

PORTFOLIO ALLOCATION WITH MANY RISKY ASSETS

4. Comparison with the multi-commodities approach

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The problems introduced by the multiplicity of assets are easy compared with those introduced by the multiplicity of commodities. It is perhaps because of these difficulties that the two strands in the pure theory of consumers behavior – the Von Neumann Morgenstern cardinal utility approach focusing on behavior towards risk and the ordinal utility theory focusing on the allocation of income among different commodities – have developed almost completely independently. Yet, it is clear that there must be close connections between the two; the purpose of this lecture is to develop some of these relations:

There are two classes of problems to be investigated:¹

(1) What is the relationship between attitudes towards risk at different sets of price ratios, and the structure of the indifference map? For instance, what are the implications for the demand curves for different commodities of the assumption that the individuals are neutral towards risk at all price ratios (in a given range)?

(2) What implications do changes in prices have for portfolio allocation?

In the first situation, in any given ‘experiment’ we keep relative prices constant. Individuals desire ‘income’ for the commodities which they can buy with it, and what they can buy with it depends, of course, on prices. Hence, attitudes towards ‘income’ – in particular, towards gambles involving income – will in general depend on the set of prices confronting an individual. That is, the individual

$$\text{maximizes } U(x) \tag{1}$$

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¹ The first question is discussed at greater length in Stiglitz [1]. The second question is discussed in Stiglitz [2], and for the particular problems arising from changes in interest rates, in Stiglitz [3].

subject to the budget constraint

$$\mathbf{p}\mathbf{x} = y, \quad (1')$$

where \mathbf{p} and \mathbf{x} are the price and consumption vectors, y is the total income. The solution to (1) yields the indirect utility function (which will be of use throughout this lecture),

$$V(\mathbf{p}, y) = \max_{\mathbf{x}} u(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{p}\mathbf{x} = y$$

giving the level of utility attainable from an income y when prices are \mathbf{p} .

Let us consider the specific question above: What are the implications of the fact that individuals are neutral towards risk at all price ratios in a given region? Risk neutrality means that, at a fixed set of prices, individuals utility is linear in income:

$$V(p, y) = a(p) + b(p)y. \quad (2)$$

This is required to be true for all p in a given (small) neighborhood. But from a well-known property of indirect utility functions

$$x_i = -V_{p_i} / V_y \quad (3)$$

$$= \frac{(a_i + b_i y)}{b}. \quad (3')$$

The demand curve is linear in income, i.e., we require all Engel curves² to be straight lines. It is obvious that if this is required to be true globally, the indifference map must be homothetic (since an Engel curve not going through the origin in fact consists of two broken line segments—a straight line in the interior and the axis from the point where it intersects the axis to the origin). Risk neutrality at all price ratios does impose severe restrictions on the indifference map.

A converse of the above proposition is also true: if income consumption curves are linear, there exists a numerical representation of the indifference map of the form (2). To go from the demand functions to the utility func-

² An Engel curve shows the variation of the purchase level of a good or of a set of goods with respect to the income level of the consumer or of the group of consumers under investigation.

tions involves, as we indicated in our first lecture (this volume, pp. 76–88) an integration.

Our demand curves are assumed to be of the form

$$x_i = \alpha_i(p) + \beta_i(p)y . \tag{4}$$

From the symmetry of the Slutsky terms³, we require

$$\begin{aligned} \left(\frac{\partial x_i}{\partial p_j}\right) &= \frac{\partial \alpha_i}{\partial p_j} + \frac{\partial \beta_i}{\partial p_j}y + (\alpha_j + \beta_j y)\beta_i = \\ \left(\frac{\partial x_j}{\partial p_i}\right) &= \frac{\partial \alpha_j}{\partial p_i} + \frac{\partial \beta_j}{\partial p_i}y + (\alpha_i + \beta_i y)\beta_j . \end{aligned} \tag{5}$$

This must be true for all y , so that

$$\frac{\partial \beta_i}{\partial p_j} = \frac{\partial \beta_j}{\partial p_i} \tag{6}$$

and

$$\frac{\partial \alpha_i}{\partial p_j} - \alpha_i \beta_j = \frac{\partial \alpha_j}{\partial p_i} - \alpha_j \beta_i . \tag{6'}$$

