

PORTFOLIO ALLOCATION WITH MANY RISKY ASSETS

2. Alternative characterizations of portfolios

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There are commonly employed three ways of characterizing portfolios:¹

(1) A portfolio could be described by certain of its statistical properties, e.g. its mean, its variance, the mean of its rate of return, the variance of its rate of return, the range of returns, etc.

(2) A portfolio could be described by its allocation to certain assets or to certain groups of assets, e.g. to money (the safe asset) or to 'risky assets' or to 'equities' as a group and 'long-term bonds' as a group.

(3) A portfolio could be described by its certainty equivalent, e.g. we could ask: what sure rate of return would give the individual the same utility that the given portfolio gives him?

This third characterization is a much less satisfactory characterization than the other two; for it is much less obvious how one could go about verifying propositions about it. In this respect, the second characterization is the most satisfactory: we can simply see how individuals allocate their portfolio, and in particular how the portfolio allocation does changes with wealth.²

The first has the problem that the relevant statistical properties are based on subjective distributions; if these are closely related to past distributions or if they correspond closely ('on average') with the actual distributions observed, then propositions concerning statistical properties of the portfolio may be tested. Unfortunately, it is extremely difficult to test the extent to which 'subjective' estimates do correspond to the 'empirically observed' distributions. An example related to a problem which will be the subject of our fourth lec-

¹ This lecture is based on joint work with D. Cass reported at greater length in [3].

² This is not to say that even here it is easy to construct meaningful tests. Individuals at different levels of wealth probably do face different opportunity sets; it is certainly true that their costs of transactions (per unit) differ.

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ture may help illustrate this: there is an hypothesis that the long-term interest rate is equal to the product of the expected short-term interest rates. This is a statement about the *mean returns* of different assets in equilibrium. When this is not satisfied in terms of observed means do we say (a) the hypothesis is incorrect, e.g. because of risk aversion individuals will hold long- and short-term assets only if the long-term rate is greater than the product of the expected short-term rates, or (b) the hypothesis is correct, but individuals' subjective estimates systematically differed from what subsequently was observed.

Let me put the matter another way: we could observe two portfolios held by two different individuals, consisting of exactly the same assets, but in terms of their characterization by 'statistical properties' or by certainty equivalents, the two portfolios might very well differ.

It is for this reason that portfolio theory has been much more concerned with establishing propositions about demands for particular assets or groups of assets, than with either of the other two characterizations.

Arrow [4] has shown that the elasticity of demand for the safe asset is greater or less than unity, as relative risk aversion increases or decreases with wealth. This is an exciting proposition, both because of its simplicity and because it seems to provide a testable proposition. Other betting behavior, or betting experiments, might reveal whether relative risk aversion increased or decreased with wealth.

But in the two asset case, when the relative proportion of wealth held in the safe asset increases, the mean rate of return decreases and the variance of the rate of return increases. Indeed, if one wants a safer portfolio — one with a lower mean and variance of the rate of return — the only way to get it is to purchase relatively more of the safe asset. In this sense, for the two asset case, the characterization in terms of relative demands for safe and risky assets and the characterization in terms of statistical properties are perfectly equivalent.

But this equivalence no longer holds when there are three or more assets. Which of these alternative characterizations is the 'correct one' then? Which is, in this sense, the more 'fundamental' characterization?

To examine this question, we return to the special case where there are as many securities as states of nature. Corresponding to this set of securities is a set of Arrow-Debreu securities giving exactly the same opportunity set. Accordingly, if we are interested in what happens, say, to the mean and variance of the rate of return, we can as well look at the demands for the Arrow-Debreu securities as we can at the demands for the original securities.

The mean and variance of the rate of return can be written

$$\bar{r} = \sum_{\hat{a}_i}^* \hat{\rho}_i \pi_i \quad \sigma_r^2 = \sum (\hat{a}_i \hat{\rho}_i - \bar{r})^2 \pi_i \quad (1)$$

where a is the relative proportion of the portfolio allocated to the different Arrow-Debreu securities.

