

## PORTFOLIO ALLOCATION WITH MANY RISKY ASSETS

### 3. The term structure of interest rates

Joseph STIGLITZ\*

As I indicated at the end of the previous lecture [1], the purpose of this lecture is to explore some of the implications of changes in relative prices of different assets, particularly those engendered by the kinds of policy changes often discussed in macro-economics. To do this, I shall revert from the rather general discussion of the previous two lectures to some 'special cases', which I hope will illustrate the principal points at issue.

In particular, I shall focus my discussion on the case where the distribution of returns of the various assets is described by a multivariate normal distribution; this allows us to use mean-variance analysis, and to employ the familiar mean-variance diagrams.

The great advantage of the mean-variance model is that there are some very simple relations which must exist in equilibrium between the values of various assets. If  $X_i(\theta)$  is the return from the  $i$ th security in state  $\theta$  (i.e. the return from all the shares of the given security;  $X_i(\theta)$  includes both the dividend and the change in the value of the shares over the period),  $V_i$  is the value of the all the shares of the  $i$ th security, then

$$V_i = \frac{EX_i - k E \sum_j (X_i - \bar{X}_i)(X_j - \bar{X}_j)}{r} = \frac{V_i + EX_i - k E \sum_j (X_i - \bar{X}_i)(X_j - \bar{X}_j)}{1+r}, \quad (1)$$

where  $r$  is the safe rate of return and  $k$  is the 'risk discount factor'.

Let us begin our discussion by reviewing the special case of two assets: a safe, short-term bond and a consol. The opportunity locus is the straight line  $SR$ , and the equilibrium is at point  $E$ .

\* Professor of Economics, Yale University, New Haven, Conn., U.S.A.



$r$  makes the representative individual better off,<sup>2</sup> which leads him to increase or decrease his demand for the risky asset as he has decreasing or increasing absolute risk aversion. On the other hand, the increase in the return of the safe asset relative to the risky tends to increase the demand for the safe asset. Hence, it is unambiguously clear that if the individual has increasing absolute risk aversion, he increases his demand for the safe asset. In the other case, the outcome depends on the relative strength of the two effects.

Let us pursue in more detail the 'normal' case where the demand for the safe asset has increased. There is now an excess demand for the safe asset. This means that the price of the consol decreases; the wealth of the individual decreases if he already owns some consols. Hence the amount the individual *thinks* he can purchase if he purchases only safe (short-term) bonds, decreases, but the number of long-term bonds he can purchase, if he purchases only long-term bonds, increases. In figure 1  $S'$  moves down to  $S''$ , while  $R$  moves up along a ray through the origin to  $R''$ . (His mean income, if he purchased only risky assets, would be  $(\bar{X}/\bar{V})(V+B)$ , while the standard deviation would be  $(\bar{V}_X)(V+B)$ , where  $V$  is the aggregate value of risky assets and  $B$  that of bonds. As  $V$  decreases, both the mean and standard deviation increase, but their ratio remains constant at  $\bar{X}/\bar{V}_X$ .) Equilibrium (figure 1) is attained when the slope of the opportunity locus equals the slope of the indifference curve at  $E''$ . ( $E''$  is the mean and standard deviation of total income when the entire stock of wealth is taken into account; it differs from  $E$  only by the increase in the mean return resulting from the increase in  $r$ .)

Note that a change in the return on the safe asset will in general entail a change in the price (and hence return) for the long-term (risky) asset.

Can we say anything (in this simple model) about which changes more? Traditional Keynesian theorem seems to have suggested that the price of the long-term bond moves inversely proportional to the short-term rate of interest. (I shall have more to say about this later.) Under what conditions will this be true?

If the value of long-term bonds moves inversely proportional to that for short, it is equivalent to assuming that  $k$  is constant (if the distribution of  $X$  is unchanged) (in (1)). A proportionate change in  $r$  will then induce less a than a proportionate change in the mean income from an 'all safe' portfolio. More im-

<sup>2</sup> For simplicity we assume all individuals are identical, and maximize the expected utility of the return from their portfolio which they will receive next period; we are explicitly ignoring the multi-period aspects of this problem.

portantly if  $V$  moves inversely to  $r$ , the opportunity line moves out in parallel. Hence, provided that there is decreasing absolute risk aversion (provided that the wealth elasticity of demand for risky assets is positive), there will be excess demand for risky assets. Equilibrium requires that the price of the long-term bonds decrease *less than proportionately* to the increase in  $r$ .