(6a) implies that β_i is an exact differential, i.e., we can define

$$B = \oint \sum_i \beta_i dp_i \tag{7}$$

and

$$\frac{\partial B}{\partial p_i} = \beta_i . \tag{7'}$$

Similarly (6') implies that

$$e^{-B}\alpha_i$$

is an exact differential, i.e., we can define

³ See for instance [4] or [5], pp. 24–26.

$$\phi = \oint \sum e^{-B} \alpha_i \, dp_i \quad (8)$$

$$\phi_i = e^{-B} \alpha_i, \phi_{ij} = e^{-B} \left(\frac{\partial \alpha_i}{\partial p_j} - \alpha_i \beta_j \right) = \phi_{ji}. \quad (8')$$

Substituting into (4), we obtain

$$x_i = \phi_i e^B + B_i y. \quad (9)$$

Let

$$\ln b = -B;$$

then, $B_i = -b_i/b$

$$\phi_i e^B = \phi_i e^{-\ln b} = \frac{\phi_i}{b}.$$

Let

$$a_i = -\phi_i,$$

then

$$x_i = -\frac{a_i}{b} - \frac{b_i}{b} y, \quad (10)$$

which is exactly of the form (3').

The more interesting questions, however, involve what is called price uncertainty: the uncertainty about the prices at which we can buy and sell commodities in the future. Although the ultimate source of uncertainty may be in 'technology' – rainfall, inventions, etc. – variations in these technological variables are translated into variations in prices, and it is these price variations to which individuals must respond. A crop failure in Russia may increase the price of Canadian wheat, a bumper crop will decrease it.

These price variations affect portfolio allocations in a number of different ways.

First we have already noted that the utility that an individual receives from any given income depends on the price ratios.

Secondly, when prices change, not even money is a safe asset: its purchasing power, the consumption goods it can 'deliver', varies among the states of

nature. It no longer is meaningful to speak of a 'safe' asset. All the theorems derived earlier which depended on the existence of a safe asset have to be re-investigated, to see which can be modified appropriately to remain valid, and which must be rejected.

Thirdly, different securities are likely to be more closely correlated (positively or negatively) with different prices. Accordingly, the 'riskiness' of a particular asset will depend on the consumption pattern of the individual. An individual who only consumes 'housing' would find an 'asset' whose return was closely related to the price of housing to be relatively safe, while an individual who only consumes food might find a farm, an asset whose price is closely correlated with the price of food, a 'safe' investment. The effects of price variations will depend not only on the relative demands for different commodities, but the elasticities of substitution of one commodity for another. If an individual has a relatively elastic demand, then changes in price lead to a consumption reallocation, with relatively little effect on utility; but if an individual has an inelastic demand, changes in price are likely to affect him far more.

The problems introduced by having many periods are formally very similar to those introduced by having many commodities: consumption in different periods is very much like consumption of different commodities. There is one important difference arising from the sequential nature of intertemporal decisions.

In the following discussion, I shall limit myself to remarks on two topics: first, on the problem of certainty equivalents with price uncertainty; secondly, on the problem of hedging and speculation. The results obtained here will, as we shall note, have some interesting implications for the problem of the term structure of interest rates discussed at greater length in the previous lecture.

For simplicity, we shall consider a case with only two commodities, x_1 and x_2 ; x_1 will be our numeraire, the price of x_2 will be just p . $U(x_1, x_2)$ is assumed to be concave, and accordingly the indirect utility function, $V(p, y)$, is concave.

The individual has initial stocks of the two commodities, denoted by ${}_0x_1$ and ${}_0x_2$. Thus,

$$y = {}_0x_1 + p{}_0x_2. \quad (11)$$

p is a random variable, and hence y is also a random variable. We denote the mean of p by \bar{p} . It is natural to define the certainty equivalent income \hat{y} , as

$$EV(p, y) = V(\bar{p}, y) \quad (12)$$

we define $\bar{y} = {}_0x_1 + \bar{p}_0x_2$.

