Sigma a_i \rho_\theta W_0] = \Sigma a_i \rho_\theta W_0 \pi_\theta$$

subject to (2), where $\pi_\theta$ is the probability that state $\theta$ will occur. $U$ is assumed to be thrice differentiable, strictly increasing, and strictly concave function of $W$:

$$U'[W] > 0, U''[W] < 0 \text{ and } U'''[W] \text{ exists.}$$  (3)

Necessary and sufficient conditions for expected utility maximization are given by

$$EU'(\rho_\theta - \rho_M) = 0 \text{ all } i. \footnote{This assumes that no restrictions are imposed on the magnitude of $a_i$. $a_i < 0$ implies that the individual is selling short the given security. If $a_M < 0$, it means he is borrowing rather than lending. It also assumes that there exists a perfectly safe asset. If no such asset exists, we may write $EU'(\rho_\theta - \rho_M) = 0$, where $\theta$ represents any particular asset.}$$  (4)

In the analysis below, we shall describe the behaviour of the portfolio in terms of the behaviour of a simple characteristic of the utility function: the elasticity of marginal utility $U''[W]/U'$. This is called the Arrow-Pratt measure of relative risk aversion [1, 3], and will be denoted by $R[W]$.  

2. TWO ASSETS

We now consider the special case where there are two assets, money and the risky asset whose returns we denote simply by $\rho_0$ (without asset subscript). The necessary and sufficient condition for expected utility maximization (4), may now be written

$$EU'[W_0(a_M \rho_M + (1-a_M) \rho_0)](\rho_M - \rho_0) = 0.$$  (5)

The theorem making use of the first characterization of portfolios, in terms of the percentage allocation to money, is straightforward and well known.

**Theorem 1.** If there is a single risky asset and money, then the wealth elasticity of the demand for money is greater than, equal to, or less than unity as relative risk aversion is an increasing, constant, or decreasing function of wealth:

$$\frac{d a_M}{d W_0} \geq 0 \text{ as } R[W] \geq 0.$$  (6)

**Proof.** \footnote{Notice that we will use brackets to denote arguments of $U$—as in (3), parentheses to denote expressions multiplied by derivatives of $U$—as in (4) (which enables us to suppress arguments of $U$ without confusion).} (5) defines $a_M$ as an implicit function of $W_0$. We thus obtain

$$\frac{d a_M}{d W_0} = \frac{EU''(W_0(a_M \rho_M + (1-a_M) \rho_0))(\rho_M - \rho_0)}{EU''(W_0(\rho_M - \rho_0))^2}.$$
The denominator is unambiguously negative. Let \( W^* = W_0 \rho_M, R^* = R[\mathbf{W^*}] \). Then

\[
- \mathbb{E}^\alpha (W_0(a_M \rho_M + (1 - a_M) \rho_0) (\rho_M - \rho_0)) = \mathbb{E}^\alpha(\rho_M - \rho_0) \\
= \mathbb{E}^\alpha((R - R^*) U'(\rho_M - \rho_0) + R^* \mathbb{E}^\alpha(\rho_M - \rho_0)) \\
= \mathbb{E}^\alpha((R - R^*) U'(\rho_M - \rho_0) \equiv 0 \text{ as } R^* \equiv 0
\]

since \( R^* \mathbb{E}^\alpha(\rho_M - \rho_0) = 0 \) [by (5)] and since when \( \rho_0 > \rho_M, W > W^* \), so \( R[W] \equiv R[\mathbf{W^*}] \) as \( R^* \equiv 0 \) (and oppositely when \( \rho_0 < \rho_M \)).

There are many alternative statistics by which a portfolio might be characterized. We focus on three; writing the return per dollar invested for the portfolio as a whole as \( \Sigma a_i \rho_{i0} \equiv \bar{r} \), we define

(a) the mean return per dollar invested, \( \bar{r} = \mathbb{E}r_0 = \Sigma a_i \rho_{i0} \);

(b) the variance of return per dollar invested, \( \sigma_r^2 = \mathbb{E}(\Sigma a_i \rho_{i0} - \bar{r})^2 \);

(c) the range of return per dollar invested, \( [r_{\min}, r_{\max}] = [\min \rho_0, \max \rho_0] \).

We make use of Theorem 1 to establish

**Theorem 2.** If there is a single risky asset whose average return is greater than that on money,\(^1\) the mean, variance, and range of the rate of return on the portfolio as a whole are increasing, constant, or decreasing functions of wealth as relative risk aversion is a decreasing, constant, or increasing function of wealth:

\[
\frac{d \bar{r}}{d W_0} \geq 0, \quad \frac{d \sigma_r^2}{d W_0} \leq 0, \quad \frac{d r_{\min}}{d W_0} \leq 0, \quad \text{and} \quad \frac{d r_{\max}}{d W_0} \leq 0 \quad \text{as} \quad R^* \equiv 0. \quad (7)
\]

**Proof.** For two assets, we have

\[
\bar{r} = a_M \rho_M + (1 - a_M) \rho_0
\]

\[
\sigma_r^2 = (1 - a_M)^2 \mathbb{E}(\rho_0 - \rho_0)^2
\]

\[
r_{\min} = a_M \rho_M + (1 - a_M) \min \rho_0
\]

and

\[
r_{\max} = a_M \rho_M + (1 - a_M) \max \rho_0,
\]

so

\[
\frac{d \bar{r}}{d W_0} = (\rho_M - \rho_0) \frac{d a_M}{d W_0} \quad (8a)
\]

\[
\frac{d \sigma_r^2}{d W_0} = -2(1 - a_M) \mathbb{E}(\rho_0 - \rho_0)^2 \frac{d a_M}{d W_0} \quad (8b)
\]

\[
\frac{d r_{\min}}{d W_0} = (\rho_M - \min \rho_0) \frac{d a_M}{d W_0} \quad (8c)
\]

and

\[
\frac{d r_{\max}}{d W_0} = (\rho_M - \max \rho_0) \frac{d a_M}{d W_0} \quad (8d)
\]

The result for the mean and the range follows immediately from (6) and (8a), (8c) and

\(^1\) We also assume that the risky asset does not dominate the safe, i.e., that \( \min \rho_0 < \rho_M \).
(8d). For the variance, we must show that \( E\rho > \rho_M \) implies \( a_M < 1 \). Because of the concavity of \( U \), \( EU \) is a concave function of \( a_M \). But

\[
\frac{dEU}{da_M} \bigg|_{a_M = 1} = EU' \bigg|_{a_M = 1} W_0(\rho_M - \rho_0) = U'[W_0 \rho_M]EW_0(\rho_M - \rho_0) < 0,
\]

and the result is immediate.

Actually, a slightly stronger result than Theorem 2 can be established. One would like to be able to make a statement about the "dispersion" of the returns which was not dependent on any particular parametric characterization, such as variance or range. Thus, let us define

\[
v_\theta \equiv \frac{r_\theta}{\bar{r}}
\]

as the value of \( r \) in any particular state relative to its mean value. Clearly, \( E v_\theta = 1 \). Rothschild and Stiglitz [4] have studied the question of when we can say one random variable \( X \) with distribution \( F \) is "more uncertain" ("more variable") than another \( Y \) with distribution \( G \), and have shown that the following statements are equivalent:

(i) \( \int U(x)dF(x) < \int U(y)dG(x) \) for every concave \( U \). All risk averters prefer \( Y \) to \( X \).

(ii) There exists a random variable \( Z \) such that \( X \) has the same distribution as \( Y + Z \), where \( E[Z \mid Y] = 0 \) for all \( Y \). \( X \) is equal to \( Y \) plus some noise.

(iii) Suppose the points of increase of \( F \) and \( G \) are confined to a closed interval \([a, b]\), and define

\[
T(y) = \int_a^y (F(x) - G(x))dx \text{ for } a \leq y \leq b.
\]

Then \( T(y) \geq 0 \) and \( T(a) = T(b) = 0 \). \( X \) has more weight in its tails than \( Y \).

They argue that if \( X \) and \( Y \) satisfy (i)-(iii), then it is reasonable to say that \( X \) is "more uncertain", or "more variable" than \( Y \).

We shall now show

**Theorem 2a.** Under the conditions of Theorem 2, \( v_\theta \) becomes more or less variable, in the sense defined above, with increasing wealth as \( R' \leq 0 \).