We referred in the previous lecture to the question of the term structure of interest rates: the relationship between long-term and short-term interest rates.

If there were no uncertainty and no costs of transaction, then the long-term rate of interest would have to be the product of the short-term rates of interest. If a  $t$  year bond pays  $1 + R_t$  at the end of  $t$  periods, and  $r_t$  is the short-term (one period) rate of interest between  $t-1$  and  $t$

$$(1+r_1)(1+r_2)(1+r_3) \dots (1+r_t) = 1+R_t. \quad (2)$$

Now assume the left side were greater than the right. Individuals would sell, short (issue) the long-term bonds, invest the proceeds in short-term bonds, and wind up at the end of  $t$  periods with a pure profit of

$$(1+r_1)(1+r_2) \dots (1+r_t) - (1+R_t). \quad (3)$$

There would thus be an 'infinite' demand for short-terms and an infinite supply of long-term bonds, decreasing the return from the short and increasing that from the long, until equilibrium were attained. Similarly, if the right-hand side of (2) were greater than the left, individuals would borrow short and lend long, again making a pure profit. Only if (2) holds with equality can there be equilibrium.

The question arises: in the presence of uncertainty can any simple relationship of the form (2) be established? What we have said already should have made clear that in the presence of uncertainty, unless individuals are risk neutral the demands for different assets will depend not only on the mean returns, but also on variances, etc.

It would, accordingly, be rather surprising if a relationship such as (2) held with equality. Some authors have argued nonetheless that even though individuals are not neutral towards risk (i.e.  $U'' < 0$ ), there are enough opportunities for portfolio diversification that individuals will only buy a small amount of any given security, and hence equilibrium will still require (2) to hold (at least approximately) with equality. Other authors have suggested that the simple relation (2) be replaced by an inequality:

$$E(1+r_1)(1+r_2)(1+r_3) \dots (1+r_t) < 1+R_t . \tag{4}$$

The long-term rate of interest must be greater than the expected product of short-term rates, the difference depending on the degree of risk aversion. It is argued that since long-term bonds are ‘riskier’ than short, they must sell at a discount.

This formulation has ignored the coupon payments of long-term bonds. The requisite modification for, for instance, the case of a consol paying £1 every period are simple. The individual would be indifferent between holding a consol for one period or holding a short-term bond if

$$\frac{1}{p} + \frac{\Delta p}{p} = r , \tag{5}$$

where  $p$  is the price of the consol. If the expected change in the price of a consol is zero, this means that

$$p = 1/r . \tag{6}$$

Eq. (6) may be looked at in another way. The present discounted value of the coupon payments is (in the absence of uncertainty) just

$$\frac{1}{1+r_1} + \frac{1}{(1+r_1)(1+r_2)} + \frac{1}{(1+r_1)(1+r_2)(1+r_3)} \dots \tag{7}$$

If we denote  $(1 + R_\infty) = 1/p$  as the rate of return on a consol, then if the present discount value of coupon payments is to be equal to the price,

$$p = \frac{1}{1+R_\infty} = \sum_{t=1}^{\infty} \prod_{j=1}^t (1+r_j) . \tag{8}$$

In the case of static expectations, where  $r_1 = r_2 = \dots = r$ , (7) is a simple geometric series, and yields (6), or

$$p = \frac{1}{1+R_\infty} = \frac{1}{r} .$$

If because of uncertainty, long-term bonds are supposed to sell at a discount (as required by (4)), then

$$p < 1/r.$$

Finally, still other authors have argued that these markets are essentially segmented, and therefore there is no simple relation between the long-term and short-term bonds. Their argument is based essentially on institutional considerations, that buyers of long-term bonds tend to be insurance companies, buyers of short-term assets tend to be banking institutions who require greater 'liquidity'.

This is not the place for an extended discussion of these various positions, which we have presented all too briefly. There is, undoubtedly, some truth in the segmentation hypothesis; yet the real question is whether at the margin there are enough individuals and institutions who do purchase both; for if there are such individuals, and if, say, the long-term rate exceeds the expected product of the short-term rates by 'very much', they would move into long-term bonds, and, conversely, if the long-term rate were less than the expected product of the short-term rates, then these individuals could ensure that an inequality such as (4) were maintained, or indeed, even that an equality of the form (3) were maintained, at least approximately. Moreover, the demand for long-term assets by insurance companies and of short-term assets by other institutions are derived demands: derived from the demand for the securities offered for sale by these institutions. And the demand for these securities will in turn depend on the returns which they can offer, which, in a competitive economy, will depend in turn on the returns these institutions receive from their investments. Hence, if the long-term rate is low, it may be true that insurance companies may not switch to short-term bonds, but individuals may do relatively less of their saving through insurance programs and more through banks.