We wish to know, what is the relationship between \hat{y} and \bar{y} . To find out, we take a Taylor series expansion of both sides, to obtain (if the variations in p are small)

$$\begin{aligned} V(\bar{p}, \bar{y}) + V_y(\hat{y} - \bar{y}) &= V(\bar{p}, \bar{y}) + V_p(\bar{p}, \bar{y})E(p - \bar{p}) + V_y(\bar{p}, \bar{y})E(y - \bar{y}) \\ &\quad + V_{py}E(p - \bar{p})(y - \bar{y}) + \frac{V_{yy}}{2}E(y - \bar{y})^2 + \frac{V_{pp}}{2}E(p - \bar{p})^2. \end{aligned} \quad (13)$$

Hence

$$\bar{y} - \hat{y} \approx \frac{1}{2V_y} (V_{pp} + 2V_{py}{}_0x_2 + V_{yy}({}_0x_2)^2) E(p - \bar{p})^2. \quad (14)$$

To calculate V_{py} and V_{pp} , we differentiate (3), which in this case can be written

$$-V_p = x_2 V_y \quad (15)$$

to obtain

$$-V_{pv} = x_2 V_{yp} + V_y \frac{\partial x_2}{\partial p} \quad (16)$$

From the first order conditions for utility maximization we have

$$\begin{bmatrix} U_{11} & U_{12} & -1 \\ U_{21} & U_{22} & -p \\ -1 & -p & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dV_y \end{bmatrix} = \begin{bmatrix} 0 \\ V_y dp \\ x_2 dp - dy \end{bmatrix}. \quad (17)$$

From (17), (11), and the budget constraint

$$x_1 + px_2 = y$$

we obtain the following results:

$$V_{yp} = -V_{yy}x_2 - V_y \frac{\partial x_2}{\partial y}$$

$$V_{pp} = V_{yy}x_2^2 + V_y \left(x_2 \frac{\partial x_2}{\partial y} - \frac{\partial x_2}{\partial p} \right)$$

Let us consider some special cases of (14):

(I) ${}_0x_2 = 0$: all the initial endowment is in the numeraire. Then, without loss of generality, letting $p = 1$,

$$\begin{aligned} \frac{\bar{y} - \hat{y}}{y} &= \left\{ \left(\frac{x_2}{y} \right)^2 R + \left(\frac{d \ln x_2}{d \ln p} - \frac{d x_2}{d y} \right) \frac{x_2}{y} \right\} \frac{\sigma_p^2}{2} \\ &= \left\{ \left(\frac{x_2}{y} \right)^2 (R - 2\eta_{2y}) - \frac{x_2}{y} \eta_{2p} \right\} \frac{\sigma_p^2}{2} \end{aligned} \quad (18)$$

$$\text{where } -\eta_{2p} = \left(\frac{\partial \ln x_2}{\partial \ln p} \right)_u \text{ and } \eta_{2y} = \frac{\partial \ln x_2}{\partial \ln y}, \quad R = \frac{-V_{yy} \mathcal{Y}}{V_y}.$$

Thus, the 'insurance premium' an individual is willing to pay is

- (a) an increasing function of his relative risk aversion (at fixed prices);
- (b) an increasing function of the share of his income spent on the commodity;
- (c) a decreasing function of the compensated price elasticity, i.e. if the compensated price elasticity is small, it is hard to 'substitute' for other commodities, price variations are more costly, and hence the premium the individual is willing to pay to get rid of risk is greater;
- (d) a decreasing function of the income elasticity for the commodity.