**Proof.**

\[
v_\theta = \frac{a_M \rho_M + (1 - a_M) \rho_0}{\rho_M + (1 - a_M) \rho_0},
\]

so that

\[
\frac{dv_\theta}{dW_0} = \frac{(E\rho - \rho_0) \rho_M}{\bar{r}^2} \frac{da_M}{dW_0} \geq 0 \text{ as } R'(E\rho - \rho_0) \geq 0.
\]

Hence, because \( E v_\theta = 1 \), an increase in \( W_0 \) must result in a more or less variable \( v \), as \( R \leq 0 \) (appealing to the particular characterization in statement (iii) above), as illustrated in Figure 1, (where the states of nature have been arranged in order of increasing \( \rho_0 \)).

We now turn to the third characterization, in terms of certainty equivalents. As noted in the introduction, we may define \( r^* \) as that certain return which would make the individual investor indifferent between his (optimally chosen) portfolio and investing all his wealth at \( r^* \):

\[
U[r^*W_0] = \max_{\Sigma a_t = 1} EU[\Sigma a_t \rho_0 W_0].
\]
Theorem 3. The certainty equivalent return is an increasing, constant, or decreasing function of wealth as relative risk aversion is a decreasing, constant, or increasing function of wealth:

\[
\frac{d\pi^*}{dW_0} \geq 0 \quad \text{as} \quad R' \geq 0.
\] (10)

Since the proof for the many asset case is no more difficult than that for the two asset case, we postpone the proof until Section 4.

![Graph](image)

**Figure 1**
Relation between relative risk aversion and variability of returns.

3. NUMBER OF SECURITIES EQUALS NUMBER OF STATES

The case of two assets is very special indeed. In that situation, for instance, the mean, variance, and range of the rate of return can increase if and only if the proportion of wealth allocated to money decreases;\(^1\) the first and second characterizations of a portfolio are perfectly equivalent. This is not true when there are more than two assets. For then, the investor can change the proportions in which he holds the risky assets as well. This raises the question, which of the two characterizations is the more "fundamental", i.e., which one can be generalized to cases of more than two assets (or can both be generalized)?

In this section we investigate this question for the special case where there are as many securities as states of nature.

Before proceeding we first note that there is an essential difference between the first characterization, pertaining to the demand for a particular type of asset in the portfolio, and the second (as well as third) characterization, relating to properties of the total portfolio. In particular, when there is money and several risky assets, there are an infinite number of alternative but quite different sets of money and risky assets which provide the same opportunity set for terminal wealth, i.e., which are equivalent from the investor's viewpoint. Consequently, though the composition of portfolios corresponding to these equivalent opportunities may also be quite different, the mean, variance and range of return per dollar invested (as well as certainty equivalent return) will obviously be identical.

This suggests that it may not be possible to prove a theorem analogous to Theorem 1 for the many asset case. And indeed, such turns out to be true; examples can be constructed in which an individual increases (decreases) his relative holdings of money as his

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\(^1\) In the case of an interior solution, where \(0 < a_{it} < 1\).
wealth increases even though he has decreasing (increasing) relative risk aversion. Surprisingly enough, however, even in these cases the statistical measures—mean, variance, and range of the return per dollar invested increase, remain constant, or decrease with wealth as the individual has decreasing, constant or increasing relative risk aversion. Moreover, the returns become unambiguously more (less) variable, in the sense defined above, as the individual has decreasing (increasing) relative risk aversion.

Before turning to the "negative" results, we first prove the latter assertion.

**Theorem 2'.** If there are as many assets as states of nature, then the mean, variance and range of the return per dollar invested increase, remain constant, or decrease with wealth as the individual has decreasing, constant or increasing relative risk aversion. Moreover, the returns become unambiguously more (less) variable, in the sense defined above, as the individual has decreasing (increasing) relative risk aversion.

In order to arrive at this result, we begin by looking at the structure of a market in which the number of assets and states is the same.

For such a market, the set of available assets is described by an \( n \times n \) matrix

\[
P = \begin{bmatrix}
    \rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\
    \vdots & \ddots & \ddots & \vdots \\
    \rho_{n1} & \rho_{n2} & \cdots & \rho_{nn}
\end{bmatrix},
\]

the rows of which represent the different assets, the columns the different states. (If there is a safe asset, its returns appear in the first row.)

Geometrically, the opportunity set of returns per dollar invested corresponding to (11) may be represented by an \( n - 1 \) dimensional hyperplane defined parametrically by

\[
r_\theta = \Sigma a_i \rho_{i\theta}, \Sigma a_i = 1.
\]

This representation makes apparent the fact that we can define a new set of securities which yields the identical opportunity set, but each of which yields a return in only one state of nature—the so-called Arrow-Debreu securities market—simply by choosing

\[
\rho_{i\theta} = \rho_\theta = \text{intersection of the hyperplane (11') with the } i\text{th axis } i = \theta \\
= 0 \quad i \neq \theta.
\]

Diagrammatically, this is illustrated in Figure 2, where the opportunity set of returns per dollar for the case of three states and three securities (one safe and two risky) is depicted. \( M \) denotes the returns in different states when all wealth is invested in money, while 2 and 3 denote the points corresponding to plunging in the two risky assets. The plane defined by the three points is the opportunity set. The dotted line joining the three points gives the set of returns feasible with holding non-negative amounts of each of the assets. The intersection of this plane with the \( i \)th axis gives us \( \rho_i \), i.e., the set of Arrow-Debreu securities yielding the same opportunity set.

The numerical calculation of \( \rho_\theta \), given \( \rho_{i\theta} \), is straightforward. The value of the \( i \)th security is equal to the sum of its returns in each state of nature times the implicit price for returns in that state of nature. But \( 1/\rho_\theta \) is the implicit price for returns in state \( \theta \). Thus,

\[
1 = \sum_i \rho_{i\theta}/\rho_\theta \quad \text{all } i.
\]

To obtain the \( \rho_i \), we need simply solve these equations (which are essentially linear). Since the opportunity sets corresponding to (11) and (12) are identical, but calculations in terms of the Arrow-Debreu securities market much simpler, we shall in the following analysis make use of the more convenient description.¹

¹ For a more complete discussion of the interrelations between Arrow-Debreu markets and arbitrary asset markets in which the number of assets equals the number of states—and, in particular, an answer to the question of when treating them interchangeably makes sense in the face of unrestricted short sales—see Appendix I of [2].
The theorem is a result of the following lemma.

**Lemma 1.** In an Arrow-Debreu securities market the percentage change in the proportion of wealth allocated to a given security as wealth increases is an increasing, constant, or decreasing function of the mean return of that security (per dollar invested) as there is decreasing, constant, or increasing relative risk aversion:

\[
\frac{d \ln n_i}{d W_0} \equiv \frac{d \ln n_j}{d W_0} \text{ as } (\pi_i \beta_i - \pi_j \beta_j)R' \equiv 0.
\]

**Figure 2**

Opportunity set of returns per dollar invested: number of securities and states the same.

**Proof.** For an Arrow-Debreu securities market, the necessary and sufficient conditions for expected utility maximization may be written parametrically

\[
U'[\alpha \beta \lambda W_0] \pi \beta = U'[W_0] \pi \beta = \lambda.
\]  
(13)

Differentiating totally, we obtain

\[
U'W_0 \pi \beta d \ln \alpha + U'W_0 \pi \beta d \ln W_0 = d \lambda.
\]  
(14)

Dividing (14) by (13), we then have

\[-R_0(d \ln \alpha + d \ln W_0) = d \ln \lambda.
\]  
(15)
Since $\Sigma a_\theta = 1$, $\Sigma da_\theta = 0$. Hence, multiplying (15) by $a_\theta/R_\theta$ and summing over $\theta$, we obtain

$$0 = \Sigma da_\theta = -d\ln \lambda \Sigma (a_\theta/R_\theta) - d\ln W_0.$$ 

Thus

$$- \frac{d\ln \lambda}{d\ln W_0} = 1/\Sigma (a_\theta/R_\theta).$$

Substituting into (15), we obtain

$$\frac{d\ln a_\theta}{d\ln W_0} = \frac{1/R_\theta}{1/\Sigma (a_\theta/R_\theta)} - 1.$$

(16)

From (13), since $U'' < 0$,

$$\beta_\theta a_\theta \geq \beta_j a_j \text{ as } \pi_\theta \beta_\theta \geq \pi_j \beta_j.$$ 

But $\pi_\theta \beta_\theta$ is the expected return of the $i$th asset, and the lemma follows directly.