One of the major objects of this lecture is to establish that simple relations of the form (3) or (4) need not hold; but that we need not appeal to the segmentation hypothesis for the rejection of these hypotheses.

Ultimately, of course, these are empirical questions, but unfortunately, as we pointed out in the previous lecture, empirical questions which are extremely difficult to verify.

Our concern here is primarily deductive: what can we say about the term 'structure of interest rates' (i.e. the relationship between long-term and short-term assets) on the basis of simple assumptions about portfolio allocation under the hypothesis of expected utility maximization?

To return to our simple model: we have established the following two results:

(1) If the long-term bond is to be held in the portfolio, it must have a higher mean return than the short-term bond. This must mean that on average the price of a consol must be less than  $1/r$  when capital gains are expected to be zero. Eq. (1) makes this clear: if the risk discount factor is positive,  $V < X/r$ . If the expected change in the price of a consol (on an average annual basis) is small, (which it is likely to be over any long-term period when calculated on an annual basis)  $X$  is just the coupon (= £1).

(2) As  $r$  increases,  $V$  decreases less than in proportion (if the relative supplies of the two assets remain unchanged).

But these results were predicated on the two asset — one safe and one risky (the long-term bond) — model, and we shall now show that these results must be seriously modified as soon as we take into account the fact that there are also equities.

Empirically, it has been observed that the price of long-term bonds is negatively correlated with the business cycle — the prices of bonds go up in depressions and down in booms. Indeed, the whole basis of Keynes' argument for a liquidity trap [2] is based on this observed phenomenon.<sup>3</sup> The consequence of this is that long-term bonds and equities are negatively correlated.

It then becomes clear from (1), that whether  $V_c \leq X_c/r$  (where  $V_c$  and  $X_c$  are the value of mean return to consols) depends on whether the correlation of long-term bonds with the total collection of risky assets (equities plus long-term bonds) is negative or positive, i.e. whether the own variance is greater or smaller than the negative correlation with equities. Since income from long-term bonds is relatively small compared to that from the stock market, and the negative correlation with the rest of the market seems fairly large, the total effect seems likely to be negative (Merton Miller's results [3] tend to corroborate this). Accordingly, the mean variance diagram for the three asset case probably looks something like as depicted in figure 2, with  $R$  the optimal proportion of equities and long-term bonds.

Now, when  $r$  increases, if the risk discount factor  $k$  remained unchanged, so that the value of both long-term bonds and equities moved inversely, then just as before, the new opportunity locus would have moved up in parallel, and there would be an excess demand for risky assets. This means that the value of risky assets as a whole must decrease by less than inversely proportionately to  $r$ , i.e.  $k$  the risk discount factor, must decrease. The decrease in  $k$  means that (from (1)) the price of long-term bonds must decrease more than proportionately to the increase in  $r$ .

<sup>3</sup> Certain recent experiences in the U.S.A. through some doubt on the inevitability of this inverse relationship.

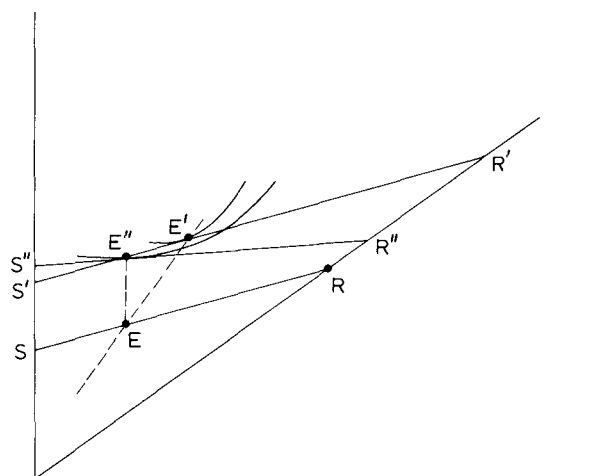


Fig. 2.