The risk premium may be written in an even simpler way when the indifference map is homothetic. We define the elasticity of substitution as

$$\left(\frac{d \ln x_2 / x_1}{d \ln p} \right)_{u=\bar{u}} = \left(\frac{d \ln x_2}{d \ln p} \right)_u - \left(\frac{d \ln x_1}{d \ln p} \right)_u = -\epsilon.$$

But

$$\left(\frac{d \ln x_1}{d \ln p} \right)_u = \frac{p x_2}{-x_1} \left(1 - \left(\frac{d \ln x_2}{d \ln p} \right)_u \right).$$

Hence letting $p = 1$

$$-\epsilon = \frac{y}{x_1 x_2} \left(\frac{\partial x_2}{\partial p} \right)_u - \frac{x_2}{x_1}.$$

Hence

$$\frac{-V_{pp}}{yV_y} = \left(\frac{x_2}{y}\right)^2 \left[R - 1 - \frac{\epsilon x_1}{x_2} \right]. \quad (19)$$

We see quite explicitly that the smaller the elasticity of substitution the greater the risk premium the individual is willing to pay.

(II) The other simple polar case is where $x_2 = 0$. Then

$$\bar{y} - \hat{y} = \frac{1}{2} \left(\frac{dx_2}{dp} \right)_{\bar{p}} E(p - \bar{p})^2. \quad (20)$$

The premium appears to be *negative*.⁴ But if we turn to our indifference curve, this is just what we would expect. Both increases and decreases from $p = \bar{p}$ increase utility. It is not that the individual is not risk averse. Rather, there is both an income and a price effect of a price variation. For prices below $p = \bar{p}$, the individual is a net consumer, so further price decreases make him better off. For prices above $p = \bar{p}$, the individual is a supplier, so further price increases make him better off.

Now consider the problem of the individual trying to allocate a fixed wealth, w_0 , among the two commodities. He stores them for one period, after which he can trade them. Initially the price is unity. If he allocates a of his wealth to x_1 and $(1-a)$ to x_2 , his wealth (income) at the end of the period is just

$$y = [a + (1-a)p] w_0. \quad (21)$$

Thus, he chooses a to maximize

$$EV(p, (a + (1-a)p)w_0) \quad (22)$$

i.e.

$$E V_y(1-p) = 0. \quad (23)$$

⁴ Except if the indifference curves are L-shaped, when it is zero.

Let us define a *pure hedging position* as that allocation which maximizes the minimum level of utility attained in any state of nature. It is found by setting

$$U_1 = U_2 . \tag{24}$$

It has the property that, if the individual were not allowed to enter the market again, his utility would be maximized with the pure hedging position. In fact, the position described above, with $x_2 = {}_0x_2$ is, in fact the hedging position. There we have shown that any deviation of price from $p = \bar{p}$ makes him better off or at least not worse off. Hence the minimum level of utility attained is in fact U^* in figures 1 and 2. Note that all allocations allocating more of initial wealth to x_1 increases utility in all states of nature when $p < 1$, but decreases utility (relative to the pure hedging position) when $p > 1$.

This is seen more clearly for the case where there is zero elasticity of substitution between commodities. Then, in fact at the pure hedging position, the level of utility attained, is completely independent of p .

Now we establish that if $E_p = 1$ individuals will, in general, speculate on x_2 , i.e., purchase more x_1 than \hat{x}_1 , the 'hedging position'. To see this, we observe that at $a = \hat{a}$ (where \hat{a} denotes the hedging allocation)

$$E V_y(1-p) = E(V_y(p, y) - V_y(1, \hat{y})) (1-p) , \tag{25}$$

but

$$\frac{1}{V_y} \frac{dV_y}{dp} \sim V_{py} + V_{yy} {}_0x_2 \sim R \frac{(x_2 - {}_0x_2)}{x_2} - \eta_{2y} . \tag{26}$$

(where \sim denotes "is of the same sign as").

Clearly, for small variations of p around unity, this is negative. If R is less than unity, and the indifference map is homothetic, again this is negative. Finally, we note that if the utility function is additive,

$$x_1 \geq \hat{x}_1 \text{ as } p \geq 1 ,$$

so

$$V_y - V_y(1, \hat{y}) \geq 0 \text{ as } p \geq 1 . \tag{27}$$

The consequence of $(dV_y/dp) < 0$ or of (27) is that (25) is unambiguously positive, which in turn implies that, since $EV(p, y)$ is a concave function of \hat{a} , $a > \hat{a}$, i.e., the individual speculates on x_1 .