An immediate implication of Lemma 1 is the following: ordering the states of nature so that $\pi_1 \beta_1 \geq \pi_2 \beta_2 \geq \ldots \geq \pi_\theta \beta_\theta$, we must have $d\ln a_\theta/d\ln W_0 \leq -d\ln a_\theta/d\ln W_0$. Thus, the range of the return per dollar invested increases, is constant, or decreases with wealth as there is decreasing, constant, or increasing relative risk aversion.

The fact that the mean behaves in the same way follows by differentiating

$$\bar{r} = \Sigma a_\theta \beta_\theta \pi_\theta$$

with respect to $W_0$, to obtain

$$\frac{d\bar{r}}{dW_0} = \Sigma \beta_\theta \pi_\theta \frac{da_\theta}{dW_0}$$

$$= \Sigma \frac{1}{W_0} \left( \frac{d\ln a_\theta}{d\ln W_0} \right) a_\theta \beta_\theta \pi_\theta.$$ 

(17)

When $R$ is constant, (17) together with Lemma 1 already entails the desired result. When $R$ is strictly monotone, define

$$1/R[W^*] = \Sigma a_\theta/R_\theta \text{ or } W^* = R^{-1}[1/\Sigma a_\theta/R_\theta]$$

and

$$(\pi \rho)^* U[W^*] = \lambda \text{ or } (\pi \rho)^* = \lambda/U[W^*],$$

i.e., the expected return which would result in an income in that state of nature such that the reciprocal of relative risk aversion would equal the average value of the reciprocal of relative risk aversion (and thus make the RHS of (16) zero). Also, for simplicity, let

$$\beta_\theta = \frac{d\ln a_\theta}{d\ln W_0}.$$ 

(18)

Then (17) may be rewritten

$$\frac{d\bar{r}}{d\ln W_0} = \Sigma a_\theta \beta_\theta (\pi_\theta \beta_\theta - (\pi \rho)^*) + (\pi \rho)^* \Sigma a_\theta \beta_\theta.$$ 

From Lemma 1, and the definition of $(\pi \rho)^*$, when $\pi_\theta \beta_\theta > (\pi \rho)^*$, $\beta_\theta \geq 0$ as $R' \leq 0$, and similarly when $\pi_\theta \beta_\theta < (\pi \rho)^*$. Hence the first term is unambiguously negative or positive as $R' \geq 0$. But from the definition of $\beta_\theta$, (18), it is clear that

$$\Sigma a_\theta \beta_\theta = \Sigma da_\theta/d\ln W_0 = 0.$$

Thus we obtain the desired result for these cases as well.

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1 Here and in the subsequent discussion of the other statistical measures considered, we assume that $\pi_i \beta_i \neq \pi_j \beta_j$ some $i, j$, i.e., that there is some variability in the distribution of returns per dollar invested.
Similarly, direct calculation verifies that
\[
\frac{d\sigma_i^2}{dW_0} = 2\Sigma\pi_\phi(a_0\bar{\rho}_0 - \bar{\pi})(\hat{\rho}_0 \frac{d\alpha_0}{dW_0} - \frac{d\bar{\pi}}{dW_0})
\]
\[
= 2\Sigma\pi_\phi(a_0\bar{\rho}_0 - \bar{\pi}) \frac{d\ln a_0}{dW_0} a_0 \bar{\rho}_0
\]
(19)
since
\[
\Sigma\pi_\phi(a_0\bar{\rho}_0 - \bar{\pi}) = \Sigma\pi_\phi a_0\bar{\rho}_0 - \bar{\pi}\Sigma a_0 = \bar{\pi} - \bar{\pi} = 0.
\]
To ascertain the sign of (19), we use Lemma 2:

**Lemma 2.** If \(\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n\) and \(\alpha_i > \alpha_{i+1}\) for some \(i\), \(\gamma_i > 0\) for \(i < i'\) and \(\gamma_i \leq 0\) for \(i \geq i'\), and \(\Sigma\gamma_i = 0\), then \(\Sigma\gamma_i > 0\).

**Proof.** Define \(\alpha'_i = a_i + \epsilon\) so that
\[
\alpha'_i \geq 0, \quad i < i'
\]
\[
\leq 0, \quad i \geq i'.
\]
Then
\[
\Sigma\gamma_i = \Sigma\alpha'_i - \epsilon\Sigma\gamma_i
\]
\[
= \Sigma\alpha'_i > 0.
\]
If we again order the states of nature so that
\[
\pi_1\bar{\rho}_1 \geq \pi_2\bar{\rho}_2 \geq \ldots \geq \pi_n\bar{\rho}_n,
\]
then
\[
a_1\bar{\rho}_1 \geq a_2\bar{\rho}_2 \geq \ldots \geq a_n\bar{\rho}_n,
\]
and, if \(R' < 0\),
\[
\frac{d\ln a_1}{dW_0} \geq \frac{d\ln a_2}{dW_0} \geq \ldots \geq \frac{d\ln a_n}{dW_0}
\]
while if \(R' > 0\)
\[
- \frac{d\ln a_1}{dW_0} \geq - \frac{d\ln a_2}{dW_0} \geq \ldots \geq - \frac{d\ln a_n}{dW_0}
\]
Hence, if \(R' < 0(>0)\), and we let
\[
\alpha_i = \frac{d\ln a_i}{dW_0} a_i\bar{\rho}_i \left( = - \frac{d\ln a_i}{dW_0} a_i\bar{\rho}_i \right)
\]
\[
\gamma_i = \pi_i (a_i\bar{\rho}_i - \bar{\pi})
\]
with
\[
\Sigma\gamma_i = 0,
\]
then the result concerning the variance of the return per dollar invested follows immediately from (19) and Lemma 2.

Finally, again ordering the states of nature as above, recalling the definition \(v_\theta = \frac{r_\theta}{r}\), and applying Lemma 1 once more, we have
\[
\frac{d\ln(v_{\theta \theta+1})}{d\ln W_0} = \frac{d\ln a_{\theta+1}}{d\ln W_0} + \frac{d\ln a_{\theta}}{d\ln W_0} \approx 0 \quad \text{as} \quad (\pi_\theta\bar{\rho}_\theta - \pi_{\theta+1}\bar{\rho}_{\theta+1})R' \approx 0.
\]
In this situation too we can depict the new distribution of \(v_\theta\) as intersecting the old only once, as in Figure 1, and can say the variability of \(v_\theta\) increases or decreases with \(W_0\) as
$R'$ is decreasing or increasing. (Notice that the result displayed is even stronger; not only does the distribution of $v_{0}$ become, say, more variable with wealth, but it does so "increasingly" towards the tails.)

We have thus established that the theorem describing the effects of wealth changes in terms of statistical properties of the portfolio may be extended to the case where there are as many securities as states of nature. The theorem concerning the certainty equivalent characterization may also be extended to this case, but once again, since the proof is simply a specialization of that to be presented in the next section, we shall not detail it here.

We are able to extend Theorem 1 only under two special circumstances:  

**Theorem 1'.** If the proportions in which the different risky assets are purchased do not change with wealth, then the elasticity of the demand for money is greater than, equal to or less than unity as relative risk aversion is an increasing, constant or decreasing function of wealth.

In this case, it is as if there is only a single risky security (the mutual fund holding the different risky securities in the given proportions), and the analysis of Section 2 applies directly.

The question naturally arises then, under what circumstances will the proportions in which different risky assets are purchased not change with the level of wealth; we have provided necessary and sufficient conditions elsewhere [2], and these turn out to be very stringent.

**Theorem 1".** If there are as many securities as states of nature, each of the risky securities yields returns in only one state of nature, and all assets are held in positive amounts, then the elasticity of the demand for money is greater than, equal to, or less than unity as relative risk aversion increases, remains constant, or decreases with wealth.