We shall now show that the mean and variance of the rate of return do increase or decrease as relative risk aversion is a decreasing or increasing function of wealth: the ‘statistical characterization’ generalizes to the many asset case, when the number of assets is equal to the number of states of nature.

The result follows immediately from the following

Lemma:

$$\frac{d\hat{a}_i}{\hat{a}_i dW_0} \leq \frac{d\hat{a}_j}{dW_0 \hat{a}_j} \text{ as } R' \quad (\pi_i \hat{\rho}_i - \pi_j \hat{\rho}_j) \leq 0, \quad (2)$$

where $R'(W)$ is the measure of relative risk aversion.

Proof: Recall from the previous lecture [1] that for the Arrow-Debreu securities market

$$U'(\hat{a}_i \hat{\rho}_i W_0) \pi_i \hat{\rho}_i = \lambda. \quad (3)$$

Taking the total differential of (3), we have

$$U'' W_i \left(\frac{d\hat{a}_i}{\hat{a}_i} + \frac{dW_0}{W_0} \right) \pi_i \hat{\rho}_i = d\lambda. \quad (4)$$

Dividing (4) by (3), we obtain

$$\frac{U'' W_i}{U'} \left(\frac{d\hat{a}_i}{\hat{a}_i} + \frac{dW_0}{W_0} \right) = \frac{d\lambda}{\lambda}. \quad (5)$$

Recalling that

$$-U'' W/U' = R$$

(and denoting $-U''(\hat{a}_i \hat{\rho}_i W_0) \hat{a}_i \hat{\rho}_i W_0 / U'(\hat{a}_i \hat{\rho}_i W_0)$ by R_i), this may be rewritten

$$\frac{d\hat{a}_i}{dW_0} = -\frac{d\lambda}{\lambda dW_0} \frac{\hat{a}_i}{R_i} + \frac{\hat{a}_i}{W_0}. \quad (6)$$

Summing over i ,

$$0 = \Sigma \frac{d\hat{a}_i}{dW_0} = -\frac{d\lambda}{\lambda dW_0} \left(\Sigma \frac{\hat{a}_i}{R_i} \right) + \frac{\Sigma \hat{a}_i}{W_0}$$

or

$$\frac{W_0}{\lambda} \frac{d\lambda}{dW_0} = \frac{1}{\Sigma \hat{a}_i/R_i} \quad (7)$$

Substituting back into (6), we obtain

$$\frac{W_0}{\hat{a}_i} \frac{d\hat{a}_i}{dW_0} = 1 - \frac{1/R_i}{\Sigma \hat{a}_j/R_j} \leq 0, \quad (8)$$

as $1/R_i$ is less than or greater than the *weighted* average of $1/R_i$, where the weights are the ratios in which the different assets are purchased.

But from (3)

$$\hat{a}_i \hat{\rho}_i \leq \hat{a}_j \hat{\rho}_j \text{ as } \pi_i \hat{\rho}_i \leq \pi_j \hat{\rho}_j, \quad (9)$$

so

$$R_i \leq R_j \text{ as } R'(\pi_i \hat{\rho}_i - \pi_j \hat{\rho}_j) \leq 0, \quad (10)$$

(2) follows immediately from (8) and (10). The proof that

$$\frac{d\bar{r}}{dW_0} \leq 0 \text{ as } \frac{d\sigma_r}{dW_0} \leq 0 \text{ as } R' \leq 0 \quad (11)$$

follows immediately; we shall sketch the proof for the mean, and leave the variance as an exercise.