Will the individual ever 'specialize in x_1 ', i.e. purchases only x_1 ? Then

$$y = {}_0x_1 = w_0$$

and

$$\frac{dV_y}{dp} = V_{py} + V_{yy} {}_0x_2 \sim R - \eta_2 y.$$

If the indifference map is homothetic, then

$$a \geq 1 \quad \text{as} \quad R \leq 1.$$

If the individual is very risk averse, then he sets $\hat{a} < a^* < 1$, he speculates some, but if $R < 1$, the individual is not very risk averse, and he 'specializes' in x_1 , even though on average the future price is just equal to the present price.

As we noted above, neither of the two assets (${}_0x_1$ or ${}_0x_2$) is really safe. We can show under fairly weak assumptions for the case of homothetic indifference maps, however, that the commodity in which the individual is speculating acts like the 'risky' asset in the following sense: as wealth increases whether the individual speculates more or less depends on whether he has decreasing or increasing relative risk aversion; if he speculates more, he allocates a larger fraction of his total wealth to the commodity on which he is speculating.

It is clear that

$$\frac{da}{dw_0} \sim E V_{yy} y (1-p) = -E R V_y (1-p) \geq 0 \quad (28)$$

as

$$dR/dp \geq 0. \quad (28')$$

But

$$\begin{aligned} \frac{dR}{dp} &= \frac{\partial R}{\partial y} \frac{dy}{dp} + \frac{\partial R}{\partial p} \\ \frac{dy}{dp} &= {}_0x_2 \\ -\frac{\partial R}{\partial p} &= \frac{V_{yyp} y}{V_y} + \frac{R V_{yp}}{V_y} \end{aligned}$$

Recalling the results derived above, p. 114, we have

$$\frac{y}{x_2} \frac{dx_2}{dy} = - \left(\frac{V_{py}}{V_y} \right) \frac{y}{x_2} + R.$$

Hence, for a homothetic indifference map, we have

$$0 = - \frac{V_{pyy}y}{V_y} - \frac{V_{py}R}{V_y} + x_2 \frac{\partial R}{\partial y}.$$

Hence

$$\frac{dR}{dp} = ({}_0x_2 - x_2) \frac{\partial R}{\partial y}.$$

If the individual has L-shaped indifference curves, it is clear that if the individual speculates on, say, x_2 , then ${}_0x_2 > x_2$, regardless of the price. In that case

$$\frac{da}{dW_0} \geq 0 \text{ as } \frac{\partial R}{\partial y} (a - \bar{a}) \leq 0. \quad (29)$$

More generally, if the variation in price is small, or if the utility function is additive (so $R = -U_{11}y/U_1$, and all we need to compare is the levels of consumption of x_1 for $p > 1$ with those for $p < 1$), then (29) also obtains.⁵

As I suggested earlier, these results have a natural application to the problem of the term structure of interest rates.

The individual has an initial wealth w_0 . He can purchase short-term bonds, which mature in one period, paying $(1+r_1)$, or 'long-term' bonds, which mature in two periods, paying then $(1+R)$. The price of the long-term bond next period is uncertain, i.e. the short-term rate of interest that will prevail in the

⁵ When there are variations in both price and income, although the certainty equivalents (risk premiums) that we have defined above are undoubtedly the most natural, there are other certainty equivalents that could be defined. For instance, in comparing the situation with and without uncertainty, we have allowed the individual to take different actions. Some of the 'riskiness' of the portfolio is a result of the deliberate policy of speculation on the part of the individual. If, as we increase the variance, we change the mean of p in such a way that the individual continues to hold the pure hedging portfolio, we can obtain an expression for the resulting value of the risk premium.

future is unknown. His wealth at the end of the first period is given by

$$y = \left((1+r_1)a + \frac{(1+R)}{1+r_2} (1-a) \right) w_0 \quad (30)$$

where a is the proportion of his wealth allocated to short-term bonds, and $(1+R)/(1+r_2)$ is the price of a long-term bond next period (see figure 2). The individual then takes this wealth, and either consumes it then, or invests it in (short-term) bonds, consuming it at the end of the second period.⁶

He does this so as to maximize his two period utility function, $U = U(C_1, C_2)$, where C_i is consumption the i th period, and U is concave and monotone increasing in both of its arguments.