Here the matrix of returns is given by

$$
P = \begin{bmatrix}
\rho_{M} & \rho_{M} & \rho_{M} & \cdots \\
0 & \rho_{2} & 0 & \cdots \\
0 & 0 & \rho_{3} & \cdots \\
& & & \ddots
\end{bmatrix}.
$$

Solving equations (12'), the matrix of returns for the corresponding Arrow-Debreu securities market is given by

$$
\tilde{P} = \begin{bmatrix}
1/(1/\rho_{M} - 1/\rho_{2} - 1/\rho_{3} \cdots) & 0 & 0 & \cdots \\
0 & \rho_{2} & 0 & \cdots \\
0 & 0 & \rho_{3} & \cdots \\
& & & \ddots
\end{bmatrix} = \begin{bmatrix}
\tilde{\rho}_{1} & 0 & 0 & \cdots \\
0 & \tilde{\rho}_{2} & 0 & \cdots \\
0 & 0 & \tilde{\rho}_{3} & \cdots \\
& & & \ddots
\end{bmatrix}.
$$

The allocations to the original securities are related to the allocations to the corresponding Arrow-Debreu securities, $\tilde{a}$, by the equations

$$
aP = \tilde{a}\tilde{P}
$$

or

$$
a = \tilde{a}\tilde{P}P^{-1}
$$

1 While still assuming as many securities as states of nature, we now consider markets in which one of the securities is money (i.e., we no longer consider Arrow-Debreu markets).

2 Indeed, they require that the utility function satisfy either $U'' = (a + bW)'$ (which yields constant relative risk aversion for the special case where $a = 0$) or $U'' = aeW$ (which yields constant absolute risk aversion).
or

\[ a_M = \left[ \frac{1}{\rho_M} \left( \frac{1}{\rho_M} - \sum_{i=1}^{1} \frac{1}{\rho_i} \right) \right] a_1 \]

and

\[ a_i = \hat{a}_i - \left[ \frac{1}{\rho_i} \left( \frac{1}{\rho_M} - \sum_{i=1}^{1} \frac{1}{\rho_i} \right) \right] \hat{a}_1 \quad i > 1. \tag{21} \]

From (21) it follows that (i) \( a_M > 0 \) implies \( \rho_M < 1/ \sum_{i=1}^{1} (1/\rho_i) \) (as all Arrow-Debreu securities are held in positive amounts) and (ii) \( a_i \geq 0 \) for \( i > 1 \) implies

\[ \hat{a}_i \geq [(1/\rho_i) - (1/\rho_M)] a_1 = (\rho_i/\rho_M) \hat{a}_1 \text{ for } i > 1, \text{ or } \min_{i > 1} \rho_i \hat{a}_i \geq \rho_1 \hat{a}_1. \]

Hence, appealing to Lemma 1, if \( a_i > 0 \) for all \( i \), then

\[ \frac{da_M}{dW_0} \sim \frac{da_1}{dW_0} \equiv 0 \text{ as } R' \equiv 0. \]

Unfortunately, stronger theorems do not seem available. Indeed we shall now show that with a slightly more complicated pattern of returns, the wealth elasticity of demand for money may be greater than or less than unity, regardless of whether there is increasing or decreasing relative risk aversion.²

Counterexample 1. Money and Two Risky Assets. Consider the matrix of returns

\[ P = \begin{bmatrix} \rho_M & \rho_M & \rho_M \\ 0 & \rho_{22} & \rho_{23} \\ \rho_{31} & 0 & \rho_{33} \end{bmatrix}. \]

The analysis of this example will involve six steps:

(i) We first solve for the corresponding Arrow-Debreu securities.

(ii) We then establish the relationship between the demands for the Arrow-Debreu securities and the original securities.

(iii) We check what restrictions on \( P \) must be imposed to ensure non-negativity of demands.³

(iv) An example confirms that all of these restrictions may be satisfied simultaneously.

(v) We derive a general expression for \( da_M/dW_0 \) and show that for, say \( R' < 0 \), it may be of either sign, even when the restrictions discussed in (iii) are satisfied. For a particular numerical example, utility functions for which \( R' < 0 \) and \( da_M/dW_0 \) is positive, zero, or negative are exhibited.

The reader who is uninterested in the details of the counterexample should skip ahead to Section 4.

1 This is equivalent to the statement that there is no mutual fund (i.e., linear combination, with weights adding to one, of the original securities) which dominates any of the original securities, which in turn is equivalent to the statement that for the market (20), there is a corresponding Arrow-Debreu market with securities yielding positive returns. For a fuller discussion, the reader is again referred to Appendix 1 of [2].

2 As noted above, the case of constant relative risk aversion is covered by Theorem 1.

3 We require \( a_i \geq 0 \) simply to demonstrate that the "pervasive" behavior illustrated by the example doesn't depend on the possibility of selling short.
(i) The Arrow-Debreu returns $\hat{\rho}_i$ corresponding to the matrix of returns $P$ are the solution to

\[
\begin{align*}
\rho_M \frac{1}{\hat{\rho}_1} + \rho_M \frac{1}{\hat{\rho}_2} + \rho_M \frac{1}{\hat{\rho}_3} &= 1 \\
\rho_{22} \frac{1}{\hat{\rho}_2} + \rho_{23} \frac{1}{\hat{\rho}_3} &= 1 \\
\rho_{31} \frac{1}{\hat{\rho}_1} + \rho_{33} \frac{1}{\hat{\rho}_3} &= 1.
\end{align*}
\]

Writing $\rho_{23} = \alpha \rho_{22}$ and $\rho_{33} = \beta \rho_{31}$ with $\alpha, \beta > 0$, we can solve for $1/\hat{\rho}_1$ in terms of $\rho_M$, $\rho_{22}$ and $\rho_{33}$:

\[
\frac{1}{\hat{\rho}_1} = \frac{1}{\rho_{31}} - \beta \frac{1}{\hat{\rho}_3}
\]

\[
\frac{1}{\hat{\rho}_2} = \frac{1}{\rho_{22}} - \alpha \frac{1}{\hat{\rho}_3}
\]

and

\[
\frac{1}{\hat{\rho}_3} = \left(\frac{1}{\rho_M} - \frac{1}{\rho_{22}} - \frac{1}{\rho_{31}}\right)(1-\alpha-\beta).
\]

(ii) Since 

\[
aP = a\hat{P},
\]

where again

\[
\hat{P} = \begin{bmatrix}
\hat{\rho}_1 & 0 & 0 \\
0 & \hat{\rho}_2 & 0 \\
0 & 0 & \hat{\rho}_3
\end{bmatrix},
\]

we then also have

\[
a = \hat{a} = \begin{bmatrix}
\hat{\rho}_1 & 0 & 0 \\
0 & \hat{\rho}_2 & 0 \\
0 & 0 & \hat{\rho}_3
\end{bmatrix}^{-1} \begin{bmatrix}
\rho_{22}\rho_{31}\alpha & -\rho_M\rho_{31}\beta & \rho_M\rho_{22}(\alpha-1) \\
\rho_{22}\rho_{31}\alpha & \rho_M\rho_{31}(\beta-1) & -\rho_M\rho_{22}\alpha \\
-\rho_{22}\rho_{31} & \rho_M\rho_{31} & \rho_M\rho_{22}
\end{bmatrix}.
\]

(iii) Thus, if for instance $\alpha + \beta > 1$, then in order for $\alpha > 0$ we require

\[
\hat{a}_1\hat{\rho}_1 + \hat{a}_3\hat{\rho}_3 > \hat{a}_3\hat{\rho}_3,
\]

\[(22a)\]

and

\[
\hat{a}_1\hat{\rho}_1 + (\alpha-1)\hat{a}_3\hat{\rho}_3 > \beta \hat{a}_1\hat{\rho}_1.
\]

\[(22b)\]

(iv) Let, for instance $\alpha = \frac{1}{2}$. Then (22c) requires

\[
\hat{a}_1\hat{\rho}_1 = (\hat{a}_3\hat{\rho}_2 + \hat{a}_1\hat{\rho}_1)/2,
\]

\[(23)\]

$\hat{a}_3\hat{\rho}_2$ must be greater than the average of $\hat{a}_1\hat{\rho}_1$ and $\hat{a}_2\hat{\rho}_2$. On the other hand, (22a) requires

\[
\beta > \frac{\hat{a}_1\hat{\rho}_1 - \frac{1}{2}\hat{a}_3\hat{\rho}_3}{\hat{a}_1\hat{\rho}_1}.
\]

\[(24)\]