When the relationship between equities and long-term bonds is taken into account, both of the propositions established earlier are reversed: if the price next period is expected to be the same as the price this period, then *the price of a consol is greater than  $1/r$  and the price is more volatile than the short-term rate of interest.*

So far, we have assumed that the expected return from long-term bonds and from the equities will remain unchanged as  $r$  changes. But as Keynes argued, when the price of the consol goes up as a result of the decrease in the rate of interest, individuals may expect in the future that the price will go down, so the expected return may decrease. Let us take the extreme case, the expected price next period is independent of the price today; i.e. a increase in the price today by a unit decreases the expected return by a unit. Then if  $k$ , the risk discount factor, were constant,

$$\left( \frac{d \ln V_R}{d \ln r} \right)_{k=\bar{k}} = \frac{r}{1+r},$$

where  $V_R$  is the total value of bonds plus equities.

The percentage change in the value of any risky asset from a percentage change in  $r$  goes to zero as  $r$  goes to zero.

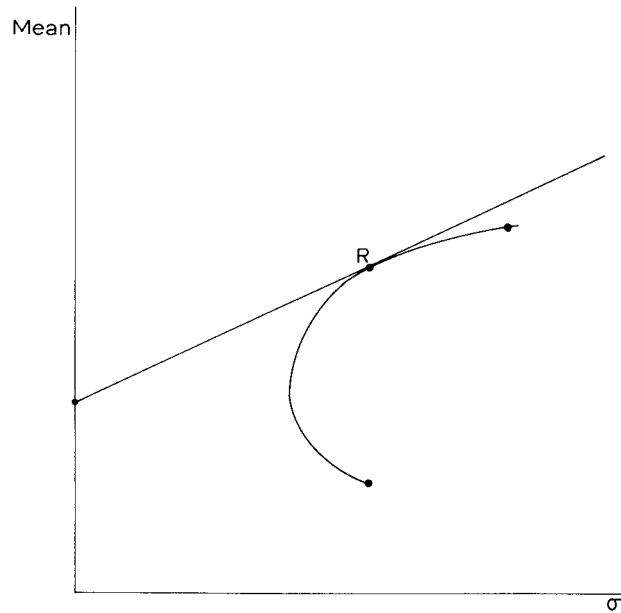


Fig. 3.

For small  $r$ , then, a decrease in  $r$ , keeping  $k$  constant, would change the opportunity set<sup>4</sup>, as depicted in figure 3.

It is still possible if not likely, however, that this will increase the demand for the risky assets, leading to an excess supply of bonds, an excess demand for the risky assets, which in turn leads to a lower  $k$  and a still lower price of long-term bonds.

**Concluding comments**

Because Keynes [2] lumped together long-term bonds and equities, what happened to the long-term rate of interest was crucial for the determination of the level of investment. Now that we have disaggregated, we see that the two may move in opposite directions. The example here is a case in point: the decrease in  $k$  may more than offset the effects of an increase in  $r$ , resulting in an increase in the value of equities. For investment purposes, the return to equity is probably more relevant than the long-term rate of interest. It is possible, in the above situation, for an increase in the short-term rate of interest to lead to an increased demand for investment.

<sup>4</sup> We let  $p_t$  be described by the stochastic process  $p_t = p^* + \epsilon_t$  where  $p^*$  is the long run normal price. Then the mean return to a consol is  $p^* - p_t$ , and the standard deviation is  $\sigma_\epsilon$ .

However, we have shown that the short term rate and the long term rate need not move together; that the price of a consol will not in general be equal to the reciprocal of the short rate of interest even if individuals have static expectations (provided they are risk averse; and that many of the conclusions of the two asset model (as in Tobin [4]) must be seriously modified when a third asset, e.g. the price of a consol may now exceed the reciprocal of the short rate, whereas in the two asset case it was always less than  $Y_r$ .

There are innumerable other questions which can be posed in terms of the simple model that we have presented here: what, for instance, are the consequences of a change in an open market operation? But rather than attempt to answer these questions here, let me note that this is still a one period model; it may have given us insight into the problem of the term structure of interest rates, but that problem necessarily involves several periods, and it is probably more worthwhile to attempt to extend our framework than to develop this model further.

This will be the subject of our next lecture.

### References

- [1] J. Stiglitz: 'Portfolio allocation with many risky assets. Lecture 2: alternative characterization of portfolios, in the present work, pp. 89–98.
- [2] J.H. Keynes, *The general theory of employment, interest and money*, New York, Harcourt Brace, 1936.
- [3] M. Miller, 'Portfolio theory and the term structure of interest rates', unpublished manuscript.
- [4] J. Tobin, 'Liquidity preference as behavior towards risk', *Review of Economic Studies* 25 (1958), pp. 65–86.