His budget constraint is given by

$$y = C_1 + \frac{C_2}{1+r_2} . \quad (31)$$

The pure hedging position is given by

$$\frac{U_2}{U_1} = \frac{(1+r_1)}{1+R} . \quad (32)$$

We now return to the question of the pattern of portfolio allocation when the long rate is equal to the product of the expected short, i.e.

$$E(1+r_1) (1+r_2) = (1+R) . \quad (33)$$

This says that if we were interested only in consumption the second period, on average we would have the same wealth at the end of the second period if we invested in a sequence of short-term bonds as we would have if we invested in a long-term bond.

One might have alternatively formulated the expectations hypothesis in terms of a 'one period holding period'. That is, the price of a long-term bond at the end of the period on average is equal to the gross return we would have obtained from holding a short-term bond

⁶ We could allow the individual at that point to make another portfolio decision; this complicates the analysis without altering the basic qualitative properties.

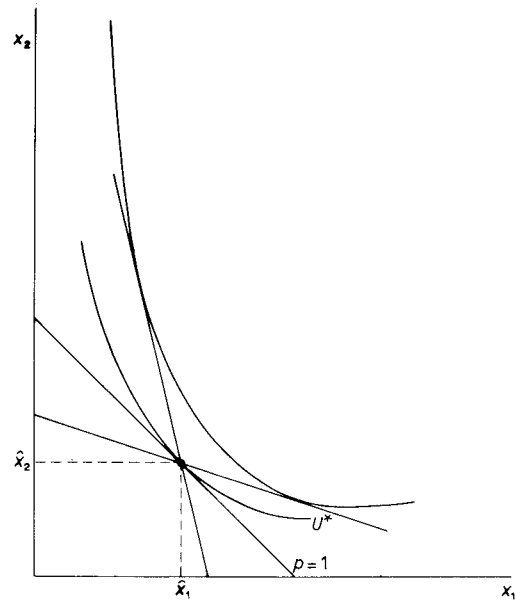


Fig. 1a.

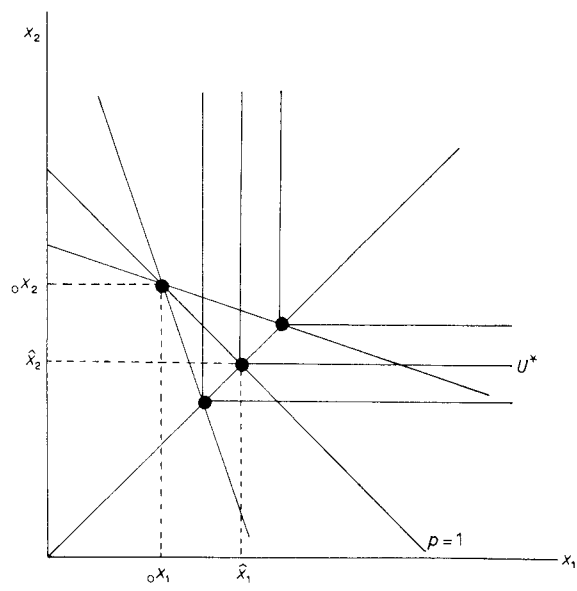


Fig. 1b.

(Since $V_Y = V_Y(1+r_2)$, it is clear that (35) and (35') are identical.) Now, if $E(1+r_2) = (1+R)/(1+r_1)$, i.e. (33) obtains, then by exactly analogous arguments to those used above, individuals will speculate on long-term bonds, and specialize in long-term bonds if they are not very risk averse ($R < 1$).

Let us reiterate what these results show: even though the long rate is equal to the product of the expected short rates, the individual speculates in long-term bonds and may even specialize in them, contrary to the presumption of much of the literature, which suggests that if the long rate is just equal to the product of the expected short rates no one would want to purchase long-term bonds.