\footnote{This phrase will be used to signal the fact that a particular assumption is being adopted for the purposes of the example.}
Finally, if
\[ \dot{\alpha}_1 \dot{\beta}_1 < \dot{\alpha}_2 \dot{\beta}_2, \]
(22b) requires
\[ \beta > \frac{\dot{\alpha}_2 \dot{\beta}_2 - \dot{\alpha}_3 \dot{\beta}_3}{\dot{\alpha}_2 \dot{\beta}_2 - \dot{\alpha}_1 \dot{\beta}_1}. \]
(26)
Both (23) and (25) are consistent with
\[ \dot{\alpha}_2 \dot{\beta}_2 > \dot{\alpha}_3 \dot{\beta}_3 > \dot{\alpha}_1 \dot{\beta}_1. \]
(27)
(v) Now consider
\[
\frac{d\alpha_M}{dW_0} \sim \frac{1}{\dot{\beta}_3} \left( \dot{\rho}_1 \dot{\beta}_1 \frac{d\dot{\alpha}_1}{dW_0} + \dot{\rho}_2 \frac{d\dot{\alpha}_2}{dW_0} \right) - \frac{d\ddot{\alpha}_3}{dW_0} \\
= \left( \frac{\dot{\beta}_1}{\dot{\beta}_3} \beta + 1 \right) \frac{d\ddot{\alpha}_1}{dW_0} + \left( \frac{\dot{\beta}_2}{\dot{\beta}_3} \alpha + 1 \right) \frac{d\ddot{\alpha}_2}{dW_0}
\]
(28)
and assume \( R' < 0 \).
From (16) we know that if (27) is satisfied, then
\[ \frac{d\ddot{\alpha}_2}{dW_0} > 0 \]
while
\[ \frac{d\ddot{\alpha}_1}{dW_0} < 0. \]
But (16) places no restrictions on their relative absolute values, so \( d\alpha_M/dW_0 \) may be positive, negative, or zero.
Suppose, for instance, \( \alpha = 0.5 \) and \( \beta = 2 \). Then, (26) is satisfied whenever (27) is.
Now let, for instance,
\[ \ddot{\alpha}_2 \ddot{\beta}_2 = 3 \ddot{\alpha}_3 \ddot{\beta}_3 = 4 \ddot{\alpha}_1 \ddot{\beta}_1 \]
(29)
Then both (24) and (27) (also (23) and (26)) are satisfied, i.e., \( a_i > 0 \) for all \( i \). But these values of \( \ddot{\alpha}_i \ddot{\beta}_i \) could be generated by any set of values for \( \ddot{\beta}_i \) (and hence \( \rho_{1i} \)), for any utility function, by choosing appropriate \( \pi_i \).
Let, for instance, \( \rho_M = \frac{1}{3}, \rho_{22} = \frac{2}{5}, \rho_{11} = \frac{3}{5}. \) Then
\[ \frac{1}{\dot{\beta}_1} = 3 - 1 = 2 \]
\[ \frac{1}{\dot{\beta}_2} = \frac{3}{5} - \frac{2}{5} = \frac{1}{5} \]
and
\[ \frac{1}{\dot{\beta}_3} = \frac{3}{5} - \frac{\frac{2}{5}}{1 - \frac{1}{5}} = \frac{4}{5}. \]
For these values of \( 1/\dot{\beta}_i \), the solution of (2) and (29) is
\[ \ddot{\alpha}_1 = \frac{3}{2}, \ddot{\alpha}_2 = \frac{2}{3}, \text{ and } \ddot{\alpha}_2 = \frac{1}{2}. \]
Thus, from (28), if \( R' < 0 \)
\[ \frac{d\alpha_M}{dW_0} \leq 0 \text{ as } \frac{d\ln \ddot{\alpha}_1/dW_0}{d\ln \ddot{\alpha}_2/dW_0} = \frac{1/R_1 - \Sigma \ddot{\alpha}_i/R_i}{1/R_2 - \Sigma \ddot{\alpha}_i/R_i} \leq 1. \]
Let, for instance,
\[ \frac{1}{R_1} = 1 \text{ and } \frac{1}{R_2} = 2. \]
Then
\[ \frac{d\alpha_M}{dW_0} \leq 0 \text{ as } \frac{1}{R_3} \geq 1.5. \]
4. MORE STATES OF NATURE THAN SECURITIES

The structure of the problem is considerably altered if there are more states of nature than securities. This is perhaps best illustrated in terms of our three state example. The opportunity set now consists of (say) only two securities, depicted by the points 1 and 2 in Figure 3, and their linear combinations, i.e., the points on the line through them. The solid part of the line represents the opportunity set holding non-negative amounts of the two securities. One way of viewing this line is as the intersection of two planes, i.e., the opportunity set that would have been generated by an Arrow-Debreu securities market in which each security had two prices, a dollar price and a "rationing point" price. A securities market with fewer securities than states of nature is equivalent to an Arrow-Debreu securities market with point rationing.¹ Not surprisingly, this complicates the structure

¹ If \( n \) is the number of states and \( m \) the number of securities, then each Arrow-Debreu security has associated with it \( n-m+1 \) "prices". As our example makes clear, whereas when \( n = m \) there is a unique Arrow-Debreu securities market corresponding to the given securities, when \( n > m \), this is not true. The equivalence of a securities market with fewer securities than states of nature to a restricted Arrow-Debreu securities market has been observed independently by Diamond and Yarri ("Implications of the Theory of Rationing for Consumer Choice under Uncertainty," mimeo, 1970), who exploit some well-known theorems on demand under rationing to derive properties of the structure of demand for securities.
of demand considerably, and Theorem 2 no longer obtains. This may be looked at another way. Theorem 2 depended on being able to find an equivalent set of securities in terms of which the utility function was additive. This independence property no longer obtains when there are more states of nature than securities.

As a result, not even the theorems concerning the statistical properties of the portfolio can be generalized. The construction of a counterexample with money and two risky assets requires, however, that there be at least four states of nature; we have not yet found any such counterexample. It is easy, however, to construct a counterexample in which either all assets are risky or in which there exists a risky wage income which cannot be insured. Since in fact money is not perfectly safe (changes in price level affect the consumption rate of return) and since wage income cannot be insured, these take care of the relevant cases.

**Counterexample 2. Two Risky Assets.** Assume that there are three states of nature and two securities, with a matrix of returns

\[
P = \begin{bmatrix}
\rho_1 & 0 & 0 \\
0 & \rho_2 & \rho_3 \\
\end{bmatrix}.
\]

The analysis of this example will proceed in the following steps:

(i) Expressions for the mean, variance and range of the rate of return per dollar, and how they depend on the portfolio allocation are derived.

(ii) The first order conditions for expected utility maximization are written out explicitly, and thence a simple expression for the effects of wealth changes on portfolio allocation is derived.

(iii) It is shown that with, say, \( R' < 0 \), the proportion of the portfolio allocated to the first asset may increase or decrease, and hence the statistical measures may move in either direction.

(iv) In fact, the example shows that the mean may decrease while variance increases (for the optimally chosen portfolio, of course).

As before, the reader may skip the details without loss of continuity, and go ahead to the presentation of counterexample 3.

(i) The mean return from the securities is (letting \( a \) represent the proportion of wealth allocated to the first asset)

\[
\bar{r} = a\pi_1 \rho_1 + (1-a)(\pi_2 \rho_2 + \pi_3 \rho_3)
\]

so that

\[
\frac{d\bar{r}}{dW_0} = \left\{ \pi_1 \rho_1 - (\pi_2 \rho_2 + \pi_3 \rho_3) \right\} \frac{da}{dW_0}.
\]

Thus, to determine what happens to the mean return, we need only determine the sign of \( da/dW_0 \). Similarly, the variance of the return is

\[
\sigma^2 = Er^2 - \bar{r}^2
\]

\[
= a^2 \pi_1^2 \rho_1^2 + (1-a)^2(\rho_2^2 \pi_2^2 + \rho_3^2 \pi_3^2) - \bar{r}^2
\]

and

\[
\frac{d\sigma^2}{dW_0} = 2\{a\pi_1 \rho_1(1-\pi_1)-(1-a)[\rho_2^2 \pi_2(1-\pi_2) + \rho_3^2 \pi_3(1-\pi_3) - 2\pi_2 \pi_3 \rho_2 \rho_3]\}
\]

\[
-(1-2a)\pi_1 \rho_1(\pi_2 \rho_3 + \pi_3 \rho_3)\} \frac{da}{dW_0}.
\]

(31)
Finally, the range of returns depends on

\[ r_{\text{min}} = \min \left[ a \rho_1, (1-a) \rho_2, (1-a) \rho_3 \right] \]

and

\[ r_{\text{max}} = \max \left[ a \rho_1, (1-a) \rho_2, (1-a) \rho_3 \right]. \]

In order to determine what happens to the variance of return and range of returns we must know both the sign of \( da/dW_0 \) and the value of \( a \).