$$\begin{aligned} W_0 \frac{d\bar{r}}{dW_0} &= \Sigma \hat{\rho}_i \pi_i \frac{d\hat{a}_i}{dW_0} W_0 = \Sigma \hat{\rho}_i \pi_i \hat{a}_i \left(1 - \frac{1/R_i}{\Sigma \hat{a}_j/R_j} \right) \\ &= \Sigma \hat{\rho}_i \pi_i \hat{a}_i \left(\frac{1/\tilde{R} - 1/R_i}{\Sigma \hat{a}_j/R_j} \right) - \Sigma \hat{\rho}_i \pi_i \hat{a}_i \left(1 - \frac{1/\tilde{R}}{\Sigma \hat{a}_j/R_j} \right) \\ &= \left(\Sigma_i (\hat{\rho}_i \pi_i - \tilde{X}) \hat{a}_i \right) \left(\frac{1/\tilde{R} - 1/R_i}{\Sigma \hat{a}_j/R_j} \right), \end{aligned} \quad (12)$$

where $1/\hat{R}(\hat{W}) \equiv \Sigma \hat{a}_i / R_{i\hat{W}}$ and $\hat{X} \equiv \lambda / U'(\hat{W})$; from (9) and (10), when $\pi : \rho : > \hat{X}$ and $\hat{X} \equiv \lambda / U'(\hat{W})$; $R_i \leq \hat{R}$ as $R' \leq 0$. The result is immediate.

What about the second – what he have argued above as the most interesting – characterization, that in terms of the demands for particular assets. To see how for instance the demand for money changes with wealth, we must reconvert our demands for Arrow-Debreu securities into demands for the original securities: letting ρ be the matrix of returns for the original securities and a_i be the proportion of wealth allocated to the i th security

$$a \rho = \hat{a} \hat{\rho} \tag{13}$$

$$a = \hat{a} \hat{\rho} \rho^{-1}$$

$$\frac{da}{dW_o} = \frac{d\hat{a}}{dW_o} \hat{\rho} \rho^{-1} \tag{14}$$

where it will be recalled from the previous lecture [1]

$$1/\hat{\rho} = \rho^{-1} \cdot e$$

(when e is the unit vector).

a_i is a kind of weighted average of $\{\hat{a}\}$, so changes in a are weighted averages of changes in \hat{a} . Since some of the $d\hat{a}_i/dW_o$ are positive and some are negative, it would seem possible that regardless of the sign of R' , da_m/dW_o could be positive, zero, or negative. And in fact, it is possible to construct examples showing this.²

Even under the highly restrictive assumption of as many states as securities, it is not possible to extend Arrow's theorem. The import of this result should be clear: only under the highly restrictive assumptions presented in lecture 1, when an aggregate of the risky securities may be formed, is it possible to obtain a general theorem about the relation between the elasticity of demand for the safe asset and the dependence of relative risk aversion on wealth.

The attempt to derive from portfolio analysis general theorems about the demands for particular assets without imposing severe restrictions on either the asset structures and/or the utility functions seems to have come to a dead

²I shall not discuss these examples at length here; their construction is rather tedious for several reasons: first, as (14) and (15) indicate, they require several matrix inversions; secondly, we must check in (13) that $a > 0$. The interested reader is referred to the Cass-Stiglitz paper [3] for a detailed exposition.

end: nor is it surprising that general theorems are not to be had. The property of the utility function which made it possible to derive simple results about Arrow-Debreu securities was that the utility function was additive in these securities. But for general securities, which may be viewed as linear combinations of these Arrow-Debreu securities, the utility function is not additive in these securities.