There are three reasons that our results differ from these of the conventional analysis. First, we have formulated a consumption oriented, rather than a capital valuation oriented model of the demand for financial assets. The capital value of a long-term bond, say a consol, is variable — but the income stream which a consol delivers is not. It is, we have argued, consumption with which individuals are concerned, not capital values for their own sake. In terms of consumption one period hence, a one period bond is safe and a long-term (two period) bond is risky, but in terms of consumption two periods hence, the sequence of one period bonds is risk and the long-term bond is safe. Secondly, we have noted in the previous lecture the negative correlation between the value of long-term bonds and the return to equities. Even if we ignore equities, long-term bonds act like insurance. If the individual were to purchase only short-term bonds, then the price of long-term bonds would be negatively correlated with utility: when the individual is badly off, long-term bonds pay off well and when the individual is well off, they pay off badly.

Thirdly, when the long rate is equal to the expected product of short rates, the expected return from holding a long-term bond one period is still greater than the safe return — the individual is still being compensated for the one period capital risk:

$$E \frac{(1+R)}{(1+r_2)} > \frac{1+R}{E(1+r_2)} = (1+r_1)$$

(since $1/(1+r_2)$ is a convex function of r_2).

Earlier, we showed how Arrow's theorem depended crucially on there being only two assets. From the results that we have derived above, it is clear that even if there are only two assets, it depends on their being only one period (or one commodity). In this context, the 'liquid' asset (money) is the short-term asset. Whether the elasticity of demand for the short-term asset is

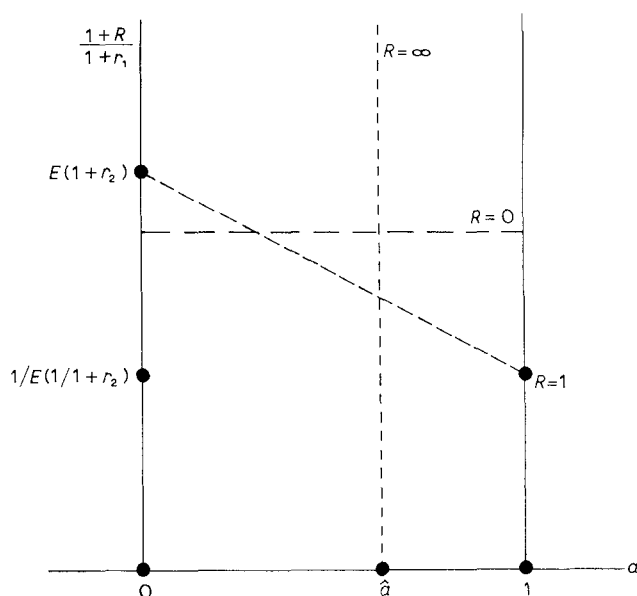


Fig. 2.

greater or less than unity depends on whether there is increasing or decreasing relative risk aversion *and* on whether the individual is speculating in long-term or short-term assets.

There is much more to say on these questions: we have yet to discuss the important concepts of 'liquidity' and 'transactions costs' and how they affect portfolio behavior, and we have yet to imbed these results on portfolio analysis into a general equilibrium model, showing how consumers attitudes affect firms investment decisions. These will have to await another occasion.

References

- [1] J.E. Stiglitz, Behavior towards risk with many commodities, *Econometrica* 37 (1969).
- [2] J.E. Stiglitz, Taxation, risk taking and the allocation of investment, in: *Studies in the theory of capital markets* (ed. by M.C. Jensen), New York, Praeger Publisher, 1972.
- [3] J.E. Stiglitz, A consumption-oriented theory of the demand for financial assets and the term structure of interest rates, *Review of Economic Studies* 37 (1970).

- [4] E.E. Slutsky, On the theory of the budget of the consumer, *Giornale degli Economisti* 51 (July 1915) pp. 1–26. Reprinted in Am. Eco., Ass., *Readings in Price Theory*, Homewood, Irwin, 1952, pp. 27–56.
- [5] J.M. Henderson and R.F. Quandt, *Microeconomic theory: a mathematical approach*, New York, McGraw Hill, 1958.