(ii) The first order conditions for this example may be written

\[ U'[W_0a \rho_1] \pi_1 \rho_1 = U'[W_0(1-a) \rho_2] \pi_2 \rho_2 + U'[W_0(1-a) \rho_3] \pi_3 \rho_3, \]

which defines \( a \) as an implicit function of \( W_0 \). Hence

\[ \frac{da}{dW_0} = \frac{U''W_0 a \rho_1 \rho_1^2 - (U''W_0(1-a) \pi_2 \rho_2^2 + U''W_0(1-a) \pi_3 \rho_3^2)}{-(U'' \pi_1 \rho_1^2 + U'' \pi_2 \rho_2^2 + U'' \pi_3 \rho_3^2)W_0^2}. \]

The denominator is unambiguously positive. Our problem is thus to determine the sign of the numerator. Dividing by the first order condition, we see that

\[ \frac{da}{dW_0} \sim R_2 \alpha + R_3 (1-a) - R_1 \]

where

\[ R_i = \text{relative risk aversion in state } i \]

and

\[ \alpha = \frac{U'[W_0(1-a)W_0 \rho_2] \pi_2 \rho_2}{U'[W_0(1-a)W_0 \rho_2] \pi_2 \rho_2 + U'[W_0(1-a)W_0 \rho_3] \pi_3 \rho_3}; \]

that is, whether \( a \) increases or decreases depends simply on whether a particular average value of relative risk aversion in those states in which the second security pays off is larger or smaller than its value in that state in which the first security pays off.

(iii) But if we choose our parameters so that

\[ (1-a) \rho_2 < a \rho_1 < (1-a) \rho_3 \]

we can make the sign of \( da/dW_0 \) be positive, negative, or even zero, by also choosing appropriate values of \( R_i \). Note that we can choose values of \( R \) independently of the parameters \( (\rho_1, \pi_i) \).

A specific numerical example may make this clearer. Let the matrix of returns be

\[
\begin{bmatrix}
2 & 0 & 0 \\
0 & 1 & 4
\end{bmatrix}
\]

and the probabilities of the three states be \( \pi_1 = 0.55 \), \( \pi_2 = 0.1 \), \( \pi_3 = 0.35 \). Assume \( W_0 = 2 \). Then the allocation \( a = 0.5 \) is utility maximizing if

\[ U'[W_0] = U'[2] = 2 \]

\[ U'[W_2] = U'[1] = 8 \]

and

\[ U'[W_3] = U'[4] = 1. \]

It is fairly easy to construct a utility function yielding these values and at the same time exhibiting decreasing relative risk aversion. This is suggested by the fact that the utility function which yields these values but piecewise constant relative risk aversion over the intervals (1, 2) and (2, 4) requires \( R \) equal 2 and 1, respectively, in these intervals; at the end points, however, it can take on any values.
The mean return to the first security is 1.1, to the second 1.5, so from (30)-(32)
\[
\frac{d\bar{r}}{dW_0} = -0.4 \frac{da}{dW_0}, \quad \frac{d\sigma^2}{dW_0} = -2.46 \frac{da}{dW_0} \quad \text{and} \quad \frac{d(r_{\max} - r_{\min})}{dW_0} = -1.5 \frac{da}{dW_0}.
\]
For this example \( \alpha = 4/11 \). Thus,
\[
\frac{da}{dW_0} \sim (4/11)R_2 + (7/11)R_3 - R_1.
\]
If, for instance, \( R_2 = 3 \) and \( R_3 = 1/2 \), then
\[
\frac{da}{dW_0} \geq 0 \quad \text{as} \quad R_1 \leq 31/22,
\]
i.e.,
\[
\frac{d\bar{r}}{dW_0} \geq 0 \quad \frac{d\sigma^2}{dW_0} \geq 0 \quad \text{as} \quad \frac{d(r_{\max} - r_{\min})}{dW_0} \geq 0 \quad \text{as} \quad R_1 \geq 31/22.
\]
Either is possible.

(iv) A slight modification of these numbers illustrates another important point: it is possible for the mean return to decrease while the variance increases and vice versa.

Assume, for instance, in the above example, \( \pi_1 = 0.6 \) and \( \pi_2 = \pi_3 = 0.2 \). It is easy to confirm that \( a = 0.5 \) is still utility maximizing. Since now \( \alpha = 2/3 \),
\[
\frac{da}{dW_0} \sim (2/3)R_2 + (1/3)R_3 - R_1.
\]
If, for instance, \( R_1 = 1.6 \) and \( R_2 = 0.6 \), then
\[
\frac{da}{dW_0} \geq 0 \quad \text{as} \quad R_2 \geq 2.1.
\]
But here
\[
\frac{d\bar{r}}{dW_0} = 0.2 \frac{da}{dW_0},
\]
so that while again,
\[
\frac{d\sigma^2}{dW_0} = -2.44 \frac{da}{dW_0} \quad \text{and} \quad \frac{d(r_{\max} - r_{\min})}{dW_0} = -1.5 \frac{da}{dW_0},
\]
\[
\frac{d\bar{r}}{dW_0} \leq 0 \quad \frac{d\sigma^2}{dW_0} \geq 0 \quad \text{as} \quad \frac{d(r_{\max} - r_{\min})}{dW_0} \leq 0 \quad \text{as} \quad R_2 \leq 2.1.
\]

**Counterexample 3. Two Assets, Wage Uncertainty.** Now assume that there are just money and a risky asset; the pattern of returns is given by
\[
P = \begin{bmatrix} \rho_M & \rho_M & \rho_M \\ \rho_M & \rho_M & \rho_M \\ \rho_M & \rho_M & \rho_M \end{bmatrix}.
\]
In addition, however, assume that the individual investor has some wage income which cannot be insured. Thus, in state 3, his wage income is \( \omega_3 \). Expected utility maximization requires
\[
EU'(\rho_{23} - \rho_M) = 0.
\]
(33)
For simplicity, let
\[
\theta = \rho_{23} - \rho_M
\]
and assume \( g_1 < 0, g_2 > 0, g_3 > 0 \). Then we can rewrite (33) as (now letting \( a \) be the proportion of wealth allocated to the risky asset)

\[
U'[\omega_1 + W_0(\rho_M + ag_1)]\pi_1 \mid a_1 = U'[\omega_2 + W_0(\rho_M + ag_2)]\pi_2g_2
\]

\[
+ U'[\omega_3 + W_0(\rho_M + ag_3)]\pi_3g_3. \tag{34}
\]

Using the same procedure as before we find that

\[
\frac{da}{dW_0} \sim R_1 \frac{\pi_1 - \omega_1}{W_0} - \left( \alpha R_2 \frac{\pi_2 - \omega_2}{W_3} + (1 - \alpha) R_3 \frac{\pi_3 - \omega_3}{W_3} \right),
\]

where

\[ R_i \text{ = relative risk aversion in state } i \]

and

\[ \alpha = U'_2 \pi_2 g_2 (U'_2 \pi_2 g_2 + U'_3 \pi_3 g_3). \]

It is clear that we can make \( da/dW_0 \) take on either sign simply by choosing our parameters appropriately, and by making income in state 1 lie between that in states 2 and 3.