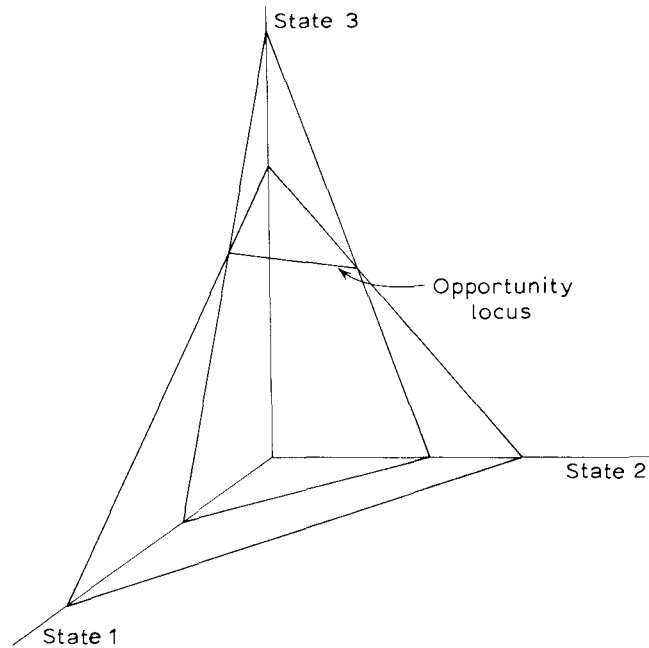


Fig. 1.

It is for very similar reasons that an attempt to extend our theorems using the first characterization – the statistical properties of the portfolio – to cases where there are more states of nature than securities also fails.

The effects of having fewer securities than states of nature are equivalent to the effects of having point rationing in addition to price rationing. For instance, if there were three securities and three states of nature, the opportunity set would be a hyperplane. If, in addition, we had point rationing, we would be restricted to the intersection of two hyperplanes, i.e. a straight line, as depicted in figure 1. Changes in wealth are equivalent to equiproportionate changes in dollar and 'point' incomes.

Utility clearly will not be additive in the individual securities. To see why simple theorems about the behavior of the statistical properties are no longer derivable, consider the simplest possible case: two securities, three states of nature. One security gives pay offs in states one and two, the other only in state three. The mean rate of return is simply

$$\bar{r} = a(\rho_1 \pi_1 + \rho_2 \pi_2) + \rho_3 (1 - \pi_1 - \pi_2)(1 - a), \quad (15)$$

where a is the proportion invested in the first security; hence

$$\frac{d\bar{r}}{dW_0} = (\rho_1 \pi_1 + \rho_2 \pi_2 - \rho_3 (1 - \pi_1 - \pi_2)) da/dW_0. \quad (16)$$

The first-order condition for utility maximization is just

$$U'(W_1) \pi_1 \rho_1 + U'(W_2) \pi_2 \rho_2 = U'(W_3) \pi_3 \rho_3. \quad (17)$$

Taking the total differential of (17), and dividing by (17) we obtain

$$\frac{da}{dW_0} \approx R_3 - (\gamma R_1 + (1 - \gamma)R_2)$$

where $\gamma = U'(W_1) \pi_1 \rho_1 / U'(W_1) \pi_1 \rho_1 + U'(W_2) \pi_2 \rho_2$, and $R_i = U''(W_i) W_i / U'(W_i)$.

Hence, whether a increases or not depends on whether the weighted average of relative risk aversion in those states when the first security pays off is less than or greater than the relative risk aversion in the state when the second security pays off. Clearly, by choosing our parameters correctly, we can make

$$a \rho_1 > (1 - a) \rho_3 > a \rho_2.$$

If $R' > 0$, then $R_1 > R_3 > R_2$. We can find utility functions for which

$$R_3 = \gamma R_1 + (1 - \gamma) R_2,$$

i.e. even though the individual has increasing relative risk aversion, the mean and variance of his portfolio is unchanged (at least at one value of W_0) as wealth changes; by slight changes in the values of the parameters, the mean and variance may increase or decrease. Indeed, it is even possible for them to move in opposite directions.

We thus come to the sad conclusions that although propositions about the

first characterisation may be extended beyond the two asset case, they do not extend to the general case where there are more states of nature than securities.

This leaves us with our third characterization in terms of certainty equivalents. Let us define the certainty equivalent rate of return:

$$EU(r^* W_0) = \max_{\{a\}} EU(\sum a_i \rho_i W_0). \quad (18)$$

We shall now show that

Theorem

$$\frac{dr^*}{dW_0} \geq 0 \quad \text{as } R' \leq 0.$$

The safe rate of return which would give us the same utility as yielded by the optimally chosen portfolio increases or decreases with wealth as relative risk aversion decreases or increases with wealth.

This theorem does not depend on there being two assets, or as many securities as states of nature. Indeed, it does not even depend on the existence of a safe asset.

Proof: Let $a^*(W_0)$ be the optimal portfolio at W_0 :

$$EU'(\sum_{j \geq 2} a_j^* (\rho_j - \rho_1) + \rho_1 W_0)(\rho_i - \rho_1) = 0, \quad i = 2, \dots, n. \quad (19)$$

Taking the total differential of (18)

$$U'(r^* dW_0 + dr^* W_0) = \frac{EU' W_0}{W_0} dW_0 + EU' (\rho_j - \rho_1) W_0 \frac{da_j^*}{dW_0}$$

or using (19)

$$W_0^2 \frac{dr^*}{dW_0} = EU' W - U' r^* W_0.$$

Since $U' > 0$, we may view W as a function of U , $W = W(U)$, and $U'(W(U))$ as a function of U . Let $U^* = U(r^* W_0)$

$$EU' W \geq E \left\{ U' r^* W_0 + (U - U^*) \frac{dU' W}{dU} \right\} = U' r^* W_0$$

as $U'W$ is a convex or concave function of U . But

$$\frac{dU'W}{dU} = \frac{\frac{dU'W}{dW}}{\frac{dU}{dW}} = \frac{U''W+U'}{U'} = 1 - R$$

$$\frac{d^2U'W}{dU^2} = -R'/U'$$

Hence

$$\frac{dr^*}{dW_0} \cong 0 \quad \text{as } R' \leq 0 .$$

A few concluding remarks:

Most of the theorems discussed in the last two lectures made use of the concept of 'relative risk aversion'. There is also a simple theorem for the two asset case was proved using absolute risk aversion: the demand for risky asset increases or decreases with wealth as the individual has decreasing or increasing absolute risk aversion. Similarly, it can be shown for the two asset case that the mean and the variance of the total return increase or decrease with wealth as the individual has decreasing or increasing absolute risk aversion, and that the difference between the mean return and the certainty equivalent return increases or decreases as the individual has decreasing or increasing absolute risk aversion. Again, it can be shown that if there are as many assets as states of nature, the theorem about the statistical properties of the portfolio remains valid, but that about the aggregate demand for risky assets does not. If there are more states of nature than securities, even the theorem about statistical properties ceases to be valid. But in all cases, the results about certainty equivalents hold.

So far, we have lifted only half of the first assumption discussed in the first lecture: we have introduced many assets, but have kept the relative price of those assets unchanged. Yet many of the policy questions in macro-economic analysis involve changes in the relative prices of different assets e.g. changes in the price of bonds relative to the price of equities.

The third lecture pursues the implications of these price changes.

References

- [1] J. Stiglitz, 'Portfolio allocation with many risky assets. Lecture 1: Portfolio separation theorem' in the present work, p. 76–88.
- [2] D. Cass and J. Stiglitz: 'The structure of investor preferences and asset returns and separability in portfolio allocation: a contribution to the pure theory of mutual funds', *Journal of Economic Theory* 2 (1970).
- [3] D. Cass and J. Stiglitz, 'Risk aversion and wealth effects on portfolios with many assets', *Review of Economic Studies*, July, 1972.
- [4] K.S. Arrow, *Essays in the theory of risk-bearing*, Amsterdam, North-Holland Publ. Co., 1970.
- [5] J. Pratt, 'Risk aversion in the small and in the large', *Econometrica*, January – April, 1964, pp. 122–136.