Consider the following numerical example. The pattern of returns is given by

\[
\begin{bmatrix}
2 & 2 & 2 \\
3 & 0 & 6
\end{bmatrix}
\]

and wage income is 0 in states 1 and 2, 1 in state 3. The probabilities of the three states are \( \pi_1 = 0.6, \pi_2 = \pi_3 = 0.2 \). Then, if \( W_0 = 3 \), and the individual allocates \( \frac{1}{3} \) of that amount to money (the first asset), \( \frac{2}{3} \) to the risky asset, his income in the three states will be

\[ W_1 = 2, \quad W_2 = 1, \quad W_3 = 4 \]

which is identical to the incomes in the previous example. Indeed, observing that

\[ g_1 = -2, \quad g_2 = 1, \quad g_3 = 4, \]

we see that if the utility function is that described in the previous example, the above allocation satisfies the first order conditions. Thus, if \( R_1 = 1.6 \) and \( R_2 = 0.8 \), then again

\[ \frac{da}{dW_0} \geq 0 \text{ as } R_2 \geq 2.1. \]

Because \( da/dW_0 \) can take on either sign, the mean and variance may decrease or increase.\(^1\)

With these examples in hand, it is hardly surprising that the only perfectly general theorem (among the possibilities we're considering) is not concerned with properties, like mean and variance, that are independent of the utility function, but rather employs certainty equivalents, which in their very definition make use of the utility function:

**Theorem 3.** The certainty equivalent rate of return is an increasing, constant, or decreasing function of wealth as relative risk aversion is decreasing, constant, or increasing.\(^2\)

**Proof.** Let the returns of the \( i \)th security in state \( \theta \) be denoted by \( \rho_{\theta i} \). Then, letting the total number of securities be \( n \), terminal wealth in state \( \theta \) can be written

\[
W_\theta = W_0 \left( \rho_{n} + \sum_{i=1}^{n-1} a_i (\rho_{\theta i} - \rho_n) \right). \tag{35}
\]

\(^1\) In the appendix, we present theorems similar to those presented in the text concerning the behaviour of the mean and variance of total income from the portfolio; in particular, we show that if there is a single risky asset or if there are as many securities as states of nature, the variance of the total returns from the portfolio increases or decreases with wealth as there is decreasing or increasing absolute risk aversion. When there is uncertain wage income, these theorems are not valid; a counterexample parallel to that given here may be constructed in which not only the variance of the return from the portfolio may increase or decrease, but also the variance of total income may increase or decrease.

\(^2\) This theorem generalizes the result of Pratt [3] to the case of more than one asset, where the assets are chosen optimally, and simplifies the proof presented there.
Optimal portfolio allocation implies that
\[ EU[W_t](\rho_{it} - \rho_{it}) = 0 \text{ for } i < n. \] (36)
Let \( a_i^* \) denote the solution to (36). Then the certainty equivalent rate of return is defined by
\[ U[r^*W_0] = \max_{x_{a_i} = 1} EU[W_0] = EU \left[ W_0 \left( \rho_{it} + \sum_{i=1}^{n-1} a_i^*(\rho_{it} - \rho_{it}) \right) \right]. \] (37)
We wish to ascertain the sign of \( dr^*/dW_0 \); by the implicit function theorem,
\[ \frac{dr^*}{dW_0} = -\frac{U' r^* E U'[W_0] - EU'[W_0] \sum_{i=1}^{n-1} (\rho_{it} - \rho_{it}) \frac{da^*}{dW_0}}{U' W_0}. \] (38)
The denominator is unambiguously positive. Our problem is to ascertain the sign of the numerator. Notice first that
\[ EU' W_0 \sum_{i=1}^{n-1} (\rho_{it} - \rho_{it}) \frac{da^*}{dW_0} = W_0 \sum_{i=1}^{n-1} \frac{da^*}{dW_0} EU'(\rho_{it} - \rho_{it}) = 0 \]
by the first order condition (36). Hence
\[ \frac{dr^*}{dW_0} \sim EU' W_0 - U' r^* W_0. \]

\( U' \) is a function of \( W \), but because (by assumption) \( U' > 0 \), \( U \) is a strictly monotonic function of \( W \). Hence, we can consider \( U' \) as a function of \( U \). Let \( W^* = r^* W_0 \), \( U^* = U[W^*] \), \( U'^* = U'[W^*] \), etc. Then
\[ W(U) U'[W(U)] \equiv W^* U'^* + (U - U^*) \frac{dU' W}{dU} \bigg|_{U = U^*} \text{ as } U' \text{ is a convex function of } U. \] (39)

(39) holds for all values of \( W \). Hence, taking expectations, we have
\[ EU'[W_t] W_0 \equiv U'^* W^* + (EU[W_t] - U[r^* W_0]) \frac{dU' W}{dU} \bigg|_{U = U^*} \text{ as } \frac{d^2 U' W}{dU^2} \equiv 0. \] (40)

But by the definition of certainty equivalents, (37),
\[ EU[W_t] - U[r^* W_0] = 0, \]
and direct calculation shows that
\[ \frac{dU' W}{dU} = \frac{dU' W}{dW} \frac{dW}{dU} = \frac{U' W + U'}{U'} = 1 - R(W), \]
so that
\[ \frac{d^2 U' W}{dU^2} \equiv 0 \text{ as } R' \equiv 0. \]

Thus
\[ \frac{dr^*}{dW_0} \equiv 0 \text{ as } EU' W \equiv U'^* W^* \text{ as } R' \equiv 0. \]

\(^1\) Or more generally (i.e., without assuming differentiability of \( R \))
\[ \frac{dr^*}{dW_0} \equiv 0 \text{ as } U' W \text{ is convex as } R \text{ is } \begin{cases} \text{decreasing} \end{cases}, \]
\[ \text{linear as } R \text{ is } \begin{cases} \text{constant} \end{cases}, \\text{and convex as } R \text{ is } \begin{cases} \text{decreasing} \end{cases}. \]
5. CONCLUDING REMARKS

The basic relations of ordinary demand theory—the Slutsky equations—are valid for the demand for risky assets as well. The work of Tobin and Arrow had suggested that one might be able to establish, for money and risky assets, general theorems which are stronger; that one might, for instance, be able to justify the Keynesian assumptions about the liquidity preference schedule or to explain, in terms of attitudes towards risk, why the elasticity of the demand for money was greater (or less) than unity. In retrospect, it is clear that this was a vain hope. We can establish stronger theorems only if we are willing to make stronger assumptions about either the utility function or the nature of the assets. It would appear that this is the direction of research in which expected returns are highest.

REFERENCES


APPENDIX

In the text, we established theorems concerning the relationship between relative risk aversion, the relative allocation to different assets, statistical properties of the rate of return, and the certainty equivalent rate of return. Here, we establish corresponding theorems concerning absolute risk aversion $-U''/U'$, the absolute allocation to certain assets, statistical properties of the total return to a portfolio, and the certainty equivalent value of terminal wealth. Since the analysis follows very much along the lines of that in the text, we present the theorems and sketch the proofs without further comment.

Theorem A.1. If there are two assets, one risky and one safe, the total purchases of risky assets increases, remains unchanged, or decreases with initial wealth as there is decreasing, constant, or increasing absolute risk aversion.

If we define

$$Z_R = (1-a_M)W_0$$  \hspace{1cm} (A.1)

as the total investment in the risky security, and

$$A = -U''/U'$$  \hspace{1cm} (A.2)

as the absolute risk aversion measure, then

$$\frac{dZ_R}{dW_0} \geq 0 \quad \text{as} \quad A' \leq 0.$$  \hspace{1cm} (A.3)

Proof: We can write terminal wealth in state $\theta$ as

$$W_\theta = W_0 \rho_M + Z_R (\rho_0 - \rho_M).$$  \hspace{1cm} (A.4)
From the first order conditions  

\[ EU'(\rho_\theta - \rho_M) = 0 \]  

(A.5)

we obtain (by implicit differentiation)  

\[ \frac{dZ_R}{dW_0} = \frac{\rho_M EU''(\rho_\theta - \rho_M)}{-EU'(\rho_\theta - \rho_M)^2}. \]  

(A.6)

The sign of the denominator is unambiguously positive; we need only ascertain the sign of the numerator:

\[
EU''(\rho_\theta - \rho_M) = E \frac{U''}{U'} U'(\rho_\theta - \rho_M)
= -EAU'(\rho_\theta - \rho_M)
= E(A^* - A)U'(\rho_\theta - \rho_M) - A^*EU'(\rho_\theta - \rho_M)
= E(A^* - A)U'(\rho_\theta - \rho_M),
\]

where

\[ A^* = A[W_0 \rho_M]. \]

If \( A' > 0 \), when \( \rho_\theta > \rho_M, A^* < A \), and when \( \rho_\theta < \rho_M, A^* > A \). Hence the numerator is negative. Conversely, if \( A' < 0 \), when \( \rho_\theta < \rho_M, A^* < A \), and when \( \rho_\theta > \rho_M, A^* > A \), so the numerator of (A.6) is positive. Thus our theorem is proved.

**Theorem A.2.** If there are two assets, one risky and one safe, the variance and range of terminal wealth increase or decrease with initial wealth as there is decreasing or increasing absolute risk aversion, provided \( Z_R > 0 \).

Defining

\[ \bar{W} = EW_\theta = W_0 \rho_M + Z_R (E \rho_\theta - \rho_M) \]  

(A.7)

\[ \sigma^2_W = E(W - \bar{W})^2 \]  

(A.8)

and

\[ \text{Range} \equiv \max_\theta W_\theta - \min_\theta W_\theta, \]  

(A.9)

then

\[ \frac{d\sigma_W^2}{dW_0} \geq 0 \text{ as } \frac{d\text{Range}}{dW_0} \geq 0 \text{ as } A' \leq 0. \]

**Proof.**

\[ \frac{d\sigma_W^2}{dW_0} = 2Z_R (E \rho_\theta - E \rho_\theta)^2 \frac{dZ_R}{dW_0} \]

and

\[ \frac{d\text{Range}}{dW_0} = (\rho_{\text{max}} - \rho_{\text{min}}) \frac{dZ_R}{dW_0} \]

where

\[ \rho_{\text{min}} = \min_\theta \rho_\theta, \rho_{\text{max}} = \max_\theta \rho_\theta. \]

Using Theorem A.1, the result is immediate.

\[ ^1 \] The mean value of terminal wealth increases with initial wealth if there is decreasing absolute risk aversion, but may either increase or decrease if there is increasing absolute risk aversion.
Lemma A.1. If there is an Arrow-Debreu securities market, then

\[ \frac{dZ_\theta}{dW_0} = \frac{1/A_\theta \rho_\theta}{\Sigma(1/A_\theta \rho_\theta)}, \]

where \( Z_\theta \) is total investment in, and \( \rho_\theta \) the only positive return from the \( \theta \)th security.

Proof. Differentiating the first order conditions

\[ U'[Z_\theta \rho_\theta] \pi_\theta \rho_\theta = \lambda \]

(A.10)

with respect to wealth yields

\[ U' \pi_\theta \rho_\theta \frac{dZ_\theta}{dW_0} = \frac{d\lambda}{dW_0}. \]

Dividing by (A.10), we have

\[ \frac{dZ_\theta}{dW_0} = -\frac{1}{A_\theta \rho_\theta} \frac{d\lambda}{dW_0}. \]

Summing over all securities,

\[ \Sigma \frac{dZ_\theta}{dW_0} = 1 = -\Sigma \frac{1}{A_\theta \rho_\theta} \frac{d\lambda}{dW_0}, \]

Hence

\[ \frac{dZ_\theta}{dW_0} = \frac{1/A_\theta \rho_\theta}{\Sigma(1/A_\theta \rho_\theta)} > 0. \]

(A.11)

Theorem A.2'. If there is an Arrow-Debreu securities market, then the mean value of terminal wealth increases with initial wealth, and its variance and range increase or decrease as absolute risk aversion decreases or increases with wealth.\(^1\)

Proof. Let

\[ \bar{W} = \Sigma \rho_\theta Z_\theta \pi_\theta \]

and

\[ \sigma_\bar{W}^2 = \Sigma(\rho_\theta Z_\theta - \bar{W})^2 \pi_\theta = \Sigma \pi_\theta(Z_\theta \rho_\theta)^2 - (\Sigma \pi_\theta Z_\theta \rho_\theta)^2. \]

Then [using (A.11)]

\[ \frac{d\bar{W}}{dW_0} = \Sigma \rho_\theta \frac{dZ_\theta}{dW_0} \pi_\theta > 0. \]

Similarly,

\[ \frac{d\sigma_\bar{W}^2}{dW_0} = 2 \left[ \Sigma \pi_\theta Z_\theta \rho_\theta \frac{dZ_\theta}{dW_0} - \bar{W} \Sigma \pi_\theta \rho_\theta \frac{dZ_\theta}{dW_0} \right] \]

\[ = 2 \left[ \Sigma(Z_\theta \rho_\theta - \bar{W}) \pi_\theta \rho_\theta \frac{dZ_\theta}{dW_0} \right] \]

\[ \sim \Sigma A_\theta^{-1} \pi_\theta (Z_\theta \rho_\theta - \bar{W}) \]

\[ = \Sigma(A_\theta^{-1} - A_*^{-1})(Z_\theta \rho_\theta - \bar{W}) \pi_\theta + A_*^{-1} \Sigma(Z_\theta \rho_\theta - \bar{W}) \pi_\theta \]

where

\[ A_* = \Delta[\bar{W}]. \]

\(^1\) Again, this theorem generalizes trivially to markets in which there are as many securities as states of nature.
Thus, if $A' > 0$, when $Z_\theta r_\theta > W$, $A_\theta > A^*$, and when $Z_\theta r_\theta < W$, $A_\theta < A^*$, so that $\frac{d\sigma^2_W}{dW_0} < 0$. Similarly, if $A' < 0$, $d\sigma^2_W/dW_0 > 0$. Finally,

\[
\frac{d \text{ Range}}{dW_0} = \frac{d(\max_\theta Z_\theta r_\theta)}{dW_0} - \frac{d(\min_\theta Z_\theta r_\theta)}{dW_0} = (A' [\max_\theta Z_\theta r_\theta]^{-1} - A' [\min_\theta Z_\theta r_\theta]^{-1})/\Sigma(1/A_\theta r_\theta) \geq 0 \text{ as } A' \leq 0.
\]

As in the text, it may be shown that when there are as many securities as states of nature the total demand for risky securities may either increase or decrease with, say, increasing absolute risk aversion. Similarly, Theorem A.2' cannot be extended to the case where there are more states of nature than securities.

If we define the certainty equivalent level of wealth as that level of wealth which, if invested in the safe asset, would yield the same expected utility as the optimally chosen portfolio

\[ U[\hat{W}\rho_M] = \max_{\sum_i=1} E U[W_0], \tag{A.12} \]

then we can prove a general theorem analogous to Theorem 3: 1

**Theorem A.3.**

\[ \frac{d\hat{W}}{dW_0} \equiv 1 \text{ as } A' \leq 0. \]

**Proof.** We define

\[ p = (W_0 - \hat{W}) \tag{A.13} \]

with

\[ \frac{dp}{dW_0} = 1 - \frac{d\hat{W}}{dW_0} \tag{A.14} \]

and our problem is to establish the sign of $dp/dW_0$. Substituting (A.13) into (A.12), and differentiating, we obtain

\[ \frac{dp}{dW_0} = \frac{U'[\hat{W}\rho_M]\rho_M - EU'\rho_M - \sum_i E U'(\rho_{bi} - \rho_M) \frac{dZ_i}{dW_0}}{U'[\hat{W}\rho_M]\rho_M}. \]

The denominator is unambiguously positive, so our only problem is to ascertain the sign of the numerator. By the first order conditions for utility maximization, the last term is zero.

Since $U$ is a monotonic function of $W$, we can consider $U'[W(U)]$, that is, $U'$ as a function of $U$,

\[ U'[W_0] \equiv U'[\hat{W}\rho_M] + (U[W_0] - U[\hat{W}\rho_M]) \frac{dU'}{dU} \bigg|_{U = U[\hat{W}\rho_M]} \text{ as } d^2U'/dU^2 \equiv 0. \]

Taking expectations,

\[ EU'[W_0] \equiv EU'[\hat{W}\rho_M] + (EU[W_0] - EU[\hat{W}\rho_M]) \frac{dU'}{dU} \bigg|_{U = U[\hat{W}\rho_M]} = U'[\hat{W}\rho_M] \text{ as } d^2U'/dU^2 \equiv 0. \]

But

\[ \frac{dU'}{dU} = \frac{dU'/dW}{dU/dW} = \frac{U''}{U'}, \]

1 Note, however, that unlike Theorem 3, this result requires the existence of a safe asset (as otherwise there is no clear-cut notion of certainty equivalent wealth).
so that 
\[ \frac{d^2 U'}{dU^2} \equiv 0 \text{ as } A' \equiv 0. \]

Hence 
\[ \frac{d\bar{W}}{d\bar{W}_0} \equiv 1 \text{ as } \frac{dp}{d\bar{W}_0} \equiv 0 \text{ as } A' \equiv 0, \]

and our theorem is proved.